Math 3331 - Ordinary Differential Equations
Sample Test 3 Solutions

1. Solve the following

$$
\begin{array}{lll}
(i) & x^2y'' - 5xy' + 8y = 0, & y(2) = 32,\ \ y'(2) = 0, \\
(ii) & x^2y'' + 5xy' + 4y = 0, & y(1) = 2,\ \ y'(1) = 1, \\
(iii) & x^2y'' - 3xy' + 5y = 0, & y(1) = 2,\ \ y'(1) = 5.
\end{array}
$$

(i) The characteristic equation is

$$
\begin{aligned}
m(m-1) - 5m + 8 &= 0, \\
m^2 - 6m + 8 &= 0, \\
(m-2)(m-4) &= 0 \quad \Rightarrow \quad m = 2, 4.
\end{aligned}
$$

The solution is

$$y = c_1x^2 + c_2x^4.$$

With the boundary conditions $y(2) = 32$ and $y'(2) = 0$ gives $c_1 = 16$ and $c_2 = -2$. Thus, the solution is

$$y = 16x^2 - 2x^4.$$

(ii) The characteristic equation is

$$
\begin{aligned}
m(m-1) + 5m + 4 &= 0, \\
m^2 + 4m + 4 &= 0, \\
(m+2)^2 &= 0 \quad \Rightarrow \quad m = -2, -2.
\end{aligned}
$$

The solution is

$$y = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2}.$$

With the boundary conditions $y(1) = 2$ and $y'(1) = 1$ gives $c_1 = 1$ and $c_2 = 3$. Thus, the solution is

$$y = \frac{1}{x^2} + \frac{3 \ln x}{x^2}.$$

(iii) The characteristic equation is

$$
\begin{aligned}
m(m-1) - 3m + 5 &= 0, \\
m^2 - 4m + 5 &= 0, \quad \Rightarrow \quad m = 2 \pm i.
\end{aligned}
$$

The solution is

$$y = c_1x^2 \sin(\ln x) + c_2x^2 \cos(\ln x).$$

With the boundary conditions $y(1) = 2$ and $y'(1) = 5$ gives $c_1 = 1$ and $c_2 = 2$. Thus, the solution is

$$y = x^2 \sin(\ln x) + 2x^2 \cos(\ln x).$$

2. Solve the following using the variation of parameters

$$\begin{aligned} &(i) \qquad y'' + y = \tan x, \\ &(ii) \qquad y'' + 3y' + 2y = \frac{1}{e^x + 1}, \quad y(0) = 2, \;\; y'(0) = 1. \end{aligned}$$

(i) The complementary equation is

$$y'' + y = 0,$$

whose solution is

$$y = c_1 \sin x + c_2\, cos x.$$

For the variation of parameters we replace c_1 and c_2 with u and v. Thus

$$y = u \sin x + v\, cos x, \tag{1.1}$$

so

$$y' = u' \sin x + u \cos x + v' \cos x - v \sin x,$$

where we set

$$u' \sin x + v' \cos x = 0. \tag{1.2}$$

Therefore

$$y' = u \cos x - v \sin x,$$

and

$$y'' = u' \cos x - u \sin x - v' \sin x - v \cos x.$$

Substituting into the original equation gives

$$\begin{aligned} u' \cos x \; &- \cancel{u \sin x} - v' \sin x - \cancel{v \cos x} \\ &- \cancel{u \sin x} \qquad\qquad\quad - \cancel{v \cos x} = \tan x \end{aligned} \tag{1.3}$$

