MATH 4340 - Numerical Methods
Project 2 - Numerical Integration
Fall 2015 - Dr. Clarence Burg

In this report, you will study the error and the accuracy of several different numerical methods for estimating the value of integrals numerically. You will use the formula for the observed order of accuracy

$$p = \frac{\ln\left(\frac{N(r^2h)-N(rh)}{N(rh)-N(h)}\right)}{\ln(r)}$$

to determine the observed order of accuracy of these methods. Once the implementation has been verified by showing that the observed order of accuracy agrees with the theoretical order of accuracy, you can use Richardson extrapolation on the numerical results to increase the order of accuracy. The Richardson extrapolation formula using a refinement ratio of r on a numerical method of order p is

$$N_{RE}(h) = \frac{r^p N(h) - N(rh)}{r^p - 1}$$

The Romberg algorithm uses Richardson extrapolation as the basis for the algorithm.

Project Report

Your project report should include the following 5 sections.

Section 1: Applications.

Just as with differentiation, the calculation of the integral of a function is one of the basic concepts of Calculus, especially since it is related to the derivative through the Fundamental Theorem of Calculus. Unlike the derivative, there are functions for which the Fundamental Theorem of Calculus does not apply because those functions do not have anti-derivatives. One such example is $f(x) = e^{-x^2}$. Thus, one application of numerical integration is the approximation of integrals for which the Fundamental Theorem of Calculus does not apply. What are the other applications of numerical integration? Just as numerical differentiation is quite widely used within numerical approximations of the solution to ordinary differential equations and to partial differential equations, numerical integration can be applied in these contexts, because these equations can often be write as integral equations.

Section 2: Description of the mathematical concepts and algorithms.

You need to include the numerical approximations used within this report, along with the **leading order error term** for each of these approximations. These approximations include the following rules:

1. Composite Trapezoid Rule
2. Composite Simpson's Rule
3. Composite Boole's Rule

Please make sure to define all terms, such as h and N, within your report.

Please provide the formulas used for Richardson extrapolation, as well.

Section 3: Translation of mathematical algorithm into pseudo-code

In class, we developed the pseudo-code for the composite trapezoid rule, but not for the other two rules. Please include pseudo-code for the **composite Simpson's rule** and for the **composite Boole's rule**.

Section 4: Verification Results

For the code verification process, we will determine the observed precision of our implementation by integrating the monomial from x^0 to x^8 and comparing them with the exact answers.

Using 4 intervals, please use your code to calculate these integrals over the interval $[0, 1]$ and complete the following chart:

Monomial	Exact answer	Trapezoid Rule	Simpson's Rule	Boole's Rule
$x^0 = 1$				
x^1				
x^2				
x^3				
x^4				
x^5				
x^6				
x^7				
x^8				

What is the largest monomial that can be exactly integrated by each method? Obviously, your results should match the theoretical precision of each method.

Section 5: Validation Results

This section will focus on the observed order of accuracy of these techniques, which should agree with the theoretical order of accuracy. You will also use the Romberg Algorithm (i.e., Richardson extrapolation) to improve the numerical results.

If you are using R, C/C++, Java or some other programming language, you need to report all 16 digits of your numerical result.

Problem 1: Estimate the value of the following integral using the composite trapezoid rule, composite Simpson's rule and composite Boole's rule:

$$\int_0^4 \frac{1}{x^3+x+1}\, dx$$

using 4, 8, 16, 32, 64 and 128 intervals. Tabulate your results in the following table, one for each method. Show that the observed order of accuracy for each method is consistent with its theoretical order of accuracy.

Intervals N	h	$N(h)$	p
4	1		NA
8	0.5		NA
16	0.25		
32	0.125		
64	0.0625		
128	0.03125		

Problem 2: Use the composite trapezoid rule to estimate the value of

$$\int_0^5 \frac{20\sin^4(x)+e^x}{30-12\cos(x)+e^x}\, dx$$

using 1, 2, 4, 8 and 16 intervals. Use the composite Simpson's rule for this function, using 2, 4, 8 and 16 intervals, and use the composite Boole's rule for this function using 4, 8 and 16 intervals. Calculate the observed order of accuracy for each method, and then calculate improved results for each method, using Richardson extrapolation, using a table such as

Intervals N	h	$N(h)$	p	$N_{RE}(h)$	p_{RE}
4	1		NA	NA	NA
8	0.5		NA		NA
16	0.25				NA
32	0.125				
64	0.0625				
128	0.03125				

What can you observe about the relationship between these methods?

Problem 3: Consider the following integral:

$$\int_0^1 e^x \, dx = e^1 - 1$$

We obviously know the value of this integral, so we will use this value to study the number of terms needed by the composite trapezoid rule, the composite Simpson's rule and the composite Boole's rule to reduce the error below 10^{-12}. Using the error terms for each method and using the fact that all of the derivatives of $f(x) = e^x$ are e^x, which is an increasing function, determine the number of intervals needed to guarantee that the error is below 10^{-12}. Then, use this number to estimate the value of the integral and the error in this calculation. Finally, show that the actual error is below 10^{-12} and determine how far below the error bound the calculation actually is (as a percentage of the error bound). In other words, complete the following steps for each of the three numerical integration methods:

1. Determine the theoretical number of intervals N to guarantee that the error is below 10^{-12}.

2. Use that N value within the numerical integration method to estimate the value of the integral numerically.

3. Calculate the absolute error in the numerical calculation.

4. Calculate the percentage relative difference between the absolute error and the error bound of 10^{-12}.

In other words, determine the following ratio:

$$\frac{|\text{absolute error} - 10^{-12}|}{10^{-12}} * 100\%$$

The Project Report for Project 2 is due on Monday, October 13, 2015.