Chapter 5. Separation of Variables

At this point we are ready to now resume our work on solving the three main equations: the heat equation, Laplace's equation and the wave equation using the method of separation of variables.

4.1 The heat equation

Consider, for example, the heat equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{4.1}$$

subject to the initial and boundary conditions

$$u(x,0) = x - x^2, \quad u(0,t) = u(1,t) = 0. \tag{4.2}$$

Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{4.3}$$

shows that the heat equation (4.1) becomes

$$XT' = X''T,$$

which, after dividing by XT and expanding gives

$$\frac{T'}{T} = \frac{X''}{X}, \tag{4.4}$$

implying that

$$T' = \lambda T, \quad X'' = \lambda X, \tag{4.5}$$

where λ is a constant. From (4.2) and (4.3), the boundary conditions becomes

$$X(0) = X(1) = 0. \tag{4.6}$$

Integrating the X equation in (4.5) gives rise to three cases depending on the sign of λ but as seen in the last chapter, only the case where $\lambda = -k^2$ for some constant k is applicable which we have as the solution

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{4.7}$$

Imposing the boundary conditions (4.6) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin k + c_2 \cos k = 0, \tag{4.8}$$

which leads to

$$c_2 = 0, \quad c_1 \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots n\pi, \tag{4.9}$$

where n is an integer. From (4.5), we further deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where we have set $c_1 c_3 = b_n$. Using the initial condition gives

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

At this point, we recognize that we have a Fourier sine series and that the coefficients b_n are chosen such that

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2) \sin n\pi x \, dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2} \cos n\pi x + \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right) \cos n\pi x\right]\Bigg|_0^1 \\ &= \frac{4}{n^3\pi^3}(1 - (-1)^n). \end{aligned}$$

Thus, the solution of the PDE as

$$u(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \tag{4.10}$$

Figure 1 shows the solution at times $t = 0, 0.1$ and 0.2.

Example 1

Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{4.11}$$

subject to

$$u(x,0) = x - x^2, \quad u_x(0,t) = u_x(1,t) = 0. \tag{4.12}$$

This problem is similar to the proceeding problem except the boundary conditions are different. The last problem had the boundaries fixed at zero whereas in this problem, the boundaries are insulated (*i.e.* no flux boundary conditions). Assuming that the solutions are separable

$$u(x,t) = X(x)T(t), \tag{4.13}$$

then from the heat equation, we obtain

$$T' = \lambda T, \quad X'' = \lambda X, \tag{4.14}$$

where λ is a constant. The boundary conditions in (4.12) become, accordingly

$$X'(0) = X'(1) = 0. \tag{4.15}$$

Integrating the X equation in (4.14) gives rise to again three cases depending on the sign of λ but as seen earlier, only the case where $\lambda = -k^2$ for some constant k is relevant. Thus, we have

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{4.16}$$

Imposing the boundary conditions (4.15) shows that

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0, \tag{4.17}$$

which leads to

$$c_1 = 0, \quad \sin k = 0 \ \Rightarrow \ k = 0, \pi, 2\pi, \ldots, n\pi, \tag{4.18}$$

where n is an integer. From (4.14), we also deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x,$$

where we have set $c_1 c_3 = a_n$. Using the initial condition gives

$$\begin{aligned} u(x,0) = x - x^2 &= \sum_{n=0}^{\infty} a_n \cos n\pi x \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x. \end{aligned}$$

We again recognize that we have a Fourier cosine series and that the coefficients a_n are chosen such that

$$\begin{aligned} a_0 &= 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{3}, \\ a_n &= 2\int_0^1 (x - x^2)\cos n\pi x\,dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2}\sin n\pi x - \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right)\sin n\pi x\right]\Bigg|_0^1 \\ &= -\frac{2}{n^3\pi^3}(1 + (-1)^n). \end{aligned}$$

Thus, the solution of the PDE as

$$u(x,t) = \frac{1}{3} + \frac{2}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^(n+1) - 1}{n^3} e^{-n^2\pi^2 t} \cos n\pi x. \tag{4.19}$$

Figure 1 shows the solution at times $t = 0, 0.25$ and 0.5. It is interesting to note that even though that same initial condition are used for each of the two problems, fixing the boundaries and insulated them gives rise two totally different behaviors after $t \geq 0$.

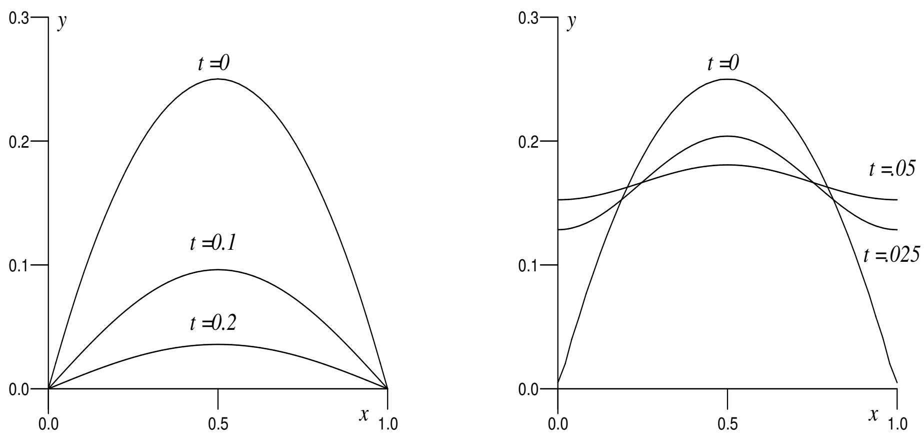

Figure 1. The solution of the heat equation with the same initial condition with fixed and no flux boundary conditions.

Example 2

Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{4.20}$$

subject to

$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \end{cases} \quad u(0,t) = u_x(2,t) = 0. \tag{4.21}$$

In this problem, we have a mixture of both fixed and no flux boundary conditions. Again, assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{4.22}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{4.23}$$

where λ is a constant. The boundary conditions in (4.21) becomes, accordingly

$$X(0) = X'(1) = 0. \tag{4.24}$$

Integrating the X equation in (4.23) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{4.25}$$

Imposing the boundary conditions (4.24) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 k \cos 2k - c_2 k \sin 2k = 0,$$

which leads to

$$c_2 = 0, \quad \cos 2k = 0 \quad \Rightarrow \quad k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots, \frac{(2n-1)\pi}{4},$$

for integer n. From (4.23), we then deduce that

$$T(t) = c_3 e^{-\frac{(2n-1)^2}{16}\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{(2n-1)}{4}\pi x,$$

where we have set $c_1 c_3 = b_n$. Using the initial condition give

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)}{4}\pi x.$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned}
b_n &= 2\int_0^1 x \sin \frac{(2n-1)}{4}\pi x \, dx + \int_1^2 (2-x) \sin \frac{(2n-1)}{4}\pi x \, dx \\
&= \left[\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x - \frac{8x}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_0^1 \\
&+ \left[-\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x + \frac{8(x-2)}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_1^2 \\
&= \frac{32}{(2n-1)^2\pi^2} \left(\sin \frac{(2n-1)\pi}{4} + \cos n\pi \right).
\end{aligned}$$

Hence, the solution of the PDE is

$$u(x,t) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{(2n-1)\pi}{4} + \cos n\pi \right)}{(2n-1)^2} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{2n-1}{4}\pi x. \quad (4.26)$$

Figure 2 shows the solution both in short time $t = 0, 0.1, 0.2$ and long time $t = 10, 20, 30$.

