APPENDIX F

THE FLOW SOLVER HIVEL2D

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F.1. The Finite-Element Implementation

HIVEL2D is a finite element solver of the two-dimensional, depth-averaged, shallow water equations as applied to open-channel flow. In particular, the code is well suited for high-velocity or supercritical flow, although it also solves the governing equations for subcritical flow. The version of HIVEL2D used in this research solves the flow equations for rectangular channels. Because of the dominant hyperbolic nature of the governing equations, information travels in the streamwise directions. Thus, to weigh the equations upstream along the characteristics of the flow, a Petrov-Galerkin test or weight function is used. Furthermore, HIVEL2D has a shock detection procedure but since it is not used in this research, the shock detection procedure is not discussed herein.

The shallow water equations are expressed in equation E.2.1, the slipwall boundary conditions are defined in equation E.2.3 and the inflow and outflow boundary conditions are described in Table F.3.1. The system of equations can be expressed in conservative form as

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + H = 0 \qquad \text{(F.1.1)}$$

where

$$
\begin{aligned}
Q &= \begin{bmatrix} h \\ p = hu \\ q = hv \end{bmatrix} \\
F &= \begin{bmatrix} p \\ \frac{p^2}{h} + \frac{1}{2}gh^2 - h\sigma_{xx} \\ \frac{pq}{h} - h\sigma_{yx} \end{bmatrix} \\
G &= \begin{bmatrix} q \\ \frac{pq}{h} - h\sigma_{xy} \\ \frac{q^2}{h} + \frac{1}{2}gh^2 - h\sigma_{yy} \end{bmatrix} \\
H &= \begin{bmatrix} 0 \\ gh\frac{\partial z}{\partial x} + g\frac{n^2 p\sqrt{p^2+q^2}}{C_o^2 h^{7/3}} \\ gh\frac{\partial z}{\partial y} + g\frac{n^2 q\sqrt{p^2+q^2}}{C_o^2 h^{7/3}} \end{bmatrix}
\end{aligned}
\tag{F.1.2}
$$

and p and q are the discharge rates in the x and y-directions, respectively.

In the finite element method, the continuous system of partial differential equations is transformed into a discrete system of integral equations by the use of weight functions ψ_i. The weight function is centered at node i and is nonzero for the nodes in the neighborhood Ω_i of node i. The size of this neighborhood depends on the order of approximation and the type of weight function. By multiplying equation F.1.1 by the weight function ψ_i and integrating over the entire domain Ω yields

$$\int_{\Omega} \psi_i \left(\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + H \right) d\Omega = 0 \tag{F.1.3}$$

Since ψ_i is zero outside the neighborhood Ω_i, the integral can be reduced to

$$\int_{\Omega_i} \psi_i \left(\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + H \right) d\Omega_i = 0 \tag{F.1.4}$$

The weight function ψ_i is the sum of the Galerkin weight function ϕ_i and the Petrov-Galerkin weight function φ_i, or $\psi_i = \phi_i I + \varphi_i$. Making this substitution and integrating by parts yields

$$\begin{aligned} \int_{\Omega_i} \left[\psi_i \left(\frac{\partial Q}{\partial t} + H \right) + \varphi_i \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) - \frac{\partial \phi_i}{\partial x} F - \frac{\partial \phi_i}{\partial y} G \right] d\Omega_i \\ + \int_{\Gamma_i} \phi_i \left(F n_x + G n_y \right) d\Gamma_i = 0 \end{aligned} \tag{F.1.5}$$

where Γ_i is the boundary of Ω_i and (n_x, n_y) is the outward pointing normal. Furthermore,

