Binary Quadratic Forms over $\mathbb{F}[T]$ and PID's

Jeff Beyerl Clemson University Masters in Mathematical Sciences Defense April 14, 2009

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A function

$$f = \sum_{i_1+i_2+\cdots+i_n=2} r_{(i_1,i_2,\ldots,i_n)} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in R[x_1,x_2,\ldots,x_n].$$

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- R is a ring
- $R[x_1, x_2, ..., x_n]$ is the polynomial ring over R.

What is a Binary Quadratic Form?

A function

$$f = ax^2 + bxy + cy^2 \in R[x, y].$$

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What is a Binary Quadratic Form?

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Alternate Notation 2: f = [a, b, c]

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- Alternate Notation 2: f = [a, b, c]
- Alternate Notation 3: $f = [a, b, *]_D$

$f = [a, b, *]_D$, what is D???

• D = Disc(f) is the discriminant of f

$$D = Disc(f) := b^2 - 4ac$$

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- f is uniquely defined by a, b and either c or D.
- ...At least if R is an integral domain, $char(R) \neq 2$.

• A ring is an additive group and multiplicative semigroup such that the operations are distributive.

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- A semigroup is a set of elements with an operation which is associative (e.g. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$)
- A group is a semigroup with an identity (e.g. a + 0 = 0) and inverses (e.g. a + (-a) = 0).

• An <u>ideal</u> of a commutative ring is a subring which is closed under multiplication of ring elements.

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- ...Unfortunately

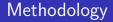
GL₂(R) is the group of 2 × 2 with entries in R which are invertible in R^{2×2}.

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- SL₂(R) is the group of 2 × 2 with entries in R and determinant 1 (and thus is invertible in R^{2×2}).
- $\langle a_1, a_2, ..., a_l \rangle_R = \{a_1r_1 + a_2r_2 + \dots + a_lr_l | r_i \in R\}$ is the ideal generated by $\{a_1, a_2, ..., a_l\}$.



Choose R.

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Choose *R*.

Do stuff.

- Choose R.
- Do stuff.
- Fail to do stuff.

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Go to step 2.

The [primary] sources of insight

My Advisers

• Primes of The Form $x + ny^2$, by David Cox.

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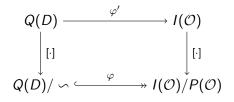
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...that's actually quite good.

Overview of my work



- Q(D) is the set of all primitive forms with discriminant D.
- $Q(D)/ \sim$ is Q(D) modulo \sim , where \sim denotes proper equivalence.
- I(O) is the group of all proper fractional ideals of a quadratic extension of 𝒫[T]
- I(O)/P(O) is the ideal class group of the same quadratic extension of 𝔅[T]

Setup (for now)

• A is a principle ideal domain

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A is a principle ideal domain 2⁻¹ exists and is in A.

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Setup (for now)

- A is a principle ideal domain
- 2^{-1} exists and is in A.
- *D* will be reserved for the discriminant of our forms.

Q(D) is the set of all primitive forms with discriminant D.
...So if [a, b, c] ∈ Q(D), then b² - 4ac = D.

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...So if [a, b, c] ∈ Q(D), then b² - 4ac = D.
If (a, b, c)_A = (1)_A, then f is said to be primitive.



• $Q(D)/ \sim = Q(D)$ modulo \sim , where $f \sim g$ is an equivalence relation called proper equivalence.



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- If $f = [a, b, c], g = [a', b', c'] \in Q(D)$, then f is said to be properly equivalent to g if there is a $\gamma \in SL_2(A)$ so that $\gamma f = g$

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- If f = [a, b, c], g = [a', b', c'] ∈ Q(D), then f is said to be properly equivalent to g if there is a γ ∈ SL₂(A) so that γf = g
 ...where γf = [p q / r s] f := f(px + qy, rx + sy) = [f(p, r), 2apq + bqr + bps + 2crs, f(q, s)].

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Proper equivalence is equivalent to saying that f and g properly represent the same things

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- If $f = [a, b, c], g = [a', b', c'] \in Q(D)$, then f is said to be properly equivalent to g if there is a $\gamma \in SL_2(A)$ so that $\gamma f = g$

• ...where
$$\gamma f = \begin{bmatrix} p & q \\ r & s \end{bmatrix} f := f(px + qy, rx + sy) = [f(p, r), 2apq + bqr + bps + 2crs, f(q, s)].$$

- Proper equivalence is equivalent to saying that f and g properly represent the same things
- ...If $f(\alpha, \beta) = m$ and $\langle \alpha, \beta \rangle_A = \langle 1 \rangle_A$, then *m* is said to be properly represented by *f*.



Theorem

 $Q(D)/\sim$ is an abelian group.



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 Given our operation, associative, identity, and inverses are all mostly easy.

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...What is this operation?