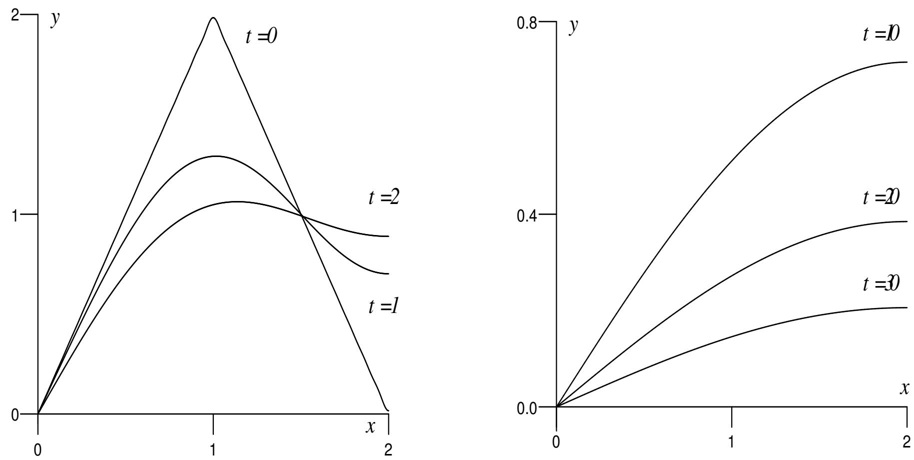

Figure 2. Short time and long time behavior of the solution (4.26).

Example 3

Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{4.27}$$

subject to

$$u(x,0) = 2x - x^2 \quad u(0,t) = 0, \;\; u_x(2,t) = -u(2,t). \tag{4.28}$$

In this problem, we have a fixed left endpoint and a radiating right end point. Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{4.29}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{4.30}$$

where λ is a constant. The boundary conditions in (4.28) becomes, accordingly

$$X(0) = 0, \quad X'(2) = -X(2). \tag{4.31}$$

Integrating the X equation in (4.30) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{4.32}$$

Imposing the first boundary condition of (4.31) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \;\; \Rightarrow \quad c_2 = 0$$

and the second boundary condition of (4.31) gives

$$\tan 2k = -k. \tag{4.33}$$

It is very important that we recognize that the solutions of (4.33) are not equally spaced as seen in earlier problems. In fact, there are an infinite number of solutions of this equation. Figure 3 shows graphically the curves $y = -k$ and $y = \tan 2k$. The first three intersection points are the first three solutions of (4.33).

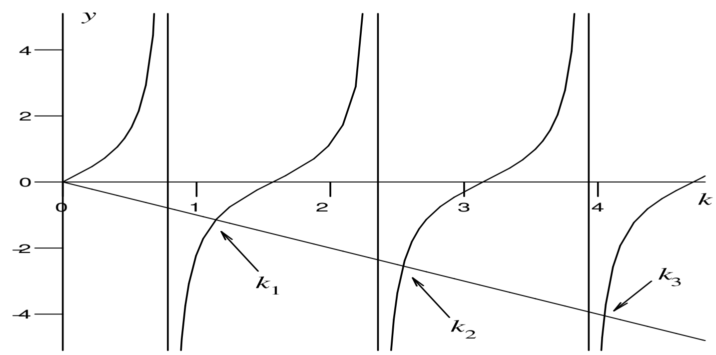

Figure 3. The graph of $y = \tan 2k$ and $y = -k$.

Thus, it is necessary that we solve for k numerically. The first 10 solutions are given in table 1.

n	k_n	n	k_n
1	1.144465	6	8.696622
2	2.54349	7	10.258761
3	4.048082	8	11.823162
4	5.586353	9	13.389044
5	7.138177	10	14.955947

Table 1. The first ten solution of $\tan 2k = -k$.

Therefore, we have

$$X(x) = c_1 \sin k_n x. \tag{4.34}$$

Further, integrating (4.30) for T gives

$$T(t) = c_3 e^{-k_n^2 t} \tag{4.35}$$

and together with X, we have the solution to the PDE as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} \sin k_n x, \tag{4.36}$$

Imposing the boundary conditions (4.28) gives

$$u(0,t) = 2x - x^2 = \sum_{n=1}^{\infty} c_n \sin k_n x, \tag{4.37}$$

It is important to know that the c_n's are **not** given by the formula

$$c_n = \frac{2}{2}\int_0^2 (2x - x^2) sin k_n x, dx$$

as usual. The reason for this is that the k_n's are not equally spaced. So it is necessary to examine (4.37) on its own. Multiplying by $\sin k_m x$ and integrating over [0,2] gives

$$\int_0^2 (2x - x^2) \sin k_m x\, dx = \sum_{n=1}^{\infty} c_n \int_0^2 \sin k_m x\, \sin k_n x\, dx,$$

For $n \neq m$, we have

$$\int_0^2 \sin k_m x\, \sin k_n x\, dx = \frac{k_m \sin 2k_n \cos 2k_m - k_n \sin 2k_m \cos 2k_n}{k_n^2 - k_m^2} \tag{4.38}$$

and imposing (4.33) for each of k_m and k_n shows (4.38) to be identically satisfied. Therefore, we obtain the following when $n = m$

$$\int_0^2 (2x - x^2) \sin k_n x\, dx = c_n \int_0^2 \sin^2 k_n x\, dx,$$

or

$$c_n = \frac{\int_0^2 (2x - x^2) \sin k_n x\, dx}{\int_0^2 \sin^2 k_n x\, dx,} \tag{4.39}$$

Table 2 gives the first ten c_n's that correspond to each k_n.

n	c_n	n	c_n
1	0.873214	6	0.028777
2	0.341898	7	-.016803
3	-.078839	8	0.015310
4	0.071427	9	-.010202
5	-.032299	10	0.009458

Table 2. The coefficients c_n from (4.77).

Having obtain k_n and c_n, the solution to the problem is found in (4.36). Figure 4 show plots at time $t = 0, 1$, and 2 when 20 terms are used.

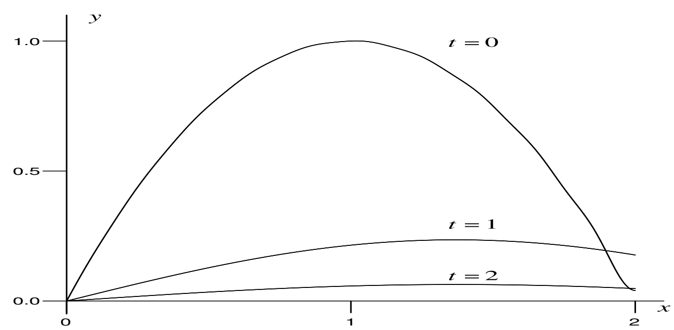

Figure 4. The solution (4.36).

4.1.1 Nonhomogeneous boundary conditions

In the preceding examples, the boundary conditions where either fixed to zero, insulted or radiating. Often, we encounter boundary condition which are non standard or nonhomogeneous. For example, the boundary may be fixed to particular constant or the flux is maintained at a constant value. The following examples illustrate.

