MATH 4340 - Numerical Methods
Project 3 - Numerical Differential Equations
Fall 2015 - Dr. Clarence Burg

In this report, you will study the error and the accuracy of several different numerical methods for estimating the solution to differential equations numerically. You will use the formula for the observed order of accuracy

$$p = \frac{\ln\left(\frac{N(r^2h)-N(rh)}{N(rh)-N(h)}\right)}{\ln(r)}$$

to determine the observed order of accuracy of these methods. Once the implementation has been verified by showing that the observed order of accuracy agrees with the theoretical order of accuracy, you can use Richardson extrapolation on the numerical results to increase the order of accuracy. The Richardson extrapolation formula using a refinement ratio of r on a numerical method of order p is

$$N_{RE}(h) = \frac{r^p N(h) - N(rh)}{r^p - 1}$$

Project Report

Your project report should include the following 5 sections.

Section 1: Applications.

Just as with numerical differentiation and numerical integration, you need to provide a justification for the use of numerical methods for the solution of differential equations. In other words, why do we need to use these methods? One obvious answer is that many interesting differential equations do not have exact solutions, which is similar to one of the justifications for numerical integration when the anti-derivative of the integrand does not exist.

Based on your past experience, please include at least three applications of differential equations which can only be solved via numerical methods. We will approximately solve the Able equation in section 5, but I do not know the physical applications from which this equation is derived. One real-world application of a differential equation that can only be solved via numerical methods is the falling body problem where the frictional drag on the object depends on its velocity in a nonlinear manner. For this problem, the differential equation is a system of two equations involving position $h(t)$ and velocity $v(t)$ or

$$h'(t) = v(t)$$
$$mv'(t) = -mg + f(v)$$

where $f(v)$ is a nonlinear function of the velocity. If $f(v) = \alpha v$ or $f(v) = \alpha\sqrt{v}$, there are techniques for solving the system exactly. But if $f(v) = \alpha v^p$ where p is a decimal such as 0.62, then there is no exact solution, and a numerical solver is required.

Please include this one, and discuss two other applications from your differential equations textbook or online sources.

Section 2: Description of the mathematical concepts and algorithms.

You need to include the following numerical methods for solving differential equations, including the local truncation error terms:

1. Euler Method
2. Modified Euler Method
3. Midpoint Method
4. Heun Method
5. 3-step Runge-Kutta Method (one-third version)
6. 4-step Runge-Kutta Method

Please make sure to define all terms within your report.

Section 3: Translation of mathematical algorithm into pseudo-code

Please include pseudo-code for the **Euler Method**, for the **Modified Euler Method** and for the **4-step Runge-Kutta Method**.

Section 4: Verification Results

For the code verification process, we will determine the observed precision of our numerical differential equation solvers for two different differential equations and compare them with their theoretical orders of accuracy. We will select differential equations for which we know an exact answer.

For each equation, use 50, 100, 200 and 400 time steps with each of the six methods listed in Section 2 to approximate the solution at the final time. Tabulate your results and calculate the observed order of accuracy in table similar to the following table:

N	$N(h)$	p
50		
100		
200		
400		

Logistic Equation for Population Growth Modeling

The first differential equation is the logistic equation from population growth modeling, which is

$$\frac{dP}{dt} = ay - by^2$$

with initial population $P(t_0) = P_0$. The solution to this initial value problem is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-a(t-t_0)}}$$

This equation has physical significance and is widely used to model population growth, when there is a limiting or carrying capacity of $P(\infty) = \dfrac{a}{b}$. However, this equation is too simple to use as a thorough verification problem because the function in the differential equation only depends on y and not on the time t.

For the numerical solution, use $a = 1$ and $b = 0.01$ and estimate the value at $t = 5$ with initial conditions of $P(0) = 3$, using the 6 different methods and tabulate your results as stated above.

Falling Objects with Linear Wind Resistance

For the second verification problem, consider the following differential equation for the velocity of a falling object

$$m\frac{dv}{dt} = -mg - kv$$

with initial conditions $v(0) = v_0$, where m is the mass of the object, g represents the influence of gravity on the object and k is the coefficient of linear wind resistance. We will assume that the upward direction is positive and that the object is falling, so that the velocity is negative. The general solution to this differential equation is

$$v(t) = -\frac{mg}{k} + \left(\frac{mg}{k} + v_0\right) e^{-kt/m}$$

Use the 6 different numerical methods to estimate the solution at $t = 10$ to the initial value problem for $v(0) = 0$, and tabulate your results as stated above, under the assumptions that $g = 9.8$ meters per second squared, $m = 10$ kg and $k = 0.5$ kg/second.

Section 5: Validation Results

Numerical Solution of the Able Equation

The Able equation is defined as

$$\frac{dy}{dt} = \frac{y+t}{yt}$$

This equation has no known exact solution, so we must estimate the solution numerically.

Using the initial condition $y(1) = 1$, use 50, 100, 200 and 400 time steps to estimate the value of the solution at $t = 10$, using the Euler method, the Modified Euler method, the 3-step Runge-Kutta method and the 4-step Runge-Kutta method. Calculate the observed order of accuracy to ensure that the results are consistent with the theoretical order of accuracy of each method.

Falling Objects with Nonlinear Wind Resistance

For the second validation problem, consider the following differential equation for the velocity of a falling object

$$m\frac{dv}{dt} = -mg - k\sqrt{|v|}$$

with initial conditions $v(0) = v_0$, where m is the mass of the object, g represents the influence of gravity on the object and k is the coefficient of linear wind resistance. We will assume that the object is falling, so that the velocity is negative.

Use the Euler method, Modified Euler method, the 3-step Runge-Kutta method and the 4-step Runge-Kutta method to estimate the solution at $t = 10$ to the initial value problem for $v(0) = 0$, and tabulate your results as stated above, under the assumptions that $g = 9.8$ meters per second squared, $m = 10$ kg and $k = 0.5$ kg/second, using 50, 100, 200 and 400 time steps. Calculate the observed order of accuracy to ensure that the results are consistent with the theoretical order of accuracy of each method.

The Project Report for Project 5 is due on Wednesday, November 3, 2015.