Chapter 4. Fourier Series

At this point we are ready to now consider the canonical equations. Consider, for example the heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0 \tag{4.1}$$

subject to

$$u(x, 0) = 2\sin x, \quad u(0, t) = u(\pi, t) = 0. \tag{4.2}$$

Here, we will assume that the solutions are separable and are of the form

$$u(x, t) = X(x)T(t). \tag{4.3}$$

Substituting into the heat equation (4.1) gives

$$XT' = X''T,$$

from which we deduce that

$$\frac{T'}{T} = \frac{X''}{X}. \tag{4.4}$$

Since each side is a function of a different variable, we can deduce that

$$T' = \lambda T, \quad X'' = \lambda X. \tag{4.5}$$

where λ is a constant. The boundary conditions in (4.2) becomes, accordingly

$$X(0) = X(\pi) = 0. \tag{4.6}$$

Integrating the X equation in (4.5) gives rise to three cases depending on the sign of λ. These are

$$X(x) = \begin{cases} c_1 e^{nx} + c_2 e^{-nx} & \text{if } \lambda = n^2, \\ c_1 x + c_2 & \text{if } \lambda = 0, \\ c_1 \sin nx + c_2 \cos nx & \text{if } \lambda = -n^2, \end{cases}$$

where n is a constant. Imposing the boundary conditions (4.6) shows that in the case of $\lambda = n^2$ and $\lambda = 0$, the only choice for c_1 and c_2 are zero and hence inadmissible as this leads to the zero solution. Thus, we focus only on the remaining case, $\lambda = -n^2$. Using the boundary conditions gives

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin n\pi + c_2 \cos n\pi = 0, \tag{4.7}$$

which leads to

$$c_2 = 0, \quad \sin nx = 0 \;\Rightarrow\; n = 0, 1, 2, \ldots. \tag{4.8}$$

From (4.5), we then deduce that

$$T(t) = c_3 e^{-n^2 t} \tag{4.9}$$

thus, giving a solution to the original PDE as

$$u = XT = c e^{-n^2 t} \sin nx \tag{4.10}$$

where we have set $c = c_1 c_3$. Finally, imposing the initial condition (4.2) gives

$$u(x, 0) = c e^0 \sin nx = 2 \sin x \tag{4.11}$$

show that $c = 2$ and $n = 1$. Therefore the solution to the PDE subject to the initial and boundary conditions is

$$u(x, t) = 2e^{-t} \sin x. \tag{4.12}$$

If the initial condition we to change, say to $4 \sin 3x$, then we would have obtained the solution

$$u(x, t) = 4e^{-9t} \sin 3x. \tag{4.13}$$

However, if the initial condition were $u(x, 0) = 2 \sin x + 4 \sin 3x$ there would be a problem as it would be impossible to choose n and c to satisfy both. However, if we were to solve the heat equation with each function separately, we can simply just add the solutions together. It is what is call the **principle of superposition**

Theorem Principle of Superposition
If u_1 and u_2 are two solution to the heat equation, then $u = c_1u_1 + c_2u_2$ is also a solution.
Proof. Calculating derivatives $u_t = c_1u_{1t} + c_2u_{2t}$ and $u_{xx} = c_1u_{1xx} + c_2u_{2xx}$ and substituting into the heat equation show it is identically satisfied if each of u_1 and u_2 satisfy the heat equation.*Q.E.D*

Therefore to solve the heat equation subject to $u(x, 0) = 2\sin x + 4\sin 3x$ we would obtain

$$u(x, t) = 2e^{-t}\sin x + 4e^{-9t}\sin 3x.$$

The principle of superposition easily extends to more than 2 solutions. Thus, if

$$u(x, t) = (a_n \cos nx + b_n \sin nx)\, e^{-n^2 t},$$

are solutions to the heat equation, then so is

$$u(x, t) = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)e^{-n^2 t}. \tag{4.14}$$

If the initial conditions were such that they involved trigonometric functions, we could choose the integers n and constants a_n and b_n according to match the terms in the initial condition. However, if the initial condition was $u(x, 0) = \pi x - x^2$, we would have a problem as in the general solution (4.14), there are no x terms. However, consider the following

$$\begin{aligned}
u_1 &= \frac{8}{\pi}e^{-t}\sin x,\\
u_2 &= \frac{8}{\pi}\left(e^{-t}\sin x + \frac{1}{27}e^{-9t}\sin 3x\right),\\
u_3 &= \frac{8}{\pi}\left(e^{-t}\sin x + \frac{1}{27}e^{-9t}\sin 3x + \frac{1}{125}e^{-25t}\sin 5x\right).
\end{aligned} \tag{4.15}$$

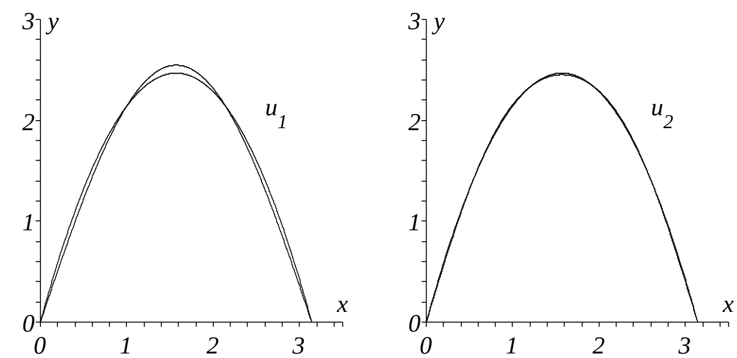

Figure 1. The solutions (4.15) with one and two terms.

From figure 1, one will notice that with each additional term added in (4.15), the solution is a better match to the initial condition. In fact, if we consider

$$u = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)^2 t} \sin(2n-1)x \tag{4.16}$$

we get a perfect match to the initial condition. Thus, we are lead to ask "How are the integers n and constants a_n and b_n chosen as to match the initial condition?"