Example 4

Solve

$$u_t = u_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{4.40}$$

subject to

$$u(x, 0) = 4x - x^2 \quad u(0, t) = 0, \;\; u(3, t) = 3. \tag{4.41}$$

If we seek separable solutions $u(x, t) = X(x)T(t)$, then from (4.41) we have

$$X(0)T(t) = 0, \quad X(3)T(t) = 3, \tag{4.42}$$

and we have a problem $-$ the second boundary condition doesn't separate. To over come this we introduce the transformation $u = v + ax + b$ and ask, "Can we choose the constants a and b as to fix both boundary conditions to zero. Upon substitution of both boundary conditions (4.41), we obtain

$$0 = v(0,t) + a(0) + b, \;\; 3 = v(3,t) + 3a + b \tag{4.43}$$

Now we require that $v(0,t) = 0$ and $v(3,t) = 0$ which implies that we must choose $a = 1$ and $b = 0$. Therefore, we have

$$u = v + x. \tag{4.44}$$

We notice that under the transformation (4.44), the original equation doesn't change form, *i.e.*

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx}$$

however, the initial condition does change, it becomes

$$v(x,0) = 3x - x^2.$$

Thus, we have the new problem to solve

$$v_t = v_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{4.45}$$

subject to

$$v(x,0) = 3x - x^2 \quad v(0,t) = 0, \;\; v(3,t) = 3. \tag{4.46}$$

At this point, we seek the usual separable solutions $V(x,t) = X(x)T(t)$ which lead to the and the systems $X'' = -k^2 X$ and $T' = -k^2 T$ with the boundary conditions $X(0) = 0$ and $X(3) = 0$. Solving for X gives

$$X(x) = c_1 \sin kx + c_2 \cos kx, \tag{4.47}$$

and imposing both boundary conditions gives

$$X(x) = c_1 \sin \frac{n\pi}{3} x, \tag{4.48}$$

and

$$T(t) = c_3 e^{-\frac{n^2\pi^2}{9} t}$$

where n is an integer. Therefore, we have the solution of (4.45) as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3}x.$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3x - x^2) \sin \frac{n\pi}{3}x\, dx \\ &= \left[-\frac{6(2x-3)}{n^2\pi^2} \sin \frac{n\pi}{3}x + \frac{3(n^2\pi^2x^2 - 3n^2\pi^2x - 18)}{n^3\pi^3} \cos \frac{n\pi}{3}x \right]\Bigg|_0^3 \\ &= \frac{32(1-(-1)^n)}{n^3\pi^3}. \end{aligned}$$

This gives

$$v(x,t) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3}x.$$

and since $u = v + x$, we obtain the solution for u as

$$u(x,t) = x + \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3}x. \tag{4.49}$$

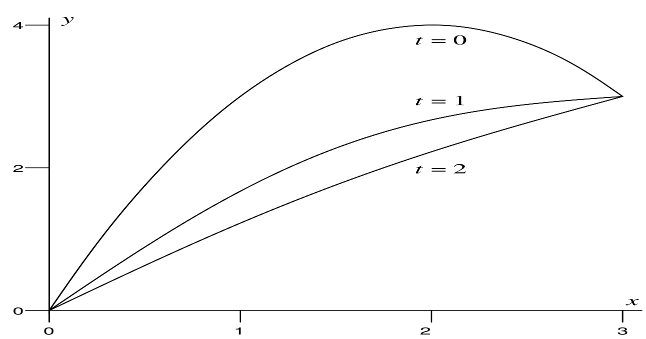

Figure 5. The solution (4.49) at time $t = 0, 1$ and 2.

Example 5

Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{4.50}$$

subject to

$$u(x,0) = 0 \quad u_x(0,t) = -1, \;\; u_x(1,t) = 0. \tag{4.51}$$

Unfortunately, the trick $u = v + ax + b$ won't work since $u_x = v_x + a$ and choosing a to fix the right boundary to zero only makes the left boundary nonzero. To overcome this we might try $u = v + ax^2 + bx$ but the original equation changes

$$u_t = u_{xx}, \;\Rightarrow\; v_t = v_{xx} + 2a \tag{4.52}$$

As a second attempt, we try

$$u = v + a(x^2 + 2t) + bx \tag{4.53}$$

noting now

$$u_t = u_{xx}, \;\Rightarrow\; v_t = v_{xx}. \tag{4.54}$$

Since $u_x = v_x + 2ax + b$, then choosing $a = 1/2$ and $b = -1$ gives the the new boundary conditions as $v_x(0,t) = 0$ and $v_x(2,t) = 0$ and the transformation becomes

$$u = v + \frac{1}{2}(x^2 + 2t) - x, \tag{4.55}$$

Finally, we consider the initial condition. From (4.55), we have $v(x,0) = u(x,0) - \frac{1}{2}x^2 + x = x - \frac{x^2}{2}$ and our problem is transformed to the new problem

$$v_t = v_{xx}, \quad 0 < x < 1, \;\; t > 0 \tag{4.56}$$

subject to

$$v(x,0) = x - \frac{1}{2}x^2, \;\; v_x(0,t) = v_x(1,t) = 0 \tag{4.57}$$

A separation of variables $v = XT$ leads to $X'' = -k^2 X$ and $T' = -k^2 T$ from which we obtain

$$X = c_1 \sin kx + c_2 \cos kx, \quad X = c_1 k \cos kx - c_2 k \sin kx, \tag{4.58}$$

and imposing the boundary conditions (4.57) gives

$$c_1 = 0, \quad k = n\pi, \tag{4.59}$$

where n is an integer. This then leads to

$$X(x) = c_2 \cos n\pi x \tag{4.60}$$

and further

$$T(t) = c_3 e^{-n^2\pi^2 t}. \tag{4.61}$$

Finally we arrive at

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x.$$

noting that we have chosen $a_n = c_1 c_3$. Upon substitution of $t = 0$ and using the initial condition (4.57), we have

$$x - \frac{1}{2}x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

a Fourier cosine series. The coefficients are obtained by

$$\begin{aligned}
a_0 &= \frac{2}{1}\int_0^1 \left(x - \frac{1}{2}x^2\right), dx = x^2 - \frac{1}{3}x^3\bigg|_0^1 = \frac{2}{3}, \\
a_n &= \frac{2}{1}\int_0^1 (x - \frac{1}{2}x^2)\cos n\pi x\, dx \\
&= \left[\frac{-2(x-1)}{n^2\pi^2}\cos n\pi x - \left(\frac{(2x-x^2)}{n\pi} + \frac{2}{n^3\pi^3}\right)\sin n\pi x\right]\bigg|_0^1 \\
&= -\frac{2}{n^2\pi^2}.
\end{aligned}$$

Thus, we obtain the solution for v as

$$v(x,t) = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t}\cos n\pi x.$$

and this, together with the transformation (4.55) gives

$$u(x,t) = \frac{1}{2}(x^2 + 2t) - x + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2 t}\cos n\pi x. \tag{4.62}$$

Figure 6 shows plots at time $t = 0.01, 0.5, 1.0$ and 1.5. It is interesting to note that at the left boundary $u_x = -1$ and since the flux $\phi = -ku_x$ implies that $\phi = k > 0$ which gives that the flux is position and that heat is being added at the left boundary. Hence the profile increase at the left while insulated at the right boundary (no flux).

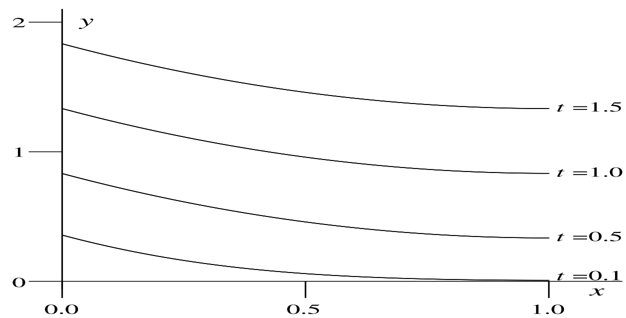

Figure 6. The solution (4.62) at time $t = 0, 1$ and 2.

