ODEs - Sample Test 2 Solutions

1. From the problem statement we see that $c_i = 2$ lb/gal., $r_i = 5$ gal/min., and $r_o = 10$ gal/min. Since $r_i \neq r_o$, then the volume changes and at a rate of $V = 500 - (10-5)t$. If $A(t)$ is the amount of salt at any time then the differential equation is

$$\begin{aligned}
\frac{dA}{dt} &= r_i c_i - r_o c_o, \\
&= 10 - 10\frac{A}{500-5t} \\
&= 10 - 2\frac{A}{100-t}
\end{aligned}$$

with an initial condition of $A(0) = 0$. This is linear with an integrating factor of $(100-t)^{-2}$. So

$$\begin{aligned}
\frac{d}{dt}\left(\frac{A}{(100-t)^2}\right) &= \frac{10}{(100-t)^2}, \\
\frac{A}{(100-t)^2} &= \frac{10}{100-t} + c
\end{aligned}$$

and with $A(0) = 0$ gives $c = -1/10$. Thus, the solution is

$$A = 10(100-t) - \frac{1}{10}(100-t)^2.$$

The tank is empty when $t = 100$ and at this time $A(100) = 0$. There is no salt in the tank.

2. If $P(t)$ is the size of the population at any time then the differential equation is

$$\frac{dP}{dt} = kP(1000-P), \quad P(0) = 100, \quad P(1) = 120.$$

Upon separating, we get

$$\begin{aligned}
\frac{dP}{P(1000-P)} &= k\,dt, \\
\frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP &= k\,dt \\
\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP &= k\,dt \quad \text{we absorbed the 1000 into } k \\
\ln P - \ln 1000 - P &= kt + \ln c \\
\frac{P}{1000-P} &= c\,e^{kt} \qquad (1.1)
\end{aligned}$$

The initial conditions gives $c = 1/9$ and $k = .20479$. Solving (1.1) gives

$$P = \frac{1000e^{.20479\,t}}{9 + e^{.20479\,t}}. \tag{1.2}$$

After 2 days $P(2) = 143.3617$ so 143 and we solve $P(t) = 900$ and obtain $t = 21.45$ so $t = 22$

3. If $x(t)$ is the amount of C formed a time t. The reaction is

$$\mathrm{A} + \mathrm{B} \rightarrow \mathrm{C} \quad \text{so} \quad 2 + 1 \rightarrow 3 \quad \text{so} \quad \frac{2x}{3} + \frac{x}{3} \rightarrow x.$$

The differential equation is

$$\begin{aligned} \frac{dx}{dt} &= k\left(10 - \frac{2x}{3}\right)\left(20 - \frac{x}{3}\right), \quad x(0) = 0, \quad x(20) = 6, \\ \frac{dx}{dt} &= \frac{2k}{9}\,(15 - x)\,(60 - x)\,, \end{aligned}$$

After separating, we obtain

$$\begin{aligned} \frac{1}{45}\left(\frac{1}{15 - x} - \frac{1}{60 - x}\right)\,dx &= \frac{2k}{9}\,dt \\ \ln\frac{60 - x}{15 - x} &= k\,t + \ln c \quad \text{(new k)} \\ \frac{60 - x}{15 - x} &= c\,e^{k\,t}. \end{aligned}$$

The initial conditions $x(0) = 0, \quad x(20) = 6$ gives $c = 4$ and $k = .02027$. The final solution is given by

$$x = \frac{60\left(e^{.02027\,t} - 1\right)}{4\,e^{.02027\,t} - 1}.$$

4(i). Solve $y'' - 5y' + 6y = 0, \quad y(0) = 1, \;\; y'(0) = 0$

The characteristic equation is

$$m^2 - 5m + 6 = 0 \quad \Rightarrow \quad (m - 2)(m - 3) = 0 \quad \Rightarrow \quad m = 2, 3.$$

The solution is

$$y = c_1 e^{2x} + c_2 e^{3x}.$$

Imposing the initial condition gives $c_1 = 3, \;\; c_2 = -2$ so

$$y = 3e^{2x} - 2e^{3x}.$$

4(ii) Solve $y'' + 2y' + 10y = 0, \quad y(0) = -1, \;\; y'(0) = 4$

The characteristic equation is

$$m^2 + 2m + 10 = 0 \quad \Rightarrow \quad m = -1 \pm 3i.$$

The solution is

$$y = c_1 e^{-x} \cos 3x + c_2 e^{-x} \sin 3x.$$

Imposing the initial condition gives $c_1 = -1, \;\; c_2 = 1$ so

$$y = -e^{-x} \cos 3x + e^{-x} \sin 3x.$$

4(iii) Solve $4y'' - 4y' + y = 0, \quad y(0) = 0, \;\; y'(0) = 1$

The characteristic equation is

$$4m^2 - 4m + 10 = 0 \quad \Rightarrow \quad m = 1/2, \; 1/2.$$

The solution is

$$y = c_1 e^{x/2} + c_2 x e^{x/2}.$$

Imposing the initial condition gives $c_1 = 0, \;\; c_2 = 1$ so

$$y = x e^{x/2}.$$

5. Solve $\left(x^2 - 2x\right) y'' - \left(x^2 - 2\right) y' + 2(x-1)y = 0, \quad y_1 = x^2$

If we let

$$y = x^2 u,$$

then

$$\begin{aligned} y' &= x^2 u' + 2xu, \\ y'' &= x^2 u'' + 4xu' + 2u. \end{aligned}$$

Substituting into the ODE gives

$$x^2 \left(x^2 - 2x\right) u'' - \left(x^4 - 4x^3 + 6x^2\right) u' = 0.$$

If we let $u' = v$, then

$$x^2 \left(x^2 - 2x\right) v' - \left(x^4 - 4x^3 + 6x^2\right) v = 0$$

Separating gives

$$\frac{v'}{v} = \frac{x^4 - 4x^3 + 6x^2}{x^2 \left(x^2 - 2x\right)} = \frac{x^2 - 4x + 6}{x^2 - 2x},$$

and integrating gives

$$v = \frac{(x-2)e^x}{x^3}.$$

And since $u' = v$, we integrate once more giving

$$u = \frac{e^x}{x^2}.$$

The second solution is

$$y = x^2 u = x^2 \frac{e^x}{x^2} = e^x.$$

The general solution is

$$y = c_1 x^2 + c_2 e^x.$$

6. Solve $y'' - y' = 2x - 3x^2$

We first consider the complimentary equation $y'' - y' = 0$. The characteristic equation is $m^2 - m = 0$ giving $m = 0$ and 1. Then, the complimentary solution is

$$y_c = c_1 + c_2 e^x,$$

If we were to guess the particular solution it would be

$$y_p = Ax^2 + Bx + C,$$

but as we see part of the particular solution is in the complimentary solution. So we need to "bump" up the solution. So we seek a particular solution of the form

$$y_p = Ax^3 + Bx^2 + Cx.$$

Substituting into the ODE gives

$$6Ax + 2B - 3Ax^2 - 2Bx - C = 2x - 3x^2.$$

Comparing coefficients gives

$$\begin{aligned} -3A &= -3, \\ 6A - 2B &= 2, \\ 2B - C &= 0, \end{aligned}$$

giving $A = 1, \;\; B = 2, \;\; C = 4$. This gives

$$y_p = x^3 + 2x^2 + 4x,$$

and the general solution as

$$y = c_1 + c_2 e^x + x^3 + 2x^2 + 4x.$$