4.1 Fourier Series

It is well known that infinitely many functions can be represented by a power series

$$f(x) = \sum_{i=1}^{\infty} a_n (x - x_0)^n ,$$

where x_0 is the center of the series and a_n, constants determined by

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \quad i = 1, 2, 3, \ldots .$$

For functions that require different properties, say for example, fixed points at the endpoints of an interval, a different type of series is required. Example of such a series is called Fourier series.

For example, suppose that $f(x) = \pi x - x^2$ has a Fourier series

$$\pi x - x^2 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x \ldots . \tag{4.17}$$

How do we choose b_1, b_2, b_3 etc such that the Fourier series looks like the function? Notice that if multiply (4.17) by $\sin x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2)\sin x\,dx = b_1\int_0^\pi \sin^2 x\,dx + b_2\int_0^\pi \sin x \sin 2x\,dx + \ldots.$$

then we obtain

$$4 = b_1.\frac{\pi}{2} \quad\Rightarrow\quad b_1 = 8/\pi, \tag{4.18}$$

since

$$\int_0^\pi (\pi x - x^2)\sin x\,dx = 4, \quad \int_0^\pi \sin x^2\,dx = \frac{\pi}{2},$$
$$\int_0^\pi \sin x \sin nx\,dx = 0, \quad n = 2, 3, 4, \ldots$$

Similarly, if multiply (4.17) by $\sin 2x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2)\sin 2x\,dx = b_1\int_0^\pi \sin x\sin 2x\,dx + b_2\int_0^\pi \sin^2 2x\,dx + \ldots.$$

then we obtain

$$0 = b_2.\frac{\pi}{2} \quad\Rightarrow\quad b_2 = 0.$$

Multiply (4.17) by $\sin 3x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2)\sin 3x\,dx = b_1\int_0^\pi \sin x\sin 3x\,dx + b_2\int_0^\pi \sin 2x\sin 3x\,dx + \ldots,$$

then we obtain

$$\frac{4}{27} = b_3.\frac{\pi}{2} \quad\Rightarrow\quad b_3 = \frac{8}{27\pi}. \tag{4.19}$$

Continuing in this fashion, we would obtain

$$b_4 = 0, \;\; b_5 = \frac{8}{125\pi}, \;\; b_6 = 0, \;\; b_7 = \frac{8}{7^3\pi}. \tag{4.20}$$

Substitution of (4.18), (4.19) and (4.20) into (4.17) gives (4.15).

4.2 Fourier Series on $[-\pi,\pi]$

Consider the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) \tag{4.21}$$

where a_0, a_n and b_n are constant coefficients. The question is: "How do we choose the coefficients as to give an accurate representation of $f(x)$?" Well, we use the following properties of $\cos n\pi x$ and $\sin n\pi x$

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx\, dx = 0, \tag{4.22}$$

$$\int_{-\pi}^{\pi} \cos nx \ \cos mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{4.23}$$

$$\int_{-\pi}^{\pi} \sin nx \ \sin mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{4.24}$$

$$\int_{-\pi}^{\pi} \sin nx \ \cos mx\, dx = 0. \tag{4.25}$$

First, if we integrate (4.21) from $-\pi$ to π, then by the properties in (4.22), we are left with

$$\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} a_0 dx = \pi a_0,$$

from which we deduce

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Next we multiply the series (4.21) by $\cos mx$ giving

$$f(x)\cos mx = \frac{1}{2}a_0 \cos mx + \sum_{n=1}^{\infty} \left(a_n \cos nx \ \cos mx + b_n \sin nx \ \cos mx\right).$$

Again, integrate from $-\pi$ to π. From (4.22), the integration of $a_0 \cos mx$ is zero, from (4.23), the integration of $\cos nx \cos mx$ is zero except when $n = m$ and further from (4.25) the integrations of $\sin nx \ \cos mx$ is zero for all m and n. This leaves

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi a_n,$$

or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

Similarly, if we multiply the series (4.21) by $\sin mx$ then we obtain

$$f(x)\sin mx = \frac{1}{2}a_0 \sin mx + \sum_{n=1}^{\infty}\left(a_n \cos nx\ \sin mx + b_n \sin nx\ \sin mx\right),$$

which we integrate from $-\pi$ to π. From (4.22), the integration of $a_0 \sin m\pi x$ is zero, from (4.25) the integration of $\sin nx\ \cos mx$ is zero for all m and n and further from (4.24) the integration of $\sin nx\ \sin mx$ is zero except when $n = m$. This leaves

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi b_n,$$

or

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where the coefficients a_n and b_n are chosen such that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{4.26}$$

for $n = 0, 1, 2, \ldots$.

Example 1

Consider

$$f(x) = x^2, \quad [-\pi, \pi]. \tag{4.27}$$

From (4.26) we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{2\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx \\
&= \frac{1}{\pi}\left.\left[\frac{x^2 \sin nx}{n} + 2\,\frac{x\cos nx}{n^2} - 2\,\frac{\sin nx}{n^3}\right]\right|_{-\pi}^{\pi} \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx \\
&= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + 2\,\frac{x\sin nx}{n^2} + 2\,\frac{\cos nx}{n^3}\right]\Bigg|_{-\pi}^{\pi} \\
&= 0
\end{aligned}
$$

Thus, the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is

$$
f_k(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{k} \frac{(-1)^n \cos nx}{n^2}. \tag{4.28}
$$

Figure 2 show consecutive plots of the Fourier series (4.28) with 5 and 10 ten on the interval $[-\pi,\pi]$ and $[-3\pi,3\pi]$.