A natural question is, can we transform

$$u_t = u_{xx}, \;\; 0 < x < L, \;\; t > 0, \tag{4.63}$$

$$u(x,0) = f(x), \;\; u(0,t) = p(t), \;\; u(L,t) = q(t) \tag{4.64}$$

to a problem with standard boundary conditions. The answer is yes. We seek a transformation of the form

$$u = v + A(t)x + B(t) \tag{4.65}$$

as to transform the nonstandard boundary conditions to standard ones. On substitution of the u and v boundary conditions, we obtain

$$p(t) = 0 + A(t)0 + B(t), \;\; q(t) = 0 + A(t)L + B(t) \tag{4.66}$$

and solving for $A(t)$ and $B(t)$ gives

$$A(t) = \frac{q(t) - p(t)}{L}, \;\; B(t) = p(t), \tag{4.67}$$

which results in the transformation

$$\begin{aligned} u &= v + \frac{q(t) - p(t)}{L}x + p(t) \\ &= v + q(t)\frac{x}{L} + p(t)\frac{(L-x)}{L}. \end{aligned} \tag{4.68}$$

However in doing so, we change not only the original equation but also the initial condition. They becomes, respectively

$$v_t = v_{xx} - q'(t)\frac{x}{L} + p'(t)\frac{(L-x)}{L},$$
$$v(x,0) = f(x) - q(0)\frac{x}{L} - p(0)\frac{(L-x)}{L}.$$

The new initial condition doesn't pose a problem but how do we solve the heat equation with a term added to the equation.

4.1.2 Nonhomogeneous equations

We now focus our attention to solving the heat equation with a source term

$$u_t = u_{xx} + Q(x), \quad 0 < x < L, \quad t > 0, \tag{4.69}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x).$$

To investigate this problem, we will considered a particular example where $L = 2$, $f(x) = 2x - x^2$ and $Q(x) = 1 - |x - 1|$. If we were to consider this problem without a source term we would have solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{2}, \tag{4.70}$$

where

$$T_n(t) = \frac{16(1-(-1)^n)}{n^3\pi^3} e^{-\frac{n^2\pi^2}{4}t} \tag{4.71}$$

We note that even if this T_n wasn't known, we could find it as substitution of (4.70) into the heat equation and isolating coefficients of $\sin \frac{n\pi x}{2}$, would lead to

$$T_n'(t) = -\frac{n^2\pi^2}{4} T_n(t),$$

leading to the solution (4.71). For the problem with a source term we look for solutions of the same form *i.e.* (4.70). However, in order that this technique works, it is necessary to expand the source term also in terms of a Fourier sine series, *i.e.*

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{2}. \tag{4.72}$$

For

$$Q(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{cases} \tag{4.73}$$

where

$$
\begin{aligned}
q_n &= \int_0^2 Q(x) \sin \frac{n\pi x}{2} dx. \\
&= \int_0^1 x \sin \frac{n\pi x}{2} dx + \int_x^2 (2-x) \sin \frac{n\pi x}{2} dx, \\
&= \left[\frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} - 2xn\pi \cos \frac{n\pi x}{2} \right] \Bigg|_0^1 \\
&+ \left[-\frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} + \frac{2x_4}{n\pi} \cos \frac{n\pi x}{2} \right] \Bigg|_1^2, \\
&= \frac{8}{n^2\pi^2} \sin \frac{n\pi x}{2}.
\end{aligned} \tag{4.74}
$$

Substituting both (4.70) and (4.72) into (4.69) gives

$$
\sum_{n=1}^{\infty} T_n'(t) \sin \frac{n\pi x}{2} = \sum_{n=1}^{\infty} - \left(\frac{n\pi x}{2} \right)^2 T_n(t) \sin \frac{n\pi x}{2} + \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{2},
$$

and re-grouping and isolating the coefficients of $\sin \frac{n\pi x}{2}$ gives

$$
T_n'(t) + \frac{n^2\pi^2}{4} T_n(t) = q_n, \tag{4.75}
$$

a linear ODE in $T_n(t)$! On solving (4.75) we obtain

$$
T_n(t) = \frac{4}{n^2\pi^2} q_n + b_n e^{-(\frac{n\pi}{2})^2 t},
$$

giving the final solution

$$
u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + b_n e^{-(\frac{n\pi}{2})^2 t} \right) \sin \frac{n\pi x}{2}. \tag{4.76}
$$

Imposing the initial condition gives (4.70)

$$
2x - x^2 = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + b_n \right) \sin \frac{n\pi x}{2}.
$$

If we set

$$
c_n = \frac{4}{n^2\pi^2} q_n + b_n,
$$

then we have

$$
2x - x^2 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2}.
$$

a regular Fourier sine series. Therefore

$$\begin{aligned} c_n &= \int_0^2 \left(2x - x^2\right) \sin \frac{n\pi x}{2} dx. \\ &= \frac{16}{n^3\pi^3} \left(1 - \cos \frac{n\pi}{2}\right), \end{aligned} \tag{4.77}$$

which in turn, gives

$$b_n = c_n - \frac{4}{n^2\pi^2} q_n,$$

and finally, the solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + \left(c_n - \frac{4}{n^2\pi^2} q_n\right) e^{-\left(\frac{n\pi}{2}\right)^2 t} \right) \sin \frac{n\pi x}{2}, \tag{4.78}$$

where q_n and c_n are given in (4.74) and (4.77), respectively. Typical plots are given in figure 7 at times $t = 0, 1, 2$ and 3.

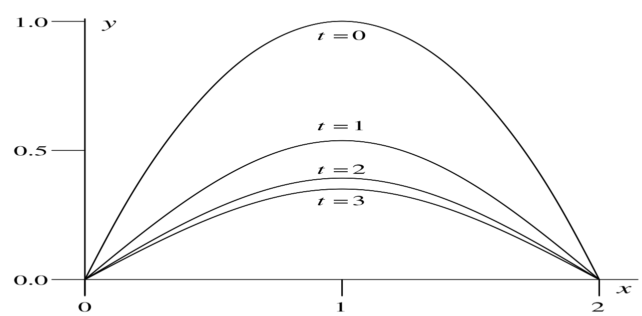

Figure 7. The solution (4.78) at time $t = 0, 1, 2$ and 3.

It is interesting to note that if we let $t \to \infty$ the solution approaches the same curve at $t = 3$. This is what is called steady state (no changes in time). It is natural to ask "Can we find this steady state solution.?" The answer is yes. For the steady state, $u_t \to 0$ as $t \to \infty$ and the original PDE becomes

$$u_{xx} + Q(x) = 0. \tag{4.79}$$

Integrating twice with $Q(x)$ given in (4.73) gives

$$u = \begin{cases} -\frac{x^3}{6} + c_1 x + c_2 & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + k_1 x + k_2 & \text{if } 1 < x < 2, \end{cases} \tag{4.80}$$

where c_1, c_2, k_1 and k_2 are constants of integration. Imposing that the solution and its first derivative are continuous at $x = 1$ and that the solution is zer0 at the endpoints gives

$$c_1 - c_1 + 1 = 0, \;\; c_1 + c_2 - k_1 - k_2 + \frac{2}{3} = 0,$$
$$c_2 = 0, \quad 2k_1 + k_2 - \frac{8}{3} = 0,$$

which gives, upon solving

$$c_1 = \frac{1}{2}, \;\; c_2 = 0, \;\; k_1 = \frac{3}{2}, \;\; k_2 = -\frac{1}{3} \tag{4.81}$$

This, in turn gives the steady state solution as

$$u = \begin{cases} -\frac{x^3}{6} + \frac{x}{2} & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + \frac{3x}{2} - \frac{1}{3} & \text{if } 1 < x < 2. \end{cases} \tag{4.82}$$