Solving (1.2) and (1.3) for u' and v' gives

$$u' = \sin x, \quad v' = -\frac{\sin^2 x}{\cos x}.$$

which, upon integrating gives

$$u = -\cos x, \quad v = \sin x - \ln \mid \sec x + \tan x \mid.$$

Substituting these into (1.1) gives

$$y = -\cos x \ln|\ \sec x + \tan x\ |,$$

which in turn gives the solution

$$y = c_1 \sin x + c_2 \cos x - \cos x \ln|\ \sec x + \tan x\ |,$$

(ii) The complementary equation is

$$y'' + 3y' + 2y = 0,$$

whose solution is

$$y = c_1 e^{-x} + c_2 e^{-2x}.$$

For the variation of parameters we replace c_1 and c_2 with u and v. Thus

$$y = ue^{-x} + ve^{-2x}, \tag{1.4}$$

so

$$y' = u'e^{-x} - ue^{-x} + v'e^{-2x} - 2ve^{-2x},$$

where we set

$$u'e^{-x} + v'e^{-2x} = 0, \tag{1.5}$$

so

$$y' = -ue^{-x} - 2ve^{-2x}.$$

Then

$$y'' = -u'e^{-x} + ue^{-x} - 2v'e^{-2x} + 4ve^{-2x}.$$

Substituting into the original equation gives

$$\begin{aligned} -u'e^{-x} + \cancel{ue^{-x}} - 2v'e^{-2x} &+ \cancel{4ve^{-2x}} \\ - \cancel{3ue^{-x}} \qquad &- \cancel{6ve^{-2x}} \\ + \cancel{2ue^{-x}} \qquad &+ \cancel{2ve^{-2x}} = \frac{1}{e^x + 1}. \end{aligned} \tag{1.6}$$

Solving (1.5) and (1.6) for u' and v' gives

$$u' = \frac{e^x}{e^x + 1}, \quad v' = -\frac{e^{2x}}{e^x + 1}.$$

which, upon integrating gives

$$u = \ln(e^x + 1), \quad v = -e^x + \ln(e^x + 1).$$

Substituting these into (1.4) gives

$$y = \left(e^{-x} + e^{-2x}\right) \ln(e^x + 1),$$

noting that the term in the particular solution e^{-x} was neglected because it appears as part of the complementary solution. This, in turn gives the solution

$$y = c_1 e^{-x} + c_2 e^{-2x} + \left(e^{-x} + e^{-2x}\right) \ln(e^x + 1),$$

3. A 10-pound weight attached to a spring stretches it 2 feet. The weight is attached to a dashpot damping device that offers resistance numerically equal to β $(\beta > 0)$ times the instantaneous velocity. Determine the values of the damping constant β so that the subsequent motion is (a) overdamped, (b) critically damped, and (c) underdamped.
The equation which governs the motion is

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

Since the 10 lb weight stretches the string 2ft, then $F = kx \quad \Rightarrow \quad 10 = 2k \quad \Rightarrow \quad k = 5.$ Further, since $F = mg$, then $10 = 32m \quad \Rightarrow \quad m = \frac{5}{16}$. So,

$$\frac{5}{16}\,\frac{d^2x}{dt^2} + \beta\,\frac{dx}{dt} + 5x = 0,$$

or

$$5\,\frac{d^2x}{dt^2} + 16\beta\,\frac{dx}{dt} + 80x = 0.$$

The characteristic equation is

$$5m^2 + 16\beta m + 80 = 0,$$

from which we obtain

$$m = \frac{8\beta \pm \sqrt{16\beta^2 - 100}}{5}.$$

The motion will be over damped if $16\beta^2 - 100 > 0$, critically damped if $16\beta^2 - 100 = 0$ and under damped if $16\beta^2 - 100 < 0$, or if $\beta > 5/2$, $\beta = 5/2$ or $\beta < 5/2$.

4. Solve the following systems

$$(i)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \bar{x} \qquad (ii)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \bar{x},$$

$$(iii)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x}, \qquad (iv)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \bar{x},$$

$$(v)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x}, \qquad (vi)\ \frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -5 \\ -5 & -5 \end{pmatrix} \bar{x}.$$

The general form is

$$(\lambda I - A)\,\bar{u} = 0. \tag{1.7}$$

and in order to have nontrivial solutions $\bar{u}$, we require that

$$|\lambda I - A| = 0. \tag{1.8}$$

(i) The characteristic equation is

$$\begin{vmatrix} \lambda-1 & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$
From (1.7) we have

$$\begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $2e_1 + e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2: $\lambda = 2$
From (1.7) we have

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $e_1 - e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

(ii) The characteristic equation is

$$\begin{vmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{vmatrix} = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = -3$.

Case 1: $\lambda = -1$
From (1.7) we have

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $e_1 - e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Case 2: $\lambda = -3$
From (1.7) we have

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $-e_1 - e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{t}} + \mathrm{c}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{-3\mathrm{t}}.$$

(iii) The characteristic equation is

$$\begin{vmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 2, 2$.

