APPENDIX E

NORMAL FLOW IN TWO DIMENSIONS

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E.1. Introduction

Normal flow can be defined as the flow for which frictional losses are exactly offset by gains in kinetic energy through a drop in bed elevation. As a result, normal flow remains unchanged as it travels down the channel. In one-dimensional flow, the frictional effects of the channel bed and walls are determined via Manning's equation. This equation relates the channel roughness, channel surface and velocity of the flow with the frictional loss coefficient S_f, via

$$Q = \frac{C_o}{n} A R^{2/3} S_f^{1/2} \tag{E.1.1}$$

where Q is the total channel discharge, n is Manning's friction coefficient, C_o is a dimensional constant, A is the channel's cross-sectional area and R is the hydraulic radius. For the rectangular channels studied in this thesis, Manning's equation can be expressed by

$$Q = \frac{C_o}{n} \frac{(bh)^{5/3}}{(b+2h)^{2/3}} S_f^{1/2} \tag{E.1.2}$$

where b is the channel width and h is the depth of flow. Manning's equation was developed via experimental data and has been used within one-dimensional and two-dimensional numerical modeling.

In one-dimensional flow, normal flow occurs in supercritical flow when S_f is equal to S_o, which is the channel slope. In two-dimensional flow, the Reynolds stresses increases the overall frictional losses across the channel. These stresses result from frictional interactions within the flow due to variations in velocity and thus are not modeled in one-dimensional flow. If Manning's equation is used to describe bed and wall friction within the two-dimensional, governing equations, the normal flow predicted by Manning's equation will be wrong, because of the additional frictional forces resulting from the Reynolds stresses.

As the frictional loss coefficient S_f increases, normal depth decreases for the same discharge Q. Thus, normal depth within the two-dimensional model should be less than for the one-dimensional model. Numerical experiments, using HIVEL2D as the simulation code for the shallow water equations, have demonstrated that Manning's equation overestimates normal depth and introduces non-physical oscillations into the flow as shown in Figure E.1

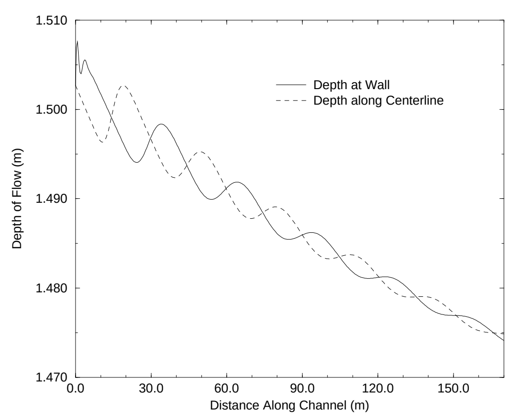

Figure E.1. Depths Using Manning's Equation.

In this appendix, the shallow water equations are reduced to a single ordinary differential equation by the use of normal flow assumptions. The resulting equation defines the viscous velocity profile across the channel. By using a power series expansion, a closed form solution is derived. This velocity profile is used as the inflow velocity for the test cases presented in this thesis.

E.2. Derivation of Differential Equation

The steady shallow water equations can be expressed as

$$\begin{aligned} \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} &= 0 \\ \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2 - h\sigma_{xx}\right) + \frac{\partial}{\partial y}\left(huv - h\sigma_{xy}\right) &= -gh\frac{\partial z}{\partial x} - \frac{gn^2u\sqrt{u^2+v^2}}{C_o^2 h^{1/3}} \\ \frac{\partial}{\partial x}\left(huv - h\sigma_{yx}\right) + \frac{\partial}{\partial y}\left(hv^2 + \frac{1}{2}gh^2 - h\sigma_{yy}\right) &= -gh\frac{\partial z}{\partial y} - \frac{gn^2v\sqrt{u^2+v^2}}{C_o^2 h^{1/3}} \end{aligned} \quad \text{(E.2.1)}$$

where h is the depth of the flow, u and v are the velocities in the x and y directions, respectively, g is gravity, and z is the bed elevation. The channel is aligned so that flow will be in the positive x direction, as shown in Figure E.2.

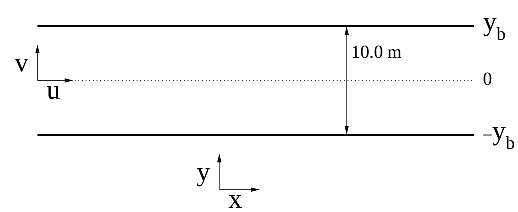

Figure E.2. Straight Wall Channel Configuration.

The Reynolds stresses σ_{xx}, σ_{xy}, σ_{yx} and σ_{yy} are expressed as

$$\begin{aligned} \sigma_{xx} &= 2\nu_t\frac{\partial u}{\partial x} \\ \sigma_{xy} = \sigma_{yx} &= \nu_t\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \sigma_{yy} &= 2\nu_t\frac{\partial v}{\partial y} \end{aligned} \quad \text{(E.2.2)}$$

where ν_t is the kinematic viscosity. For the shallow water equations, ν_t is assumed to be linearly dependent on the velocity magnitude $\sqrt{u^2+v^2}$ **[70]**. Thus, $\nu_t = \alpha\sqrt{u^2+v^2}$. For HIVEL2D, $\alpha = C_b n h^{11/6}\sqrt{8g}$.