4.1.3 Equations with a solution dependent source term

We now consider the heat equation with a solution dependent source term. For simplicity we will consider a source term that is linear. Take, for example

$$u_t = u_{xx} + \alpha u, \quad 0 < x < 1, \;\; t > 0,$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = x - x^2, \tag{4.83}$$

where α is some constant. We could try a separation of variables to obtain solutions for this problem, but for more complicated source terms like $Q(x,t)u$, a separation of variables is unsuccessful. Therefore, we try a different technique. Here we will try and transform the PDE to one that has no source term. In attempting to do so, we seek a transformation of the form

$$u(x,t) = A(x,t)v \tag{4.84}$$

and ask "Is is possible to find A such that the source term in (4.83) can be removed?" Substituting of (4.84) in (4.83) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx} v + \alpha Av,$$

and dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + 2\frac{A_x}{A}v_x + \left(\frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha\right)v.$$

In order to target the standard heat equation, we choose

$$A_x = 0, \quad \frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha = 0. \tag{4.85}$$

From the first we obtain that $A = A(t)$ and from the second we obtain $A' = \alpha A$ which has the solution $A(t) = A_0 e^{\alpha t}$ for some constant A_0. The boundary conditions becomes

$$\begin{aligned} u(0,t) = 0 \ &\Rightarrow \ A_0 e^{\alpha t} v(0,t) = 0 \ \Rightarrow \ v(0,t) = 0, \\ v(1,t) = 0 \ &\Rightarrow \ A_0 e^{\alpha t} v(1,t) = 0 \ \Rightarrow \ v(1,t) = 0, \end{aligned}$$

so the boundary conditions become unchanged. Next, we consider the initial condition, so

$$u(x,0) = x - x^2 \ \Rightarrow \ A_0 e^{\alpha 0} v(x,0) = 0 \ \Rightarrow \ A_0 v(x,0) = x - x^2,$$

and as to leave the initial condition unchanged, we choose $A_0 = 1$. Thus, under the transformation

$$u = e^{\alpha t} v \tag{4.86}$$

the problem (4.83) becomes

$$\begin{gathered} v_t = v_{xx}, \quad 0 < x < 1, \ \ t > 0, \\ v(x,0) = 0, \quad v(1,t) = 0, \quad v(x,0) = x - x^2, \end{gathered} \tag{4.87}$$

This particular problem was considered at the beginning of this chapter, (4.1) where the solution was given in (4.26), namely

$$v(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}e^{-n^2\pi^2 t}\sin n\pi x,$$

and so, from (4.86) we obtain the solution of (4.83) as

$$u(x,t) = \frac{4}{\pi^3}e^{\alpha t}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}e^{-n^2\pi^2 t}\sin n\pi x. \tag{4.88}$$

Figure 8 shows plots at times $t = 0, 0.1$ and 0.2 when $\alpha = 5$ and 12. It is interesting to note that in the case where $\alpha = 5$, the diffusion is slower in comparison with no source term (*i.e* $\alpha = 0$ see Figure 1) and there is no diffusion at all when $\alpha = 12$.

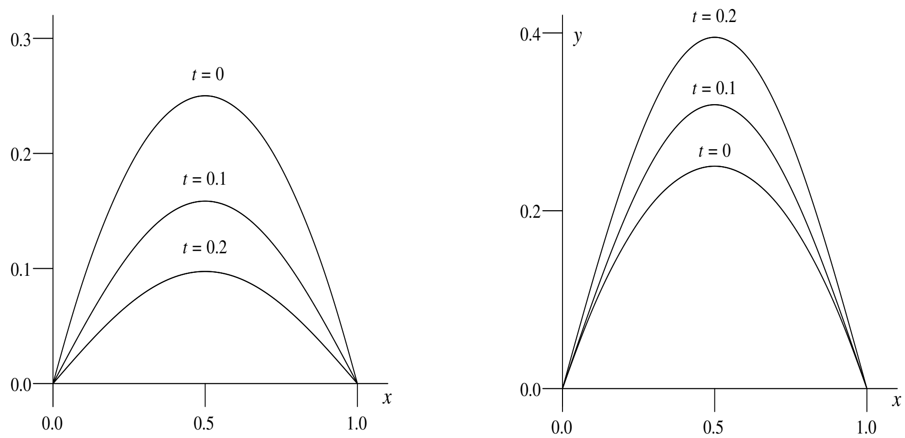

Figure 8. The solution (4.88) of the heat equation with a source (4.83) with $\alpha = 5$ and $\alpha = 12$.

It is natural to ask, for what value of α do we achieve a steady state solution. To answer this consider the first few terms of the solution (4.88)

$$u = \frac{8}{\pi^3} e^{\alpha t} \left(e^{-\pi^2 t} \sin \pi x + \frac{1}{27} e^{-9\pi^2 t} \sin 3\pi x + \dots \right). \tag{4.89}$$

Now clearly the exponential terms in $(\cdot)$ will decay to zero with the first term decaying the slowest. Therefore, it is the balance between $e^{\alpha t}$ and $e^{-\pi^2 t}$ which determine whether the solution will decay to zero or not. It is equality $\alpha = \pi^2$ that gives the steady state solution.

Example 6

Solve

$$\begin{aligned} u_t &= u_{xx} + \alpha u, \quad 0 < x < 2, \quad t > 0 \\ u(x,0) &= 4x - x^3 \quad u_x(0,t) = 0, \;\; u_x(2,t) = 0. \end{aligned} \tag{4.90}$$

Established already is that the transformation (4.86) will transform the equation to the heat equation and will leave the initial condition unchanged.

It is now necessary to determine what happens to the boundary conditions. Using (4.86) we have

$$u_x(0,t) = 0 \;\Rightarrow\; A_0 e^{\alpha t} v_x(0,t) = 0 \;\Rightarrow\; v_x(0,t) = 0, \tag{4.91}$$

$$v_x(2,t) = 0 \;\Rightarrow\; A_0 e^{\alpha t} v_x(2,t) = 0 \;\Rightarrow\; v_x(2,t) = 0, \tag{4.92}$$

and so the insulted boundary conditions also remain insulated! Thus, the problem (4.90) becomes

$$\begin{gathered} v_t = v_{xx}, \quad 0 < x < 2, \;\; t > 0 \\ v(x,0) = 4x - x^2, \quad v_x(0,t) = 0, \quad v_x(2,t) = 0. \end{gathered} \tag{4.93}$$

Using a separation of variables and imposing the boundary conditions gives (see example 1)

$$v = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}t} \cos\frac{n\pi}{2}x, \tag{4.94}$$

where

$$\begin{aligned} a_0 &= \frac{2}{2}\int_0^2 (4x - x^3)\,dx = \left[2x^2 - \frac{x^4}{4}\right]\Big|_0^2 = 4, \\ a_n &= \frac{2}{2}\int_0^2 (4x - x^3)\cos\frac{n\pi}{2}x\,dx \qquad (4.95) \\ &= \left[\left(\frac{2(4x - x^3)}{n\pi} + \frac{48}{n^3\pi^3}\right)\sin\frac{n\pi}{2}x + \left(\frac{4(4 - 3x^2)}{n^2\pi^2} + \frac{96}{n^4\pi^4}\right)\cos\frac{n\pi}{2}x\right]\Big|_0^2 \\ &= -\frac{16}{n^2\pi^2} - \frac{96}{n^4\pi^4} + \left(\frac{6}{n\pi} + \frac{48}{n^3\pi^3}\right)\sin\frac{n\pi}{2} + \left(\frac{4}{n^2\pi^2} + \frac{96}{n^4\pi^4}\right)\cos\frac{n\pi}{2}. \end{aligned}$$

Together with the transformation (4.86), the solution of (4.90) is

$$u = 2e^{\alpha t} + e^{\alpha t}\sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}t}\cos\frac{n\pi}{2}x, \tag{4.96}$$

where a_n is given in (4.95). Figure 9 show plots at $t = 0, 0.2, 0.4$ and 0.6 when $\alpha = -2$ and 2. It is interesting to note that the sign of α will determine whether the solution will grow or decay exponentially.