For $\lambda = 2$, from (1.7) we have

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $e_1 + e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For the second solution, we seek a solution of the form

$$\bar{x} = \bar{u} t \mathrm{e}^{2\mathrm{t}} + \bar{\mathrm{v}} \mathrm{e}^{2\mathrm{t}}. \tag{1.9}$$

Substitution into our system gives

$$\begin{aligned} (2I - A)\bar{u} &= \bar{0}, & (1.10) \\ (2I - A)\bar{v} &= -\bar{u}. & (1.11) \end{aligned}$$

Equation (1.10) gives the eigenvector just found, whereas (1.11) gives

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

from which we obtain upon expanding $e_1 + e_2 = -1$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

So the second solution is

$$\bar{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t \mathrm{e}^{2\mathrm{t}} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathrm{e}^{2\mathrm{t}}.$$

and general solution is

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{2t} + \mathrm{c}_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{te}^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} \right].$$

(iv) The characteristic equation is

$$\begin{vmatrix} \lambda - 5 & 4 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0,$$

from which we obtain the eigenvalues $\lambda = 3, 3$.

For $\lambda = 3$, from (1.7) we have

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $e_1 - 2e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For the second solution, we seek a solution of the form

$$\bar{x} = \bar{u}t\mathrm{e}^{2t} + \bar{\mathrm{v}}\mathrm{e}^{2t}. \tag{1.12}$$

Substitution into our system gives

$$\begin{aligned} (3I - A)\bar{u} &= \bar{0}, & (1.13) \\ (3I - A)\bar{v} &= -\bar{u}. & (1.14) \end{aligned}$$

Equation (1.13) gives the eigenvector just found, whereas (1.14) gives

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = - \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

from which we obtain upon expanding $-e_1 + 2e_2 = -1$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So the second solution is

$$\bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\mathrm{e}^{3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{3t}.$$

and general solution is

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mathrm{e}^{3t} + \mathrm{c}_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \mathrm{te}^{3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{3t} \right].$$

(v) The characteristic equation is

$$\begin{vmatrix} \lambda-6 & 1 \\ -5 & \lambda-4 \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0,$$

from which we obtain the eigenvalues $\lambda = 5 \pm 2i$.

Case 1: $\lambda = 5 + 2i$
From (1.7) we have

$$\begin{pmatrix} -1+2i & 1 \\ -5 & 1+2i \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $-(1-2i)e_1 + e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

The second eigenvector would just be the complex conjugate. Thus,

$$\bar{E_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{E_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The two solutions are

$$\begin{aligned} \bar{x_1} &= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] \mathrm{e}^{5\mathrm{t}}, \\ \bar{x_2} &= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t \right] \mathrm{e}^{5\mathrm{t}}, \end{aligned}$$

and the general solution

$$\begin{aligned} \bar{x} &= c_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] \mathrm{e}^{5\mathrm{t}}, \\ &+ c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t \right] \mathrm{e}^{5\mathrm{t}}. \end{aligned}$$

(vi) The characteristic equation is

$$\begin{vmatrix} \lambda-3 & -5 \\ 5 & \lambda+5 \end{vmatrix} = \lambda^2 + 2\lambda + 10 = 0,$$

from which we obtain the eigenvalues $\lambda = -1 \pm 3i$.

Case 1: $\lambda = -1 + 3i$
From (1.7) we have

$$\begin{pmatrix} -4+3i & -5 \\ 5 & 4+3i \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $(-4+3i)e_1 - 5e_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 5 \\ -4+3i \end{pmatrix}.$$

The second eigenvector would just be the complex conjugate. Thus,

$$\bar{E}_1 = \begin{pmatrix} 5 \\ -4 \end{pmatrix}, \quad \bar{E}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

The two solutions are

$$\begin{aligned} \bar{x}_1 &= \left[\begin{pmatrix} 5 \\ -4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t \right] e^{-t}, \\ \bar{x}_2 &= \left[\begin{pmatrix} 5 \\ -4 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right] e^{-t}, \end{aligned}$$

and the general solution

$$\begin{aligned} \bar{x} &= c_1 \left[\begin{pmatrix} 5 \\ -4 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t \right] e^{-t}, \\ &+ c_2 \left[\begin{pmatrix} 5 \\ -4 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right] e^{-t}. \end{aligned}$$