The three boundary conditions that define the slipwall are

$$
\begin{aligned}
(hu)n_x + (hv)n_y &= 0 \\
(hu^2 - h\sigma_{xx})n_x + (huv - h\sigma_{xy})n_y &= -\frac{gh^{2/3}un^2\sqrt{u^2+v^2}}{C_o^2} \\
(huv - h\sigma_{yx})n_x + (hv^2 - h\sigma_{yy})n_y &= -\frac{gh^{2/3}vn^2\sqrt{u^2+v^2}}{C_o^2}
\end{aligned}
\tag{E.2.3}
$$

Since normal flow does not change as it travels down the channel, the depth h and the velocities u and v are not dependent on x. Furthermore, to prevent a change of shape in the flow profile across the channel, the transverse velocity v is set to zero, the transverse slope $\partial z/\partial y$ is set to zero, and the depth is held constant. Under these assumptions, the longitudinal velocity u is only a function of y. Applying these assumptions, the viscous stresses reduce to

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{yy} = 0 \\
\sigma_{xy} &= \sigma_{xy} = \nu_t\frac{\partial u}{\partial y} = \alpha u\frac{\partial u}{\partial y}
\end{aligned}
\tag{E.2.4}
$$

where α is now a constant. The shallow water equations become a single ordinary differential equation

$$
\frac{d^2u^2}{dy^2} = \frac{2g}{\alpha}\frac{\partial z}{\partial x} + \frac{2gn^2u^2}{\alpha C_o^2 h^{4/3}} \tag{E.2.5}
$$

The slipwall boundary conditions under the above assumptions can be reduced to a single equation

$$\frac{du^2(\pm y_b)}{dy} = -\frac{2gn^2u^2(\pm y_b)}{\alpha C_o^2 h^{1/3}} \tag{E.2.6}$$

where $\pm y_b$ are the locations of the channel walls.

Making the substitution $f = u^2$, the ordinary differential equation becomes

$$\frac{d^2 f}{dy^2} = \frac{2g}{\alpha}\frac{\partial z}{\partial x} + \frac{2gn^2}{\alpha C_o^2 h^{4/3}} f = A + Bf \tag{E.2.7}$$

where $A = \frac{2g}{\alpha}\frac{\partial z}{\partial x}$ and $B = \frac{2gn^2}{\alpha C_o^2 h^{4/3}}$. The boundary condition states

$$\frac{df(\pm y_b)}{dy} = -hBf(\pm y_b) \tag{E.2.8}$$

E.3. Analytic Velocity Profile

Due to reflection about the centerline, it is clear that $f(y)$ is an even function. Thus, the power series expansion is

$$f(y) = a_0 + a_2 y^2 + a_4 y^4 + \cdots = \sum_{n=0}^{\infty} a_{2n} y^{2n} \tag{E.3.1}$$

Plugging into equation (E.2.7) and equating similar terms yields

$$\begin{aligned} a_2 &= \frac{A + Ba_0}{2} \\ a_4 &= \frac{2Ba_2}{4!} \\ a_{2n} &= \frac{2B^{n-1}a_2}{(2n)!} \end{aligned} \tag{E.3.2}$$

The boundary condition, which is equation (E.2.8), becomes

$$\sum_{n=0}^{\infty} 2na_{2n}y_b^{2n-1} = -hB\left(a_0 + \sum_{n=1}^{\infty} a_{2n}y_b^{2n}\right) \tag{E.3.3}$$

Solving for a_0 and using the expression for a_{2n} yields

$$a_0 = -\frac{2a_2}{hB}\left[\sum_{n=1}^{\infty}\left(h\frac{B^n}{(2n)!}y_b^{2n} + \frac{B^{n-1}}{(2n-1)!}y_b^{2n-1}\right)\right] = -\frac{2a_2}{hB}C \tag{E.3.4}$$

where C is the result of the summation. Using the first relationship in equation (E.3.2) to solve for a_0 yields

$$\begin{aligned} a_0 &= -\frac{AC}{B(h+C)} \\ a_2 &= \frac{Ah}{2(h+C)} \\ a_4 &= \frac{BAh}{4!(h+C)} \\ a_{2n} &= \frac{B^{n-1}Ah}{(2n)!(h+C)} \end{aligned} \tag{E.3.5}$$

Plugging into equation (E.3.1), the analytic expression for the velocity profile can be expressed in closed form as

$$u^2(y) = -\frac{AC}{B(h+C)} + \frac{Ah}{h+C}\sum_{n=1}^{\infty}\left(\frac{B^{n-1}}{(2n)!}y^{2n}\right) \tag{E.3.6}$$

where A, B and C are defined in equations (E.2.11), (E.2.12) and (E.3.4), respectively.

To determine normal depth from equation (E.3.6), given the total discharge Q, one can solve the following equation for h

$$Q = h\int_{-y_b}^{y_b} u(y)dy \tag{E.3.7}$$

Using this velocity profile to determine the inflow velocity profile, nearly normal flow is established immediately as shown in Figure E.3. Furthermore, these derivations show that a constant depth flow is a solution to the shallow water equations under certain circumstances. As a result, an objective function that measures the non-uniformity of the depth has the property that it can equal zero, under certain circumstances.

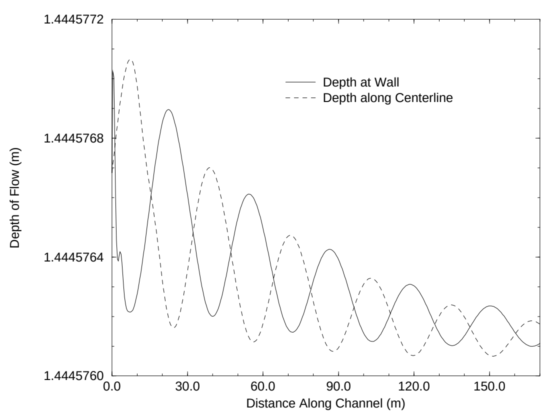

Figure E.3. Depths Using Velocity Profile.