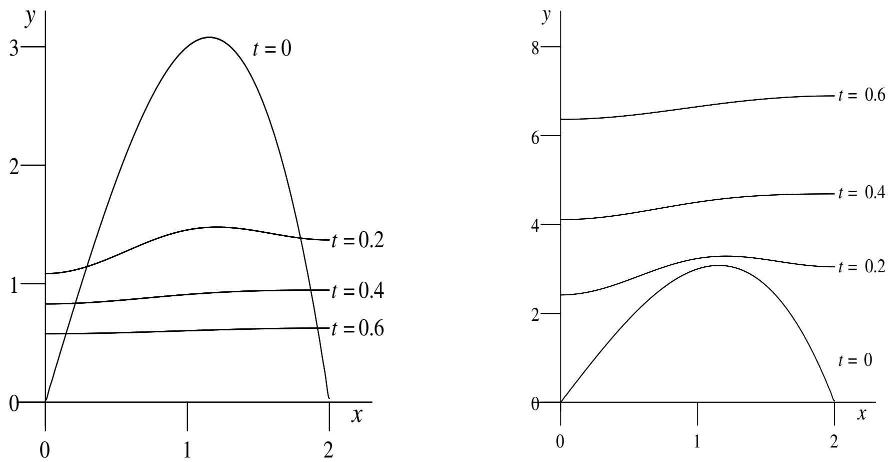

Figure 9. The solution (4.96) of the heat equation with a source (4.90) with no flux boundary condition with $\alpha = -2$ and $\alpha = 2$.

4.1.4 Equations with a solution dependent convective term

We now consider the heat equation with a solution dependent linear convection term, *i.e.*

$$
\begin{aligned}
u_t &= u_{xx} + \beta\, u_x, \quad 0 < x < 1, \;\; t > 0 \\
u(0,t) &= 0, \quad u(1,t) = 0, \quad u(x,0) = x - x^2,
\end{aligned}
\tag{4.97}
$$

where β is some constant. We consider the same initial and boundary conditions as in the previous section as it provides a means of comparing the two respective problems. Again, we could try a separation of variables to obtain solutions for this problem, but for more complicated convection terms like $P(x,t)u_x$, a separation of variable is unsuccessful. Therefore, we again try a different technique. We will try and transform the PDE to one that has no convection term. In attempting to do so, we seek a transformation of the form

$$u(x,t) = A(x,t)v \tag{4.98}$$

and ask "Is is possible to find A such that the convection term can be removed?" Substituting of (4.98) in (4.97) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx} v + \beta\left(Av_x + A_x v\right),$$

and dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + \frac{2A_x + \beta A}{A} v_x + \frac{A_{xx} - A_t + \beta A_x}{A} v. \tag{4.99}$$

In order to target the standard heat equation, we choose

$$2A_x + \beta A = 0, \quad A_{xx} - A_t + \beta A_x = 0. \tag{4.100}$$

From the first we obtain that $A(x,t) = C(t)e^{-\frac{1}{2}\beta x}$ and from the second we obtain $C' + \frac{\beta^2}{4}C = 0$ which has the solution $C(t) = A_0 e^{-\frac{1}{4}\beta^2 t}$ for some constant A_0. This then gives

$$A(x,t) = A_0 e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} \tag{4.101}$$

The boundary conditions becomes

$$u(0,t) = 0 \;\Rightarrow\; C_0 e^{-\frac{1}{4}\beta^2 t} v(0,t) = 0 \;\Rightarrow\; v(0,t) = 0, \tag{4.102}$$

$$v(1,t) = 0 \;\Rightarrow\; C_0 e^{-\frac{1}{2} - \frac{1}{4}\beta^2 t} v(1,t) = 0 \;\Rightarrow\; v(1,t) = 0, \tag{4.103}$$

so the boundary conditions become unchanged. Next, we consider the initial condition, so

$$u(x,0) = x - x^2 \;\Rightarrow\; v(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}, \tag{4.104}$$

where we have chosen $A_0 = 1$. So here, the initial condition actually changes. Thus, under the transformation

$$u = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v, \tag{4.105}$$

the problem (4.97) becomes

$$\begin{gathered} v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0, \\ v(x,0) = 0, \quad v(1,t) = 0, \quad v(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}. \end{gathered} \tag{4.106}$$

As in the previous section, the solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{4.107}$$

where b_n is now given by

$$b_n = \frac{2}{1}\int_0^1 (x-x^2)e^{\frac{1}{2}\beta x}\sin n\pi x\,dx, \tag{4.108}$$

and from (4.105)

$$u(x,t) = e^{-\frac{1}{2}\beta x-\frac{1}{4}\beta^2 t}\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin n\pi x, \tag{4.109}$$

At this point we consider two particular examples: $\beta = 6$ and $\beta = -12$. For $\beta = 6$

$$\begin{aligned} b_n &= 2\int_0^1 (x-x^2)e^{3x}\sin n\pi x\,dx \\ &= -4n\pi\frac{27+n^2\pi^2+2n^2\pi^2e^3\cos n\pi}{(9+n^2\pi^2)^3}. \end{aligned} \tag{4.110}$$

For $\beta = -12$

$$\begin{aligned} b_n &= 2\int_0^1 (x-x^2)e^{-6x}\sin n\pi x\,dx \\ &= 2n\pi\frac{7n^2\pi^2+108+(5n^2\pi^2+324)e^{-6}\cos n\pi}{(36+n^2\pi^2)^3}. \end{aligned} \tag{4.111}$$

The respective solutions for each are

$$\begin{aligned} u(x,t) &= 4\pi e^{-\frac{1}{2}\beta x-\frac{1}{4}\beta^2 t} \\ &\times \sum_{n=1}^{\infty} n\frac{27+n^2\pi^2+2n^2\pi^2e^3\cos n\pi}{(9+n^2\pi^2)^3}e^{-n^2\pi^2 t}\sin n\pi x, \end{aligned} \tag{4.112}$$

$$\begin{aligned} u(x,t) &= 2\pi e^{-\frac{1}{2}\beta x-\frac{1}{4}\beta^2 t} \\ &\times \sum_{n=1}^{\infty} n\frac{7n^2\pi^2+108+(5n^2\pi^2+324)e^{-6}\cos n\pi}{(36+n^2\pi^2)^3}e^{-n^2\pi^2 t}\sin n\pi x. \end{aligned} \tag{4.113}$$

Figure 10 shows graphs at a variety of times for $\beta = 6$ and $\beta = -12$.