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial Q} \frac{\partial Q}{\partial x} \approx A \frac{\partial Q}{\partial x} \\ \frac{\partial G}{\partial y} &= \frac{\partial G}{\partial Q} \frac{\partial Q}{\partial y} \approx B \frac{\partial Q}{\partial y} \end{aligned} \tag{F.1.6}$$

where $A = \frac{\partial F^*}{\partial Q}$ and $B = \frac{\partial G^*}{\partial Q}$ and F^* and G^* are the vectors F and G without the Reynolds stresses. Thus, the Petrov-Galerkin finite element method applied to the shallow water equations results in a system of integral equations of the form

$$\begin{aligned} \int_{\Omega_i} \left[\psi_i \left(\frac{\partial Q}{\partial t} + H \right) + \varphi_i \left(A \frac{\partial Q}{\partial x} + B \frac{\partial Q}{\partial y} \right) - \frac{\partial \phi_i}{\partial x} F - \frac{\partial \phi_i}{\partial y} G \right] d\Omega_i \\ + \int_{\Gamma_i} \phi_i \left(F n_x + G n_y \right) d\Gamma_i = 0 \end{aligned} \tag{F.1.7}$$

These integrals are approximated via Gaussian quadrature using three Gauss points in each direction for triangular elements and either four or five Gauss points in each direction for quadrilateral elements.

As is typical for finite elements, the conservative variables h, p and q are defined at the nodes. To obtain a continuous function for these variables, the Galerkin weight functions are used as the trial or basis functions via

$$\begin{aligned} h(x,y) &= \sum_i \phi_i(x,y) h_i \\ p(x,y) &= \sum_i \phi_i(x,y) p_i \\ q(x,y) &= \sum_i \phi_i(x,y) q_i \end{aligned} \tag{F.1.8}$$

Since these basis functions are bilinear functions for HIVEL2D, this flow solver is second order in space.

Finally, the boundary conditions along the walls are implemented via the integral over the boundary in equation F.1.7. If Γ_i does not coincide with the boundary of the channel, then ϕ_i is zero along the boundary, and the integral is zero. If Γ_i coincides with the channel's boundary, then the integral is evaluated, and the boundary conditions are implemented by making the appropriate substitutions within the integral.

F.2. Petrov-Galerkin Weight Function

The Petrov-Galerkin weight function is defined as

$$\varphi_i = \beta \left(\Delta x \frac{\partial \phi_i}{\partial x} \hat{A} + \Delta y \frac{\partial \phi_i}{\partial y} \hat{B} \right) \tag{F.2.1}$$

where β is a parameter that ranges from 0.1 to 0.5 and ϕ_i are the trial or basis functions. The metric terms are defined as

$$\begin{aligned} \Delta x &= 2\left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2\right]^{1/2} \\ \Delta y &= 2\left[\left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right]^{1/2} \end{aligned} \tag{F.2.2}$$

To determine $\hat{A}$, consider the matrix $A = \frac{\partial F^*}{\partial Q}$. Using the typical eigenvalue/eigenvector decomposition, the matrix A can be written as

$$A = P^{-1}\Lambda P \tag{F.2.3}$$

where Λ is the diagonal matrix whose entries are the eigenvalues of A. P is the matrix of right eigenvectors, and P^{-1} is the corresponding matrix of left eigenvectors. The matrix $\hat{A}$ is calculated as

$$\hat{A} = P^{-1}\hat{\Lambda}P \tag{F.2.4}$$

where

$$\hat{\Lambda} = \begin{bmatrix} \frac{\lambda_1}{\sqrt{\lambda_1^2+v^2}} & 0 & 0 \\ 0 & \frac{\lambda_2}{\sqrt{\lambda_2^2+v^2}} & 0 \\ 0 & 0 & \frac{\lambda_3}{\sqrt{\lambda_3^2+v^2}} \end{bmatrix} \tag{F.2.5}$$

where $\lambda_1 = u + c$, $\lambda_2 = u - c$, $\lambda_3 = u$ and $u = \sqrt{gh}$. Note that v is the velocity in the y-direction. The matrix $\hat{B}$ is determined in a similar fashion.

For a single, one-dimensional partial differential equation, such as the wave equation or Berger's equation, the linear trial or basis function at node i is the Lagrange

polynomial of degree one whose shape is shown in Figure F.2.1.

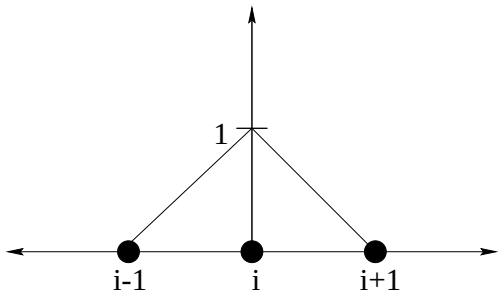

Figure F.2.1. Linear Basis Function for Wave Equation.