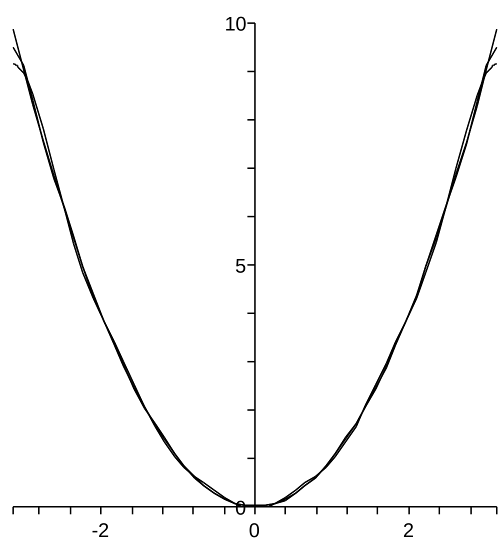

Figure 2. The solutions (4.28) with five and ten terms on $[-\pi,\pi]$.

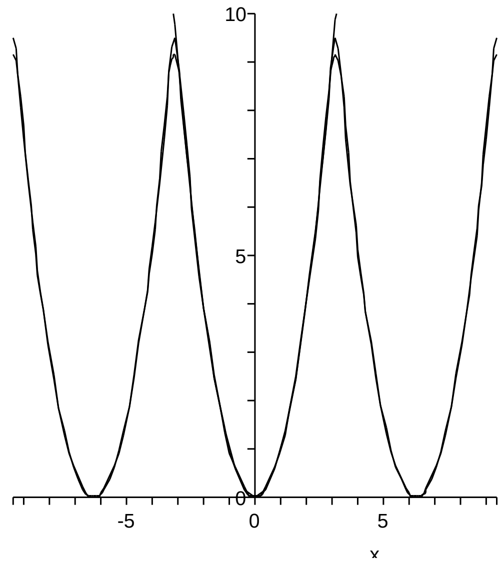

Figure 3. The solutions (4.28) with five and ten terms on [-3π,3π].

Example 2

Consider

$$f(x) = x, \quad [-\pi, \pi] \tag{4.29}$$

From (4.26) we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \\
&= \frac{1}{\pi}\left.\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\
&= \frac{1}{\pi}\left.\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]\right|_{-\pi}^{\pi} \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f_k(x) = 2 \sum_{n=1}^{k} \frac{(-1)^{n+1} \sin nx}{n}. \tag{4.30}$$

Figure 3 show consecutive plots of the Fourier series (4.30) with 5 and 50 terms on the interval $[-\pi,\pi]$ and $[-3\pi,3\pi]$.

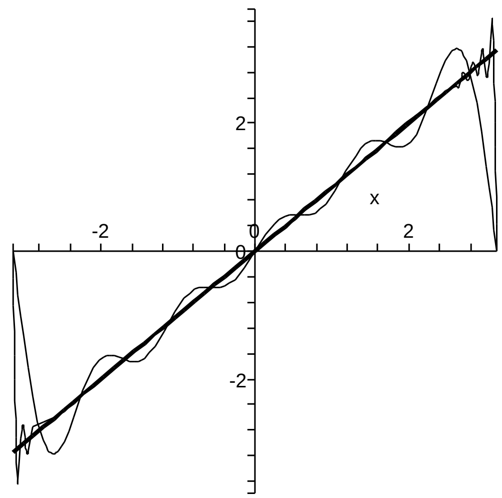

Figure 4. The solutions (4.30) with five and fifty terms on $[-\pi,\pi]$.

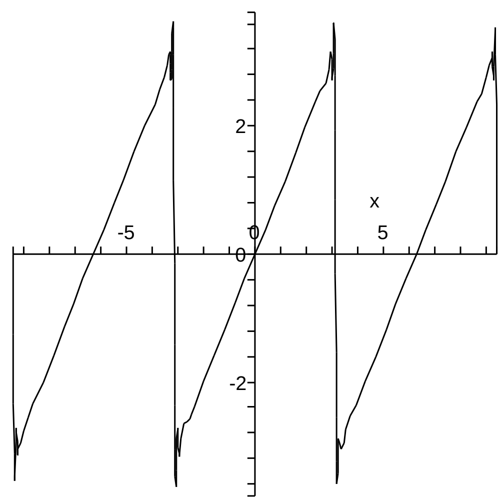

Figure 5. The solutions (4.30) with five and fifty terms on $[-3\pi,3\pi]$.

Example 3

Consider

$$f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0, \\ \text{x} + 1 & \text{if } 0 < x < \pi, \end{cases}$$

From (4.26) we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} x + 1\, dx = \frac{1}{\pi}\, x|_{-\pi}^{0} + \frac{1}{\pi}\, \frac{x^2}{2} + x\bigg|_0^{\pi} = \frac{\pi}{2} + 2 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (x+1)\cos nx\, dx \\
&= \frac{1}{\pi}\, \frac{\sin nx}{n}\bigg|_{-\pi}^{0} + \frac{1}{\pi}\left[(x+1)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]\bigg|_0^{\pi} \\
&= \frac{1}{\pi}\frac{(-1)^n - 1}{n^2}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (x+1)\sin nx\, dx \\
&= \frac{1}{\pi}\, -\frac{\cos nx}{n}\bigg|_{-\pi}^{0} + \frac{1}{\pi}\left[-(x+1)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]\bigg|_0^{\pi} \\
&= \frac{1}{\pi}\frac{(-1)^n - 1}{n} + \frac{1}{\pi}\frac{(\pi+1)(-1)^{n+1} + 1}{n} \\
&= \frac{(-1)^{n+1}}{n}.
\end{aligned}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f_k(x) = \frac{\pi}{4} + 1 + \sum_{n=1}^{k}\left(\frac{1}{\pi}\frac{(-1)^n - 1}{n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right). \tag{4.31}$$

Figure 6 shows a plot of the Fourier series (4.31) with 10 terms on the interval [$-\pi$,π].