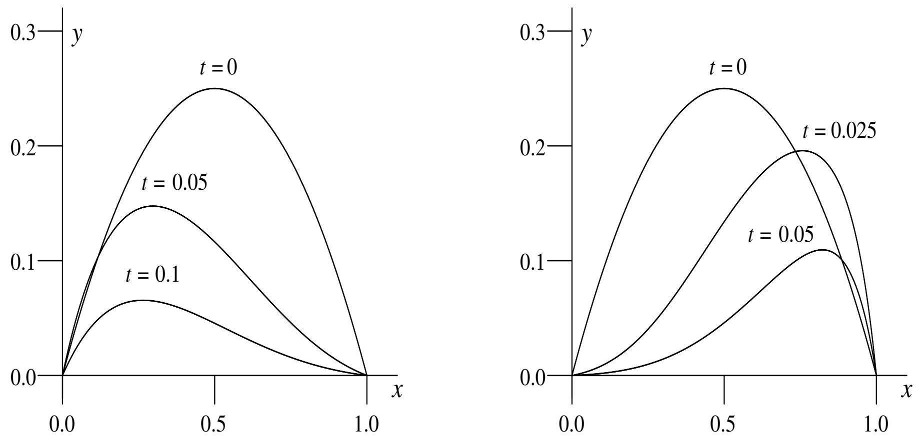

Figure 10. The solution (4.112) of the heat equation with convection with fixed boundary conditions with $\beta = 6$ and $\beta = -12$.

Example 7

As a final example, we consider

$$u_t = u_{xx} + \beta u_x, \quad 0 < x < 2, \quad t > 0 \tag{4.114}$$

subject to

$$u(x,0) = 4x - x^3 \quad u_x(0,t) = 0, \;\; u_x(2,t) = 0. \tag{4.115}$$

This problem is like problem 6 but with insulated boundary conditions. Here, we will simply transform the problem to one that is in standard form as to contrast the differences between the two problems. The transformation (4.105) transforms (4.114) to the heat equation so we will primarily focus on the boundary and initial conditions. For the boundary conditions, upon differentiating (4.105) with respect to x gives

$$u_x = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v_x - \frac{\beta}{2} e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v, \tag{4.116}$$

and so

$$u_x(0,t) = 0 \;\;\Rightarrow\;\; v_x(0,t) - \frac{\beta}{2} v(0,t) = 0, \tag{4.117}$$

$$u_x(2,t) = 0 \;\;\Rightarrow\;\; v_x(2,t) - \frac{\beta}{2} v(2,t) = 0. \tag{4.118}$$

Thus, the insulated boundary condition become radiating boundary conditions. As for the initial condition

$$u(x,0) = 4x - x^3 \quad \Rightarrow \quad v(x,0) = (4x - x^3)e^{\frac{1}{2}\beta x}, \tag{4.119}$$

so again, the initial condition changes.

4.2 Laplace's equation

The two dimensional Laplace's equation is

$$u_{xx} + u_{yy} = 0 \tag{4.120}$$

We will show that a separation of variables also works for this equation As an example consider the boundary conditions

$$u(x,0) = 0, \;\; u(x,1) = x - x^2 \tag{4.121a}$$

$$u(0,y) = 0, \;\; u(1,0) = 0. \tag{4.121b}$$

If we assume separable solutions of the form

$$u(x,y) = X(x)Y(y), \tag{4.122}$$

then substituting this into (4.120) gives

$$X''Y + XY'' = 0. \tag{4.123}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{4.124}$$

and since each term is only a function of x or y, then each must be constant giving

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda. \tag{4.125}$$

From the first of (4.121a) and both of (4.121b) we deduce the boundary conditions

$$X(0) = 0, \quad X(1) = 0, \quad Y(0) = 0. \tag{4.126}$$

The remaining boundary condition in (4.121a) will be used later. As seen the in previous section, in order to solve the X equation in (4.125) subject to the boundary conditions (4.126), it is necessary to set $\lambda = -k^2$. The X equation (4.125) as the general solution

$$X = c_1 \sin kx + c_2 \cos kx \tag{4.127}$$

To satisfy the boundary conditions in (4.126) it is necessary to have $c_2 = 0$ and $k = n\pi, \ \ k \in \mathbb{Z}^+$ so

$$X(x) = c_1 \sin n\pi x. \tag{4.128}$$

From (4.125), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y \tag{4.129}$$

Since $Y(0) = 0$ this implies $c_4 = 0$ so

$$X(x)Y(y) = a_n \sin n\pi x \sinh n\pi y \tag{4.130}$$

where we have chosen $a_n = c_1 c_3$. Therefore, we obtain the solution to (4.120) subject to three of the four boundary conditions in (4.121)

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y. \tag{4.131}$$

The remaining boundary condition is (4.121a) now needs to be satisfied, thus

$$u(x,1) = x - x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi. \tag{4.132}$$

This looks like a Fourier sine series and if we let $A_n = a_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} A_n \sin n\pi x = x - x^2. \tag{4.133}$$

which is precisely a Fourier sine series. The coefficients A_n are given by

$$\begin{aligned} A_n &= \frac{2}{1} \int_0^2 \left(x - x^2\right) \sin n\pi x \, dx \\ &= \frac{16}{n^3 \pi^3} (1 - \cos n\pi), \end{aligned} \tag{4.134}$$

and since $A_n = a_n \sinh n\pi$, this gives

$$a_n = \frac{4(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}. \tag{4.135}$$

Thus, the solution to Laplace's equation with the boundary conditions given in (4.121) is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}. \tag{4.136}$$

Figure 11 show both a top view and a $3-D$ view of the solution.

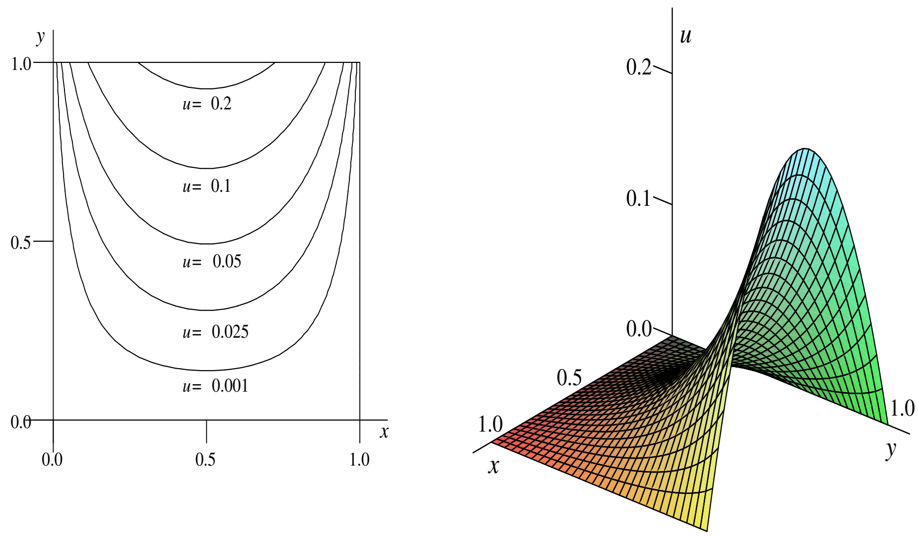

Figure 11. The solution (4.120) with the boundary conditions (4.121)

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{4.137}$$

subject to

$$\begin{aligned} &u(x,0) = 0, \;\; u(x,1) = f(x), \\ &u(0,y) = 0, \;\; u(1,y) = 0, \end{aligned}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \frac{n\pi y}{L_y}}{\sinh \frac{n\pi}{L_y}}, \tag{4.139}$$

where

$$b_n = \frac{2}{L_x} \int_0^{L_x} f(x) \sin \frac{n\pi x}{L_x} \, dx. \tag{4.140}$$

In the next three examples, we will construct solutions of Laplace's equation when we have nonzero boundary condition on each of the remaining three sides of the region $0 < 1, \quad 0 < y < 1$.