• To define an operation, we will use the following proposition:

Proposition

Let $D \in A$, $M \in A \setminus \{0\}, C_1, C_2 \in Q(D) / \sim$. Then there are $f_1 \in C_1$ and $f_2 \in C_2$ such that

$$f_1 = [a_1, B, a_2 C], f_2 = [a_2, B, a_1 C]$$

where $a_i, B, C \in A$, $a_1a_2 \neq 0$, $\langle a_1, a_2 \rangle_A = \langle 1 \rangle_A$, and $\langle a_1a_2, M \rangle_A = \langle 1 \rangle_A$. (Forms that look like this are called concordant)

• Let $[[a, b, a'c]], [[a', b, ac]] \in Q(D) / \sim$

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- ...which of course I'm not going to do here
- ...for the health and sanity of the audience



Q(D)[·] $Q(D)/\sim$

• A is any PID in which 2^{-1} exists and is in A.





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- Q(D)/ ∽ is the set of such primitive forms modulo some strange equivalence defined by properly representing the same things.



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- Q(D)/ ∽ is a group with some strange operation only defined on very special pairs of representatives from each operand.



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 …and it actually works!

Recap

Q(D) \downarrow $[\cdot]$ $Q(D)/\sim$

- A is any PID in which 2^{-1} exists and is in A.
- Q(D) is the set of all primitive forms of discriminant D.
- Q(D)/ ∽ is the set of such primitive forms modulo some strange equivalence defined by properly representing the same things.
- Q(D)/ ∽ is a group with some strange operation only defined on very special pairs of representatives from each operand.
- …and it actually works!
- I...well, up to the fact that my proof only works if A is a blasted PID...

Specializing further to $\mathbb{F}[\mathcal{T}]$

•
$$\mathbb{F} = \mathbb{F}_r = \mathbb{F}_{p^m}$$
, T an indeterminant.

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Specializing further to $\mathbb{F}[\mathcal{T}]$

- $\mathbb{F} = \mathbb{F}_r = \mathbb{F}_{p^m}$, *T* an indeterminant.
- 𝔅[𝒯] = {∑_{i=0}ⁿ a_i𝔅ⁱ | n ∈ ℤ_{≥0}, a_i ∈ 𝔅} is the polynomial ring in one variable (𝒯) with coefficients coming from 𝔅.

• $\mathbb{F}[T]$ is a PID, and much more.

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- ...In particular, we have a discrete valuation on $\mathbb{F}[T]$, $\deg_T \left(\sum_{i=0}^n a_i T^i \right) = n \left(\deg_T (0) = -\infty \right)$

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- $\mathbb{F}[T]$ is a PID, and much more.
- ...In particular, we have a discrete valuation on $\mathbb{F}[T]$, $\deg_T \left(\sum_{i=0}^n a_i T^i \right) = n \left(\deg_T (0) = -\infty \right)$
- ...Actually the negative of the degree, but same idea.

Using the degree, we are able to get

Lemma

Denote $f = [a, b, c] \in Q(*)$. Then $f \backsim f' = [a', b', c']$ where $\deg(b') < \deg(a') \le \deg(c')$.

Using the degree, we are able to get

Lemma

Denote $f = [a, b, c] \in Q(*)$. Then $f \backsim f' = [a', b', c']$ where $\deg(b') < \deg(a') \le \deg(c')$.

Which gives

Theorem

 $Q(D)/ \sim$ is finite.



• Let $D \in \mathbb{F}[T]$ be an irreducible polynomial.





Let D ∈ 𝔽[T] be an irreducible polynomial. ∂ := √D

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On $\mathbb{F}[T][\mathfrak{d}]$

Let D ∈ F[T] be an irreducible polynomial.
∂ := √D
O_K = F[T][∂], K = F(T)[∂]

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On $\mathbb{F}[T][\mathfrak{d}]$

- Let $D \in \mathbb{F}[T]$ be an irreducible polynomial.
- $\bullet \ \mathfrak{d} := \sqrt{D}$

•
$$\mathcal{O}_{K} = \mathbb{F}[T][\mathfrak{d}], \ K = \mathbb{F}(T)[\mathfrak{d}]$$

 ...there's more behind where this comes from that you saw last week.

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A subring {1} ⊆ O ⊆ 𝔅(𝔅)[𝔅] is said to be an <u>order</u> in 𝔅(𝔅)[𝔅] when O is a finitely generated 𝔅[𝔅]-submodule of 𝔅(𝔅)[𝔅], and contains a basis of 𝔅(𝔅)[𝔅] as a 𝔅(𝔅)-vector space.