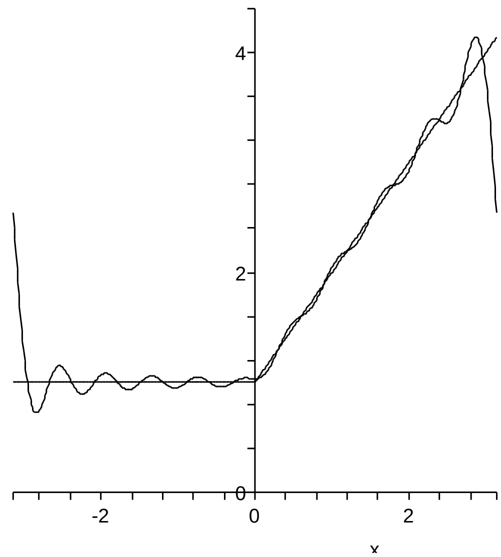

Figure 6. The solutions (4.31) with 10 terms.

4.3 Fourier Series on $[-L\ ,L]$

Consider the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{4.32}$$

where L is a positive number and a_0, a_n and b_n constant coefficients. The question is: "How do we choose the coefficients as to give an accurate representation of $f(x)$?" Well, we use the following properties of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} dx = 0, \qquad \int_{-L}^{L} \sin \frac{n\pi x}{L} dx = 0, \tag{4.33}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4.34}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4.35}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L} dx = 0. \tag{4.36}$$

First, if we integrate (4.32) from $-L$ to L, then by the properties in (4.33), we are left with

$$\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-L}^{L} a_0 dx = La_0,$$

from which we deduce

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

Next we multiply the series (**??**) by $\cos\frac{m\pi x}{L}$ giving

$$f(x)\cos\frac{m\pi x}{L} = \frac{1}{2}a_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right).$$

Again, integrate from $-L$ to L. From (4.33), the integration of $a_0\cos\frac{m\pi x}{L}$ is zero, from (4.34), the integration of $\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero except when $n = m$ and further from (4.36) the integrations of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and n. This leaves

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} dx = a_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L} dx = La_n,$$

or

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} dx.$$

Similarly, if we multiply the series (**??**) by $\sin\frac{m\pi x}{L}$ then we obtain

$$f(x)\sin\frac{m\pi x}{L} = \frac{1}{2}a_0\sin\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right),$$

which we integrate from $-L$ to L. From (4.33), the integration of $a_0\sin\frac{m\pi x}{L}$ is zero, from (4.36) the integration of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and

n and further from (4.35) the integration of $\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$ is zero except when $n=m$. This leaves

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}dx = La_n,$$

or

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where the coefficients a_n and b_n are chosen such that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{4.37}$$

for $n = 0, 1, 2, \ldots$.

Example 4

Consider

$$f(x) = 9 - x^2, \quad [-3, 3] \tag{4.38}$$

In this case $L = 3$ so from (4.37) we have

$$\begin{aligned}
a_0 &= \frac{1}{3}\int_{-3}^{3} 9 - x^2\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]\Bigg|_{-3}^{3} = 12\\
a_n &= \frac{1}{3}\int_{-3}^{3}(9 - x^2)\cos\frac{n\pi x}{3}dx\\
&= \frac{1}{3}\left[\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\sin\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\cos\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3}\\
&= \frac{36(-1)^{n+1}}{n^2\pi^2}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3}(9 - x^2)\cos\frac{n\pi x}{3}dx\\
&= \frac{1}{3}\left[-\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\cos\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\sin\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3}\\
&= 0
\end{aligned}$$

Thus, the Fourier series for (4.38) on $[-3, 3]$ is

$$f_k(x) = 6 + \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n^2\pi^2} \cos \frac{n\pi x}{3}. \tag{4.39}$$

Figure 6 show the graph of this Fourier series (4.39) with 20 terms.

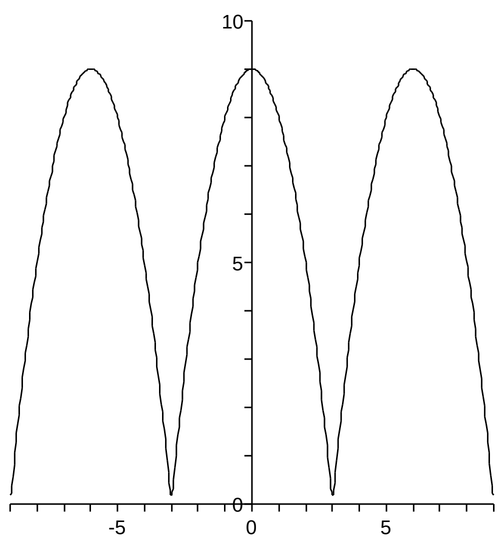

Figure 6. The solutions (4.39) with 20 terms.