Example 8

Solve

$$u_{xx} + u_{yy} = 0, \tag{4.141}$$

subject to

$$u(x,0) = 0, \;\; u(x,1) = 0, \tag{4.142a}$$

$$u(0,y) = 0, \;\; u(1,y) = y - y^2. \tag{4.142b}$$

Assume separable solutions of the form

$$u(x,y) = X(x)Y(y) \tag{4.143}$$

Then substituting this into (4.141) gives

$$X''Y + XY'' = 0. \tag{4.144}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{4.145}$$

from which we obtain

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda \tag{4.146}$$

From (4.186) we deduce the boundary conditions

$$X(0) = 0, \quad Y(0) = 0, \quad Y(1) = 0. \tag{4.147}$$

The remaining boundary condition in (4.186) will be used later. As seen the in previous problem, in order to solve the Y equation in (4.146) subject to the boundary conditions (4.147), it is necessary to set $\lambda = k^2$. The Y equation (4.146) as the general solution

$$Y = c_1 \sin ky + c_2 \cos ky \tag{4.148}$$

To satisfy the boundary conditions in (4.147) it is necessary to have $c_2 = 0$ and $k = n\pi$ so

$$Y(y) = c_1 \sin n\pi y. \tag{4.149}$$

From (4.146), we obtain the solution to the X equation

$$X(x) = c_3 \sinh n\pi x + c_4 \cosh n\pi x \tag{4.150}$$

Since $X(0) = 0$ this implies $c_4 = 0$. This gives

$$X(x)Y(y) = a_n \sinh n\pi x \sin n\pi y \tag{4.151}$$

where we have chosen $a_n = c_1 c_3$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi x \sin n\pi y. \tag{4.152}$$

The remaining boundary condition is (4.186) now needs to be satisfied, thus

$$u(1, y) = y - y^2 = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi y. \tag{4.153}$$

If we let $A_n = a_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} A_n \sin n\pi y = y - y^2. \tag{4.154}$$

Comparing with previous problem, we find that interchanging x and y interchanges the two problems and thus we con conclude that

$$A_n = \frac{16}{n^3\pi^3}(1 - \cos n\pi), \tag{4.155}$$

and further that the solution to Laplace's equation with the boundary conditions given in (4.141) subject to (4.186) is

$$u = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\frac{\sinh n\pi x}{\sinh n\pi}\sin n\pi y. \tag{4.156}$$

Figure 12 show both a top view and a $3-D$ view of the solution. In comparing the solutions (4.136) and (4.156) shows that if we interchange x and y they are the same. This should not be surprising because if we consider Laplace equations with the boundary conditions given in (4.121) and (4.186), that if we interchange x and y, the problems are transformed to each other.

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y \tag{4.157}$$

subject to

$$u(x,0) = 0, \;\; u(x,1) = 0$$
$$u(0,y) = 0, \;\; u(1,y) = g(y).$$

is

$$u = \sum_{n=1}^{\infty} b_n \frac{\sinh \frac{n\pi x}{L_x}}{\sinh \frac{n\pi}{L_x}} \sin \frac{n\pi y}{L_y} \tag{4.159}$$

where

$$b_n = \frac{2}{L_y}\int_0^{L_y} g(y)\sin\frac{n\pi y}{L_y}\,dy \tag{4.160}$$

Example 9

Solve

$$u_{xx} + u_{yy} = 0 \tag{4.161}$$

subject to

$$u(x,0) = x - x^2, \;\; u(x,1) = 0 \tag{4.162a}$$

$$u(0,y) = 0, \;\; u(1,y) = 0. \tag{4.162b}$$

Again, assuming separable solutions of the form

$$u(x,y) = X(x)Y(y) \tag{4.163}$$

leads to

$$X''Y + XY'' = 0, \tag{4.164}$$

when substituted into (4.161). Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \tag{4.165}$$

Since each term is only a function of x or y then each must be constant giving

$$\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda \tag{4.166}$$

From the first of (4.162a) and both of (4.162b) we deduce the boundary conditions

$$X(0) = 0, \quad X(1) = 0, \quad Y(1) = 0, \tag{4.167}$$

noting that the last boundary condition is different than the boundary condition considered at the beginning of this section (*i.e.* $Y(0) = 0$). The remaining boundary condition in (4.162a) will be used later. In order to solve the X equation in (4.166) subject to the boundary conditions (4.167), it is necessary to set $\lambda = -k^2$. The X equation (4.166) as the general solution

$$Y = c_1 \sin kx + c_2 \cos kx \tag{4.168}$$

To satisfy the boundary conditions in (4.167) it is necessary to have $c_2 = 0$ and $k = n\pi$ so

$$X(x) = c_1 \sin n\pi x. \tag{4.169}$$

From (4.166), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y \tag{4.170}$$

Since $Y(1) = 0$ this implies

$$c_3 \sinh n\pi + c_4 \cosh n\pi = 0 \quad \Rightarrow \quad c_4 = -c_2 \frac{\sinh n\pi}{\cosh n\pi}. \tag{4.171}$$

From (4.172) we have

$$\begin{aligned} Y(y) &= c_3 \sinh n\pi y - c_3 \cosh n\pi y \frac{\sinh n\pi}{\cosh n\pi} \\ &= -c_3 \frac{\sinh n\pi(1-y)}{\sinh n\pi}. \end{aligned} \tag{4.172}$$

so

$$X(x)Y(y) = a_n \sin n\pi x \sinh n\pi(1-y) \tag{4.173}$$

where we have chosen $a_n = -c_1 c_3 / \sinh n\pi$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y). \tag{4.174}$$

The remaining boundary condition is (4.162a) now needs to be satisfied, thus

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi. \tag{4.175}$$

At this point we recognize that this problem is now identically to the first problem in this section where we obtained

$$a_n = \frac{4(1-(-1)^n)}{n^3 \pi^3 \sinh n\pi} \tag{4.176}$$

so the solution to Laplace's equation with the boundary conditions given in (4.179) is

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi(1-y)}{\sinh n\pi}. \tag{4.177}$$

Figure 12 show both a top view and a $3-D$ view of the solution.

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y \tag{4.178}$$

subject to

$$u(x,0) = f(x), \;\; u(x,1) = 0$$
$$u(0,y) = 0, \;\; u(1,y) = 0.$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \frac{n\pi(1-y)}{L_y}}{\sinh \frac{n\pi}{L_y}} \tag{4.180}$$

where

$$b_n = \frac{2}{L_x} \int_0^{L_x} f(x) \sin \frac{n\pi x}{L_x}\, dx \tag{4.181}$$

Example 10

As the final example, we consider

$$u_{xx} + u_{yy} = 0 \tag{4.182}$$

subject to

$$u(x,0) = 0, \;\; u(x,1) = 0 \tag{4.183a}$$

$$u(0,y) = y - y^2, \;\; u(1,y) = 0. \tag{4.183b}$$

We could go through a separation of variables to obtain the solution but we can avoid many of the steps by considering the previous three problems. One will notice to transform between the first and second problem, the variables x and y only need to be interchanged. This can also be seen in their respective solutions. One will also notice to transform between this problem and problem 9, we only need to transform x and y again. Thus, to obtain the solution for this final problem, we will transform the solution given in (4.177) giving

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \frac{\sinh n\pi(1-x)}{\sinh n\pi} \sin n\pi y. \tag{4.184}$$

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y \tag{4.185}$$

subject to

$$u(x,0) = 0, \;\; u(x,1) = 0$$

$$u(0,y) = g(y), \;\; u(1,y) = 0.$$

is

$$u = \sum_{n=1}^{\infty} b_n \frac{\sinh \frac{n\pi(1-x)}{L_y}}{\sinh \frac{n\pi}{L_y}} \sin \frac{n\pi y}{L_y}. \tag{4.187}$$

where

$$b_n = \frac{2}{L_y} \int_0^{L_y} g(y) \sin \frac{n\pi y}{L_y} \, dy \tag{4.188}$$