Sample Test 1 - Solutions

1. $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{1}{y} + x + 1.$

Solution: After factoring, the equation separates

$$\begin{aligned}
\frac{dy}{dx} &= \left(\frac{1}{y}+1\right)(x+1), \\
\frac{y}{y+1}\,dy &= (x+1)dx, \\
y - \ln|y+1| &= \frac{1}{2}x^2 + x + c.
\end{aligned}$$

2. $x\,\dfrac{dy}{dx} + 2\,y = x^2y^2.$

Solution: The equation is Bernoulli, so we put in standard form

$$\begin{aligned}
x\,\frac{dy}{dx} + 2\,y &= x^2y^2, \\
\frac{dy}{dx} + \frac{2}{x}\,y &= x\,y^2, \\
\frac{1}{y^2}\,\frac{dy}{dx} + \frac{2}{x}\,\frac{1}{y} &= x.
\end{aligned}$$

We let $u = \frac{1}{y}$ so $\frac{du}{dx} = -\frac{1}{y^2}\,\frac{dy}{dx}$ and substituting gives

$$\begin{aligned}
-\frac{du}{dx} + \frac{2}{x}\,u &= x, \\
\frac{du}{dx} - \frac{2}{x}\,u &= -x, \quad \left(\text{the integrating factor is } \mu = \frac{1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{1}{x^2}\,u\right) &= -\frac{1}{x}.
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{1}{x^2}\,u &= c - \ln|\,x\,|, \\
u &= x^2\left(c - \ln|\,x\,|\right), \\
\frac{1}{y} &= x^2\left(c - \ln|\,x\,|\right), \\
y &= \frac{1}{x^2\left(c - \ln|\,x\,|\right)}.
\end{aligned}$$

3. $\dfrac{dy}{dx} - y = 2e^x, \quad y(0) = 3.$

Solution: The equation is linear and already in standard form. The integrating factor is $\mu = e^{-x}$. Thus,

$$\begin{aligned} \frac{d}{dx}\left(e^{-x}y\right) &= 2, \\ e^{-x}y &= 2x + c, \ \text{ from the IC } c = 3, \\ e^{-x}y &= 2x + 3, \\ y &= (2x+3)e^x. \end{aligned}$$

4. $\dfrac{dy}{dx} = \dfrac{1 - 2xy^2}{1 + 2x^2y}, \quad y(1) = 1.$

Solution: The equation is exact. The alternate form is

$$(2xy^2 - 1)dx + (2x^2y + 1)dy = 0,$$

and it is an easy matter to verify

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x},$$

so z exists such that

$$\begin{aligned} \frac{\partial z}{\partial x} &= M = 2xy^2 - 1 \quad \Rightarrow \quad z = x^2y^2 - x + A(y), \\ \frac{\partial z}{\partial y} &= N = 2x^2y + 1 \quad \Rightarrow \quad z = x^2y^2 + y + B(x), \end{aligned}$$

so we can choose A and B giving $z = x^2y^2 - x + y$ and the solution as $x^2y^2 - x + y = c$. Since $y(1) = 1$, this give $c = 1$ and the solution $x^2y^2 - x + y = 1$.

5. $\dfrac{dy}{dx} = (\ln y - \ln x + 1)\,\dfrac{y}{x}.$

Solution: The equation is homogeneous. We re-write it as

$$\frac{dy}{dx} = \left(\ln\frac{y}{x} + 1\right)\frac{y}{x}.$$

If we let $y = xu$ so $\frac{dy}{dx} = x\frac{du}{dx} + u$ then

$$x\frac{du}{dx} + u = (\ln u + 1)u,$$

which separates

$$\frac{du}{u\ln u} = \frac{dx}{x} \quad \Rightarrow \quad \ln\ln u = \ln x + \ln c \quad \Rightarrow \quad u = e^{cx}.$$

Therefore,

$$\frac{y}{x} = e^{cx} \quad \text{or} \quad y = xe^{cx},$$

6. $\dfrac{dy}{dx} = \dfrac{x - y + 3}{x + y - 1}.$

Solution: The equation is linear-fractional. Since $a = 1,\ b = -1,\ c = 1,\ d = 1$ then $ad - bc = 2 \neq 0$ so the original equation can be made homogeneous. If we let $x = \bar{x} + \alpha$ and $y = \bar{y} + \beta$ then

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{x} + \alpha - \bar{y} - \beta + 3}{\bar{x} + \alpha + \bar{y} + \beta - 1}.$$

and choosing

$$\alpha - \beta + 3 = 0, \quad \alpha + \beta - 1 = 0, \quad (\text{this gives } \alpha = -1,\ \beta = 2),$$

then

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{x} - \bar{y}}{\bar{x} + \bar{y}}. \tag{1}$$

If we let $\bar{y} = \bar{x}\,\bar{u}$ then (1) becomes

$$\bar{x}\frac{d\bar{u}}{d\bar{x}} + \bar{u} = \frac{1 - \bar{u}}{1 + \bar{u}},$$

or, after separating

$$\frac{\bar{u} + 1}{\bar{u}^2 + 2\bar{u} - 1}\, d\bar{u} = -\frac{d\bar{x}}{\bar{x}}.$$

Integrating gives

$$\frac{1}{2}\ln|\,\bar{u}^2 + 2\bar{u} - 1\,| = -\ln|\,\bar{x}\,| + \frac{1}{2}\ln c,$$

or

$$\bar{u}^2 + 2\bar{u} - 1 = \frac{c}{\bar{x}^2}.$$

Back substituting

$$\bar{u} = \frac{\bar{y}}{\bar{x}} = \frac{y - 2}{x + 1},$$

gives the solution as

$$(y - 2)^2 + 2(x + 1)(y - 2) - (x + 1)^2 = c.$$