Example 5

Consider

$$f(x) = \begin{cases} -2 - x & \text{if } -2 < x < -1, \\ x & \text{if } -1 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{cases} \tag{4.40}$$

In this case $L = 2$ so from (4.37) we have

$$\begin{aligned}
a_0 &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\,dx + \int_{-1}^{1} x\,dx + \int_{1}^{2}(2-x)\,dx\right\} \\
&= \frac{1}{2}\left\{\left[-2x - \frac{x^2}{2}\right]\Bigg|_{-2}^{-1} + \frac{x^2}{2}\Bigg|_{-1}^{1} + \left[2x - \frac{x^2}{2}\right]\Bigg|_{1}^{2}\right\} = 0 \\
a_n &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\cos\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx\right\} \\
&= \left[-\frac{(x+2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{-2}^{-1} \\
&+ \left[\frac{x}{n\pi}\sin\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{-1}^{1} \\
&+ \left[-\frac{(x-2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{1}^{2} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\sin\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\sin\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}\,dx\right\} \\
&= \left[\frac{(x+2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-2}^{-1} \\
&+ \left[-\frac{x}{n\pi}\cos\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-1}^{1} \\
&+ \left[\frac{(x-2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{1}^{2} \\
&= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} \frac{16}{n^2\pi^2} & \text{if } n = 1,5,9\ldots, \\ -\frac{16}{n^2\pi^2} & \text{if } n = 3,7,11\ldots, \end{cases}
\end{aligned}$$

Thus, the Fourier series for (4.40) on $[-2, 2]$ is

$$\begin{aligned}
f_k(x) &= \frac{16}{\pi^2}\left\{\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - + \ldots\right\} \\
&= \frac{16}{\pi^2}\sum_{n=1}^{k}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2} \qquad (4.41)
\end{aligned}$$

Figure 7 show the graph of this Fourier series (4.41) with 5 terms.

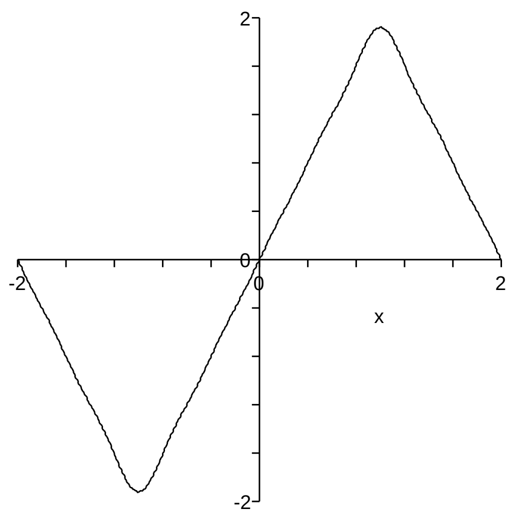

Figure 7. The solutions (4.41) with 5 terms.

Example 6

Consider

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1, \end{cases}$$

In this case $L = 1$ so from (4.37) we have

$$\begin{aligned} a_0 &= \int_0^1 dx = 1 \\ a_n &= \int_0^1 \cos n\pi x\, dx = \frac{1}{n\pi} \sin n\pi x \Big|_0^1 = 0 \end{aligned}$$

and

$$b_n = \int_0^1 \sin n\pi x\, dx - \frac{1}{n\pi} \cos\{n\pi x \Big|_0^1 = \frac{1-(-1)^n}{n\pi}$$

Thus, the Fourier series for (4.3) on $[-1, 1]$ is

$$\begin{aligned} f_k(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{k} \frac{1-(-1)^n}{n} \sin n\pi x \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{k} \frac{\sin(2n-1)\pi x}{2n-1} \end{aligned} \tag{4.42}$$

Figure 8 and 9 show the graph of this Fourier series (4.42) with 5 and 50 terms.

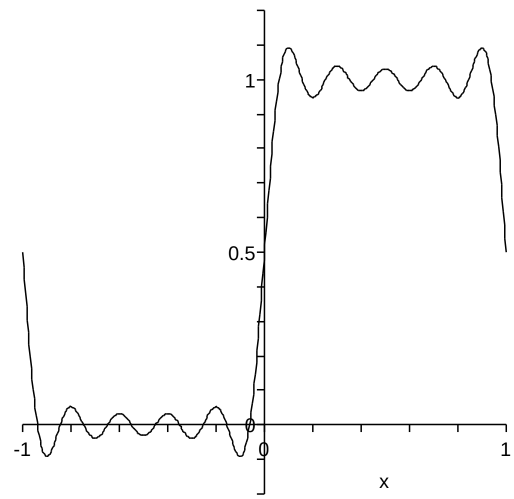

Figure 8. The solutions (4.42) with 5 terms.

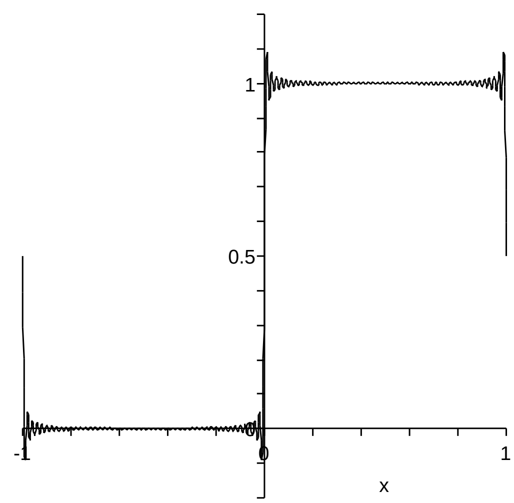

Figure 9. The solutions (4.42) with 50 terms.

It is interesting to note that regardless of the number of terms we have in the Fourier series, we cannot eliminate the spikes at $x = -1, 0, 1$ etc. This phenomena is know as **Gibb's phenomena**.

4.4 Odd and Even Extensions

Consider $f(x) = x$ on $[0,\pi]$. Here the interval is half the interval $[-\pi,\pi]$. Can we still construct a Fourier series for this? Well, it really depends on what $f(x)$ looks like on the interval $[-\pi,0]$. For example, if $f(x) = x$ on $[-\pi,0]$, then yes. If $f(x) = -x$ on $[-\pi,0]$, then also yes. In either case, as long a we are given $f(x)$ on $[-\pi,0]$, then the answer is yes. If we are just given $f(x)$ on $[0,\pi]$, then it is natural to extend $f(x)$ to $[-\pi,0]$ as either an odd extension or even extension. Recall that a function is even it $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. For example, if

$$f(x) = x, \text{then} f(-x) = -x = -f(x)$$

so $f(x) = x$ is odd. Similarly, if

$$f(x) = x^2, \text{then} f(-x) = (-x)^2 = x^2 = f(x)$$

so $f(x) = x^2$ is even. For each extension, the Fourier series constructed will contain only sine terms or cosine terms. These series respectively are called *Sine series* and *Cosine series*. Before we consider each series separately, it is necessary to establish the following lemma's.

