Differentials

If we consider approximating the change in y by moving a small amount in x, we can use the equation of the tangent. At the point (x_0, y_0), the equation of the tangent is

$$y - y_0 = f'(x_0)(x - x_0). \tag{1}$$

Now if we let $x = x_0 + dx$ and $y = y_0 + dy$, we see from (1) that

$$y_0 + dy - y_0 = f'(x_0)(x_0 + dx - x_0),$$

or

$$dy = f'(x_0)dx,$$

a relation between the differential dx and dy. We go further and define this relationship for general x as

$$dy = f'(x)dx,$$

or

$$dy = \frac{dy}{dx}\, dx,$$

which applies for all x. Does this extend to $3 - D$? Yes. We now follow the tangent plane. The tangent plane is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{2}$$

Now if we let $x = x_0 + dx$, $y = y_0 + dy$ and $z = z_0 + dz$ then from (2) we see that

$$z_0 + dz - z_0 = f_x(x_0, y_0)(x_0 + dx - x_0) + f_y(x_0, y_0)(y_0 + dy - y_0).$$

or

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy,$$

a relation between the differential dx, dy and dz. We go further and define this relationship for general x and y as

$$dz = f_x\, dx + f_y\, dy,$$

or

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy.$$

Example 1

If $z = x^2y^5$, find dz. Calculating the partial derivatives, we find that $\frac{\partial z}{\partial x} = 2xy^5$ and $\frac{\partial z}{\partial y} = 5x^2y^4$ so the differential dz is

$$dz = 2xy^5\, dx + 5x^2y^4\, dy.$$

Example 2

If $z = e^{xy} + x\sin y$, find dz. Calculating the partial derivatives, we find that $\frac{\partial z}{\partial x} = ye^{xy} + \sin y$ and $\frac{\partial z}{\partial y} = xe^{xy} + x\cos y$ so the differential dz is

$$dz = \left(ye^{xy} + \sin y\right)\, dx + \left(xe^{xy} + x\cos y\right)\, dy.$$