MATH 4305 - Ordinary Differential Equations II
Project 1 - Motion of a Falling Object with Wind Resistance
Fall 2015 - Dr. Clarence Burg

Introduction

In Calculus I, we learned about the ideal motion of a object falling under the influence of gravity, which can be described by the following system of two differential equations:

$$\frac{ds}{dt} = v$$
$$m\frac{dv}{dt} = -mg$$

where s is the position of the object, v is its velocity, m is its mass and g is the force of gravity. In this system, the upward direction is positive, so the velocity of a falling body is becoming more negative; and since the equation has $-mg$ for the gravitational force, the constant g is positive. (All constants should be positive when modeling physical quantities.) The first equation is obtained from the definition of velocity, which is the instantaneous rate of the change of position. The second equation is derived from Newton's Laws of Motion, which states that the mass times the acceleration experienced by an object is equal to the sum of the forces acting upon the object. In the ideal case, only gravity is acting upon the object.

To complete this system of equations, two initial conditions are needed, which are typically the initial position s_0 and the initial velocity v_0 of the object. With these initial conditions, the solution to the initial value problem is

$$s(t) = s_0 + v_0 t - g\frac{t^2}{2}$$
$$v(t) = v_0 - gt$$

To obtain this solution, the velocity equation is solved first and then the position equation is solved. This ideal system for a falling object does not include any terms for wind resistance.

The simplest model for wind resistance is to add a viscous force to the sum of the forces acting on the falling object. Thus, the system of equations for falling objects with wind resistance can be modeled as

$$\frac{ds}{dt} = v$$
$$m\frac{dv}{dt} = -mg - sign(v)k|v|^p$$

where $sign(v)k|v|^p$ is the model for viscous, wind resistance force and $sign(v)$ returns either positive or negative based on the sign of the velocity v. The wind resistance force should be greater when the velocity is larger and should be 0 when the velocity is 0, so a power function representation makes sense. Furthermore, we must be careful to ensure that this force is acting to slow to the object. Thus, if the velocity v is positive, the viscous force must be negative, in order to slow down the object; if the velocity is negative, then the viscous force must be positive. Hence, the velocity and the viscous force must have the opposite sign, leading to the negative for the viscous force in the equation.

The wind resistance for an object is dependent on several factors including its density, the smoothness of its surface and the shape of the object. These factors will influence the choice of the viscous coefficient k and the exponent p.

In this project, we will look at two different models for wind resistance, which are defined as

1. Linear Model ($p = 1$): Viscous force is kv
2. Square Model ($p = 2$): Viscous force is $sign(v)kv^2$

We will assume an initial velocity of $v(0) = v_0$.

There are several different methods for find the solution to the linear model. To find the solution to the square model, use separation of variables and then use the substitution $u = v\sqrt{k/(mg)}$. Recall the following two indefinite integrals

$$\int \frac{1}{u^2+1}du = \arctan(u)$$
$$\int \frac{1}{u^2-1}du = \frac{1}{2}\ln\left(\frac{1-u}{1+u}\right)$$

You may use technology to check your solution, but you must show the details of your derivations in a hand-written (or typed) appendix to your project report.

Project Report

Each section of your report must be typed. Please write in complete sentences. Describe and introduce the equations, graphs and tables that you include in the report. Make references in the typed portion of the report to the appendices, so that the reader can find further details about the derivations. The appendices can be hand-written. The project should consist of the four sections as outlined below:

Section 1: Solution for Falling Objects Without Wind Resistance

In this section, you should present the governing differential equations for falling objects without wind resistance, along with the initial conditions for both position and velocity. You should include the general solution for both position and velocity. In the appendix, include a section that provides the step-by-step solution to the system of differential equations.

Section 2: Solution for Falling Objects with Linear Wind Resistance

In this section, you should present the governing differential equations for falling objects with linear wind resistance, along with the initial conditions for velocity. You should include the general solution for velocity. In the appendix, include a section that provides the step-by-step solution to the velocity differential equation for falling objects with linear wind resistance.

Section 3: Solution for Falling Objects with Quadratic Wind Resistance

In this section, you should present the governing differential equations for falling objects with quadratic wind resistance, along with the initial conditions for velocity. For this problem, we will assume that the object is falling, so the velocity is negative and the velocity equation is

$$m\frac{dv}{dt} = -mg + kv^2$$

You should include the general solution for velocity. In the appendix, include a section that provides the step-by-step solution to the velocity differential equation for falling objects with quadratic wind resistance.

Section 4: Comparison of Models

Plot the velocity solution to the three models from $t = 0$ to $t = 10$ seconds for a 10 kg object, assuming gravity is 9.8 m/sec^2, with the initial velocity as $v(0) = 0$. For the Linear Model, assume $k = 1$. For the Quadratic Model, assume $k = 0.01$. Because of wind resistance, the velocity for the linear and quadratic model should be less negative than for the no wind resistance case.

Provide a table as shown below:

Time	No Resistance	Linear Model	Quadratic Model
0	0	0	0
2			
4			
6			
8			
10			
100			

Describe the long term behavior of the models. Is there a maximum velocity for the linear and quadratic models? If so, determine the value of the maximum velocity from the solution, algebraically, in terms of m, k and g.

The report for Project is due on Friday, September 11, 2015.