LEMMA 1 *If $f(x)$ is an odd function then*

$$\int_{-l}^{l} f(x)\,dx = 0$$

and if $f(x)$ is an even function, then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

Proof
Consider

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx.$$

Under a change of variables $x = -y$, the second integral changes and we obtain

$$\int_{-l}^{l} f(x)\,dx = -\int_{l}^{0} f(-y)\,dy + \int_{0}^{l} f(x)\,dx.$$

If $f(x)$ is odd, then $f(-y) = -f(y)$ then

$$\begin{aligned} \int_{-l}^{l} f(x)\,dx &= \int_{l}^{0} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= -\int_{0}^{l} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= 0. \end{aligned}$$

If $f(y)$ is even, then $f(-y) = f(y)$ then

$$\begin{aligned} \int_{-l}^{l} f(x)\,dx &= -\int_{l}^{0} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= \int_{0}^{l} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= 2\int_{0}^{l} f(x)\,dx, \end{aligned}$$

establishing the result. At this point we a ready to consider each series separately.

4.4.1 Sine Series

If $f(x)$ is given on $[0,L]$ we assume that $f(x)$ is an odd function which gives us $f(x)$ on the interval $[-L,0]$. We now consider the Fourier coefficients a_n and b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.43}$$

Since $f(x)$ is odd and $\cos\frac{n\pi x}{L}$ is even, then their product is odd and by lemma 1

$$a_n = 0, \quad \forall n.$$

Similarly, since $f(x)$ is odd and $\sin\frac{n\pi x}{L}$ is odd, then their product is even and by lemma 1

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.44}$$

The Fourier series is therefore

$$f_k(x) = \sum_{n=1}^{k} b_n \sin \frac{n\pi x}{L} \tag{4.45}$$

where b_n is given in (4.44).

Example 1

Find a Fourier sine series for

$$f(x) = x^2, \quad [0,1]. \tag{4.46}$$

The coefficient b_n is given by

$$\begin{aligned} b_n &= 2\int_0^1 x \sin n\pi x \, dx \\ &= \left[-\frac{x^2 \cos n\pi\, x}{n\pi} + 2\,\frac{x \sin n\pi\, x}{n^2\pi^2} + 2\,\frac{\cos n\pi\, x}{n^3\pi^3} \right]\Bigg|_0^1 \\ &= 2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi} \end{aligned}$$

giving the Fourier Sine series as

$$f = 2\sum_{n=1}^{k} \left(2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi} \right) \sin n\pi x. \tag{4.47}$$

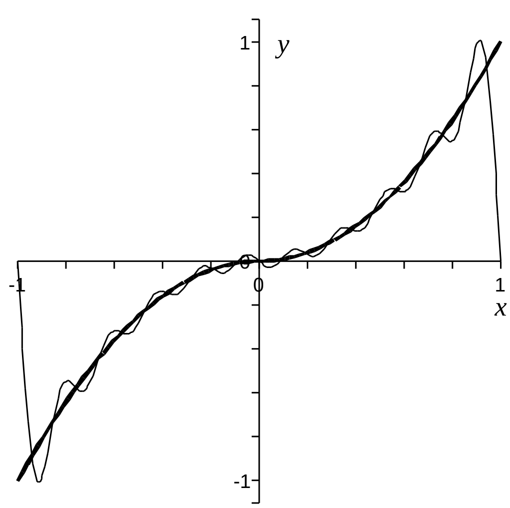

Figure 10. The function (4.46) with its odd extension and its Fourier sine series (4.47) with 10 terms .

Example 2

Find a Fourier sine series for

$$f(x) = \cos x, \quad [0, \pi] \tag{4.48}$$

Two cases need to be considered here. The case where $n = 1$ and the case where $n \neq 1$. The coefficient b_1 is given by

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin \pi x \, dx = 0$$

and the coefficient b_n is given by

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x \sin n\pi x \, dx \\ &= \left[-\frac{1}{2}\frac{\cos(n-1)x}{n-1} - \frac{1}{2}\frac{\cos(n+1)x}{n+1} \right]\Bigg|_0^{\pi} \\ &= \frac{n\,(1+(-1)^n)}{n^2-1}. \end{aligned}$$

The Fourier Sine series is then given by

$$\begin{aligned} f &= \frac{2}{\pi}\sum_{n=2}^{k} \frac{n\,(1+(-1)^n)}{n^2-1} \sin nx \\ &= \frac{8}{\pi}\sum_{n=1}^{k} \frac{n}{4n^2-1} \sin 2nx. \end{aligned} \tag{4.49}$$

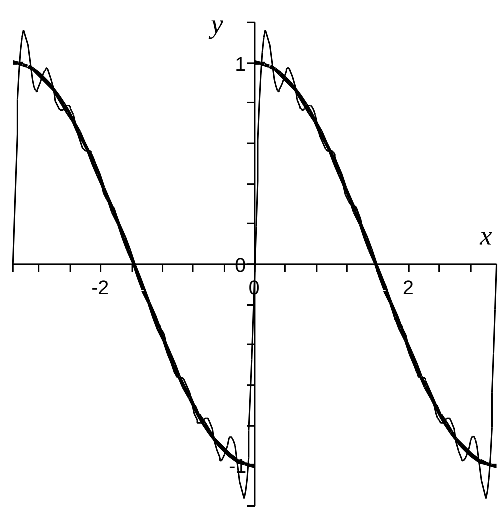

Figure 11. The function (4.48) with its odd extension and its Fourier sine series (4.49) with 20 terms .

