MATH 4305 - Applied Mathematics I
Project 3 - Militarization of Neighboring Countries
Fall 2015 - Dr. Clarence Burg

Introduction

Lewis Fry Richardson (and several other mathematicians) developed models for the growth of the military capability of countries before and during World War I and were based on conventional military encounters before the dawn of flight and the development of nuclear weapons. The development of aircraft and the atomic bomb changed the way that battles are fought, so these models are not as effective for the military development within countries in light of modern warfare capabilities.

In the simplest form, two neighboring countries gradually become more militarized (i.e., have more troops and weapons), primarily because they observe that the other country is becoming more militarized. Thus, to maintain a strong defense, each country must devote more resources to the military, creating a spirally arms race. The rate of militarization is based on the cost of the militarization, the increase in the military forces of the enemy and the motivation of the people to accept the greater militarization. Richardson and others used the term "grievances" to describe the impact of this motivation.

In mathematical terms, let x_1 be the amount of militarization of the first country and x_2 be the amount of militarization of the second country. Then, the two differential equations for the growth of each country's military are

$$\frac{dx_1}{dt} = kx_2 - \alpha x_1 + g$$
$$\frac{dx_2}{dt} = lx_1 - \beta x_2 + h$$

where α and β are the respective costs of militarization, k and l are the influence of the other country's militarization and g and h are each country's motivation to support militarization (i.e., the grievances that each country has against the other country). These coefficients are positive constants. Also, note that $x_1(t)$ and $x_2(t)$ are positive (or 0) for all $t > 0$, where $t = 0$ is the initial time.

In matrix form, the system of differential equations can be written as

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}$$

where

$$A = \begin{bmatrix} -\alpha & k \\ l & -\beta \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} g \\ h \end{bmatrix}$$

In this project, you will find the fixed point for this system of equations and determine when this fixed point is realistic. You will also determine the eigenvalues of the system of equations; from these eigenvalues, you will determine when the fixed point is stable and when it is unstable. You will also find the exact solution to the system of equations.

Your project report should include the following:

1. A brief description of the motivation of Richardson's development of this model. For what time period and for what countries did Richardson develop this model?

2. Write a four nation system of equations in variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, where each country is in an arms race with each of the other countries.

3. Assume that nations 1 and 2 are allies, that nations 3 and 4 are allies and that allies are not in an arms race with each other. How would the impact of these alliances be modeled?

4. Determine the fixed points for the two nation model, and call them (x^*, y^*). Since the coefficients are positive and the amount of militarization is positive, what constraint must be placed on these coefficients to ensure that the fixed point is positive?

5. Determine the eigenvalues of the matrix equation for the two nation model. Show that the two eigenvalues must be real. One eigenvalue is clearly negative, since it is the summation of negative values. Is the second eigenvalue positive or negative? Based on your conclusion, classify this fixed point as either stable, unstable or semi-stable.

6. Determine the eigenvectors of the matrix equation for the two nation model. Use $D = (\alpha - \beta)^2 + 4kl$, to simplify your derivation (i.e., D for discriminant).

7. Determine the Fundamental Solution Matrix e^{At} for the two nation model.

8. From this matrix, determine the exact solution for the two nation model, using the initial conditions $x_1(0)$ and $x_2(0)$.

Remember that the solution to the differential equation has the form

$$\vec{x}(t) = e^{At}\left(\vec{x}(0) + A^{-1}b\right) - A^{-1}b$$

Group the components of the solution in terms of $(e^{\lambda_1 t} - e^{\lambda_2 t})$ and $(e^{\lambda_1 t} + e^{\lambda_2 t})$.

The report for Project 3 is due on Wednesday, December 2 , 2015.