This basis function is the Galerkin weight function ϕ_i. For a one-dimensional, hyperbolic partial differential such as the wave equation, the Petrov-Galerkin weight function reduces to

$$\varphi_i = \beta \Delta x \frac{\partial \phi_i}{\partial x} \hat{A} \tag{F.2.6}$$

The matrix $\hat{A}$ is actually a scalar for this case, whose value is either positive one or negative one, dependent on the wave speed. If the wave speed is positive, then the information is traveling from left to right, and the weight function should be weighted on the left side. The Petrov-Galerkin weight function for a positive wave speed is shown in Figure F.2.2.

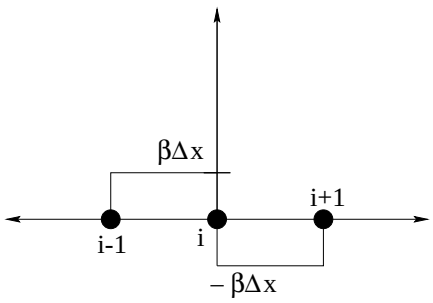

Figure F.2.2. Petrov-Galerkin Weight Function φ_i for Wave Equation.

and the composite weight function ψ_i is given in Figure F.2.3.

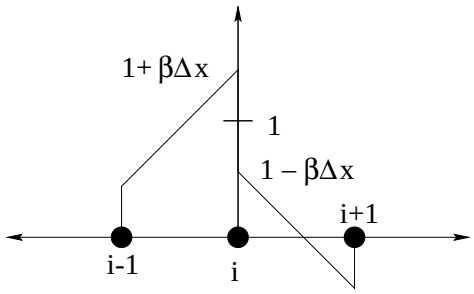

Figure F.2.3. Composite Weight Function ψ_i for Wave Equation.

F.3. Other Concerns

The temporal derivative at node i is approximated via finite differences as

$$\frac{\partial Q}{\partial t} \approx \alpha \left(\frac{Q_i^{n+1} - Q_i^n}{t^{n+1} - t^n} \right) + (1 - \alpha) \left(\frac{Q_i^n - Q_i^{n-1}}{t^n - t^{n-1}} \right) \qquad \text{(F.3.1)}$$

When $\alpha = 1$, the approximation is first order in time. When $\alpha = 1.5$, the approximation to the temporal derivative is second order. Since the other terms in equation F.2.7. are evaluated at time level $n+1$, the temporal derivative approximation terms is the only term that includes information from previous flow iterations.

At the inflow and outflow boundaries, the number of characteristics originating outside the computational domain determines the number of boundary conditions that must be specified as shown in Table F.3.1.

Table F.3.1. Relationships concerning Inflow/Outflow Boundary Conditions.

Boundary	Flow Type	Characteristics From Outside Domain	Boundary Conditions Specified
Inflow	Supercritical	3	h, p, q
Inflow	Subcritical	2	p, q
Outflow	Supercritical	0	none
Outflow	Subcritical	1	h

To be consistent with the number of integral equations, these boundary conditions are implemented implicitly by adding them to the equations generated by the finite element method at the boundary nodes. For instance, when depth is specified at boundary node i, the finite element method generates an equation for the conservation of mass at node i, called $W_{1,i} = 0$. The subscript 1 indicates the discrete implementation for the first partial differential equation of equation E.2.1., which is conservation of mass. The boundary condition is implemented by changing this equation to

$$W_{1,i} + \omega(h_i - h_{boundary}) = 0 \tag{F.3.2}$$

where ω is a large number, say 10^6. Since the influence of $W_{1,i}$ is quite small in comparison to ω, h_i is driven towards the specified boundary value $h_{boundary}$.

When the finite element method is applied to the shallow water equations, a system of integral equations is formed, three equations for each node. Since there are three unknowns at each node, the number of unknowns and the number of equations are the same, so the system is closed.

This system of equations can be expressed as $W(Q^{n+1}) = 0$ and is solved via the Newton-Raphson sub-iteration method as described in Chapter 3.3.