4.4.2 Cosine Series

If $f(x)$ is given on [0,L] we assume that $f(x)$ is an even function which gives us $f(x)$ on the interval [$-L$,L]. We now consider the Fourier coefficients a_n and b_n.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \tag{4.50}$$

Since $f(x)$ is even and $\cos\frac{n\pi x}{L}$ is even, then their product is even and by lemma 1

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \tag{4.51}$$

Similarly

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{4.52}$$

Since $f(x)$ is even and $\sin\frac{n\pi x}{L}$ is odd, then their product is odd and by lemma 1

$$b_n = 0, \quad \forall\, n. \tag{4.53}$$

The Fourier series is therefore

$$f_k(x) = \frac{1}{2}a_0 + \sum_{n=1}^{k} a_n \cos\frac{n\pi x}{L} \tag{4.54}$$

where a_n is given in (4.53).

Example 3

Find a Fourier cosine series for

$$f(x) = x, \quad [0,2]. \tag{4.55}$$

The coefficient a_0 is given by

$$a_0 \;=\; \frac{2}{2}\int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2$$

The coefficient n_n is given by

$$\begin{aligned} a_n &= \int_0^2 x\cos\frac{n\pi}{2}x\,dx \\ &= \left[\frac{2x}{n\pi}\sin\frac{n\pi\,x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi\,x}{2}\right]\Big|_0^2 \\ &= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \end{aligned}$$

giving the Fourier Cosine series as

$$f = 1 + \frac{4}{\pi^2}\sum_{n=1}^{k}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi\,x}{2}. \tag{4.56}$$

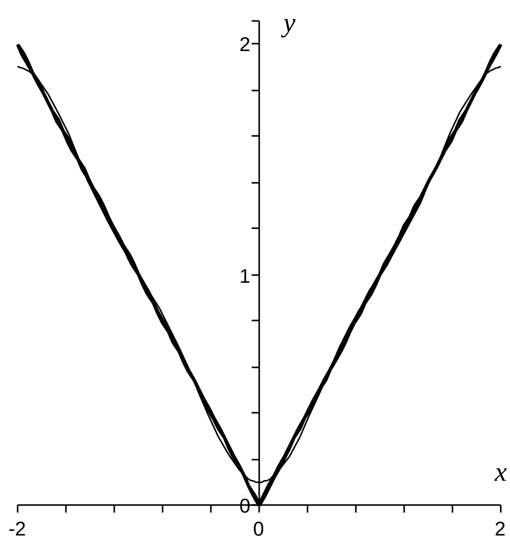

Figure 12. The function (4.55) with its even extension and its Fourier sine series (4.56) with 3 terms .

Example 4

Find a Fourier cosine series for

$$f(x) = \sin x, \quad [0, \pi]. \tag{4.57}$$

The coefficient a_0 is given by

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi \sin x\,dx \\ &= \frac{2}{\pi}\left[-\cos\ x\right]\Big|_0^\pi \\ &= \frac{4}{\pi} \end{aligned} \tag{4.58}$$

For the remaining coefficients a_n, the case a_1 again needs to be considered separately. For a_1

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\,dx = 0$$

and the coefficient $a_n,\ \ n \geq 2$ is given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos n\pi x\,dx \\ &= \left[\frac{1}{2}\frac{\cos(n-1)x}{n-1} - \frac{1}{2}\frac{\cos(n+1)x}{n+1}\right]\Bigg|_0^\pi \\ &= -\frac{n\left(1+(-1)^n\right)}{n^2-1}. \end{aligned}$$

Thus, the Fourier series is

$$\begin{aligned} f &= -\frac{2}{\pi}\sum_{n=2}^{k}\frac{n\left(1+(-1)^n\right)}{n^2-1}\sin nx \\ &= -\frac{8}{\pi}\sum_{n=1}^{k}\frac{n}{4n^2-1}\ \sin 2nx. \end{aligned} \tag{4.59}$$

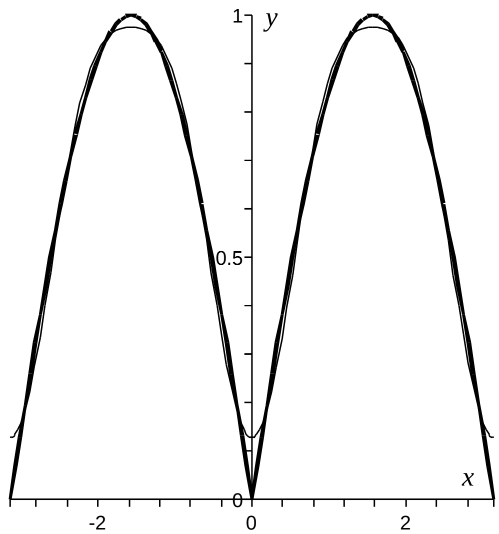

Figure 13. The function (4.57) with its even extension and its Fourier sine series (4.59) with 5 terms .

Example 5

Find a Fourier sine and cosine series for

$$f(x) = \begin{cases} 4x - x^2 & \text{for } 0 \leq x \leq 2 \\ 8 - 2x & \text{for } 2 < x < 4 \end{cases} \tag{4.60}$$

For the Fourier sine series $a_n = 0$ and b_n are obtained by

$$\begin{aligned} b_n &= \frac{2}{4}\int_0^2 (4x - x^2)\sin\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8-2x)\sin\frac{n\pi x}{4}\,dx \\ &= \left[\left(\frac{32 - 16x}{n^2\pi^2}\right)\sin\frac{n\pi x}{4} + \left(\frac{2x^2 - 8x}{n\pi} - \frac{64}{n^3\pi^3}\right)\cos\frac{n\pi x}{4}\right]\Bigg|_0^2 \\ &+ \left[\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4} - \frac{4x - 16}{n\pi}\cos\frac{n\pi x}{4}\right]\Bigg|_2^4 \\ &= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{64}{n^3\pi^3}\left(1 - \cos\frac{n\pi}{2}\right). \end{aligned}$$

Thus, the Fourier sine series is given by

$$f = \frac{16}{\pi^2}\sum_{n=1}^{k}\left(\frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{4}{n^3\pi}\left(1 - \cos\frac{n\pi}{2}\right)\right)\sin\frac{n\pi x}{4}. \tag{4.61}$$

For the Fourier cosine series $b_n = 0$ and a_0 and a_n are given by

$$
\begin{aligned}
a_0 &= \frac{2}{4}\int_0^2 4x - x^2\,dx + \frac{2}{4}\int_2^4 8 - 2x\,dx \\
&= \frac{1}{2}\left[2x^2 - \frac{x^3}{3}\right]\Bigg|_0^2 + \frac{1}{2}\left[8x - x^2\right]\Big|_2^4 \\
&= \frac{8}{3} + 2 = \frac{14}{3},
\end{aligned}
$$

and

$$
\begin{aligned}
a_n &= \frac{2}{4}\int_0^2 (4x - x^2)\cos\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8 - 2x)\cos\frac{n\pi x}{4}\,dx \\
&= \left[\left(\frac{32 - 16x}{n^2\pi^2}\right)\cos\frac{n\pi x}{4} - \left(\frac{2x^2 - 8x}{n\pi} - \frac{64}{n^3\pi^3}\right)\sin\frac{n\pi x}{4}\right]\Bigg|_0^2 \\
&+ \left[-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4} + \frac{4x - 16}{n\pi}\sin\frac{n\pi x}{4}\right]\Bigg|_2^4 \\
&= \frac{16}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{64}{n^3\pi^3}\sin\frac{n\pi}{2}.
\end{aligned}
$$

Thus, the Fourier cosine series is given by

$$
f = \frac{14}{3} + \frac{16}{\pi^2}\sum_{n=1}^{k}\left(\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{4}{n^3\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{4}, \tag{4.62}
$$

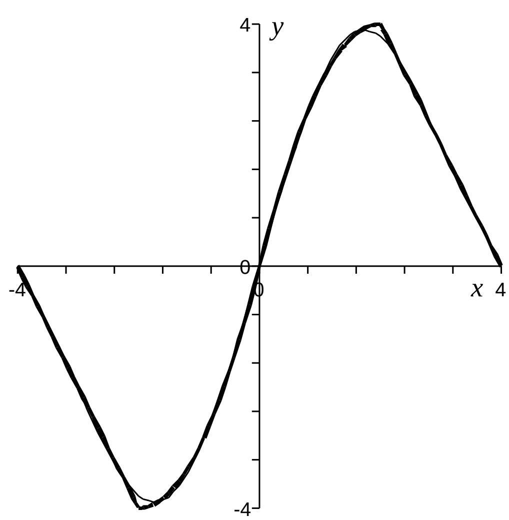

Figure 14. The function (4.60) with its odd extension and its Fourier sine series (4.61) with 3 terms .

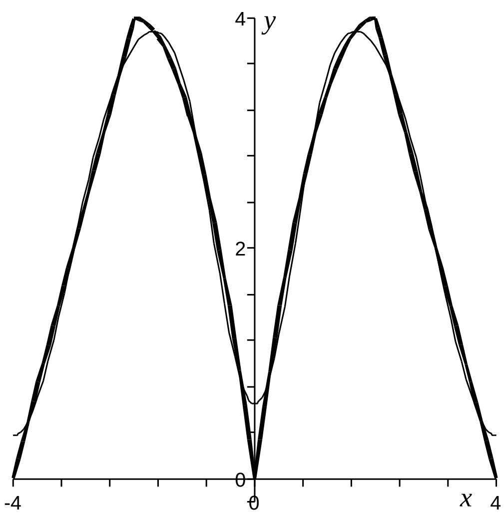

Figure 15. The function (4.60) with its even extension and its Fourier cosine series (4.62) with 3 terms .

As shown in the examples in this chapter, often only a few terms are needed to obtain a fairly good representation of the function. It is interesting to note that if discontinuity is encountered on the extension, Gibb's phenomena occurs. In the next chapter, we return to solving the heat equation, Laplace's equation and the wave equation using separation of variables as introduced at the beginning of this chapter.

Exercises

1. Find Fourier series for the following

(i) $f(x) = \mathrm{e}^{-x}$ on $[-1, 1]$

(ii) $f(x) = |\,x\,|$ on $[-2, 2]$

(iii) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ x - 1 & \text{if } \ \ 0 < x < 1, \end{cases}$

(iv) $f(x) = \mathrm{e}^{-x^2}$ on $[-5, 5]$.

2. Find Fourier sine and cosine series for the following and illustrate the function and its corresponding series on $[-L, L]$ and $[-2L, 2L]$

(i) $f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 < x < 2, \\ 3 - x & \text{if } 2 < x < 3, \end{cases}$

(ii) $f(x) = x - x^2$ on $[0, 2]$

(iii) $f(x) = \begin{cases} x + 1 & \text{if } 0 < x < 1, \\ 4 - 2x & \text{if } 1 < x < 2, \end{cases}$,

(iv) $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{cases}$,

3. Find the first 10 terms numerically of the Fourier series of the following

(i) $f(x) = \mathrm{e}^{-x^2}$, on $[-5, 5]$

(ii) $f(x) = \sqrt{x}$, on $[0, 4]$

(iii) $f(x) = -x \ln x$ on $[0, 1]$.