- A subring {1} ⊆ O ⊆ F(T)[∂] is said to be an <u>order</u> in F(T)[∂] when O is a finitely generated F[T]-submodule of F(T)[∂], and contains a basis of F(T)[∂] as a F(T)-vector space.
- …For a subring, {1} ⊆ O ⊆ 𝔽(𝒯)[𝑌], this is equivalent to saying that O is a free 𝒴[𝒯]-submodule of rank 2

- A subring {1} ⊆ O ⊆ F(T)[∂] is said to be an <u>order</u> in F(T)[∂] when O is a finitely generated F[T]-submodule of F(T)[∂], and contains a basis of F(T)[∂] as a F(T)-vector space.
- …For a subring, {1} ⊆ O ⊆ 𝔽(𝒯)[𝔅], this is equivalent to saying that O is a free 𝒴[𝒯]-submodule of rank 2
- ...So for instance, every order looks like $\langle 1, f \mathfrak{d} \rangle_{\mathbb{F}[T]}$ for some $f \in \mathbb{F}[T]$.



• $I(\mathcal{O})$ is the group of all proper fractional ideals of \mathcal{O} .





I(O) is the group of all proper fractional ideals of O.
...What's a fractional ideal?

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On $I(\mathcal{O})$

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-What's a fractional ideal?
- ...What's it mean to be proper?



• A <u>fractional ideal</u> \mathfrak{a} of \mathcal{O} is a nonzero \mathcal{O} -submodule of $\mathbb{F}(\mathcal{T})[\mathfrak{d}]$ such that there is an $a \in \mathcal{O}$ such that $a\mathfrak{a} \subseteq \mathcal{O}$.

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More on $I(\mathcal{O})$

- A <u>fractional ideal</u> \mathfrak{a} of \mathcal{O} is a nonzero \mathcal{O} -submodule of $\mathbb{F}(\mathcal{T})[\mathfrak{d}]$ such that there is an $a \in \mathcal{O}$ such that $a\mathfrak{a} \subseteq \mathcal{O}$.
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- Also note that $\mathcal{O} \subseteq \mathcal{O}_{\mathcal{K}}$.



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- ... $IJ = \{\sum_{k=0}^{m} i_k j_k | i_k \in I, j_k \in J, m \in \mathbb{Z}_{\geq 1}\}.$
- ...And the identity is \mathcal{O} .



• $P(\mathcal{O})$ is the subgroup of all principal proper fractional ideals.



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- ...But this is all just like algebraic number theory, except O is typically not a Dedekind Domain.
-(A Dedekind Domain's is an integral domain in which all the fractional ideals are invertible).
-Hence why we introduced the notion of proper fractional ideals.



We have a group, and a normal subgroup, would anyone not take a gander at their quotient?

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We have a group, and a normal subgroup, would anyone not take a gander at their quotient?

• $I(\mathcal{O})/P(\mathcal{O})$ is called the ideal class group of \mathcal{O} .





• On the left we have quadratic forms over $\mathbb{F}[\mathcal{T}]$





- On the left we have quadratic forms over $\mathbb{F}[T]$
- On the right we have proper fractional ideals of an order $\mathcal{O} = \langle 1, f \mathfrak{d} \rangle_{\mathbb{F}[T]}$

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- $I(\mathcal{O})/P(\mathcal{O})$ is also an abelian group.
- ...And I drew them next to each other.

• Define $\varphi' : Q(D) \to I(\mathcal{O})$ by $[a, b, c] \mapsto \langle a, \tau \rangle_{\mathbb{F}[T]}$

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■ ...where $\tau = \frac{-b + \sqrt{D}}{2a}$

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• ...where
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• ...doesn't
$$\tau$$
 look familiar?

But we don't really care about Q(D) and I(O), so φ' is not our concern.

- But we don't really care about Q(D) and I(O), so φ' is not our concern.
- Instead consider Q(D)/ ∽ and I(O)/P(O) and the induced map:

$$arphi : Q(D) / \leadsto I(\mathcal{O}) / P(\mathcal{O})$$

 $[[a, b, c]] \mapsto \left[\langle a, \tau \rangle_{\mathbb{F}[\mathcal{T}]} \right]$

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• If
$$[[a, b, c]] = [[a', b', c']]$$
, does $\left[\langle a, \tau \rangle_{\mathbb{F}[T]}\right] = \left[\langle a', \tau' \rangle_{\mathbb{F}[T]}\right]$?

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Well ves

- **Is** φ well defined?
- If [[a, b, c]] = [[a', b', c']], does $[\langle a, \tau \rangle_{\mathbb{F}[\mathcal{T}]}] = [\langle a', \tau' \rangle_{\mathbb{F}[\mathcal{T}]}]$?

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- Well, yes.
- ...But again, I'm not going to torture the audience with the details.

In fact, φ has some other* nice properties.

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• φ is injective.

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- φ is injective.
- φ is surjective.

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- φ is injective.
- φ is surjective.
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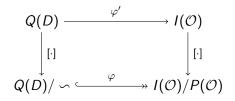
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- φ is injective.
- φ is surjective.
- φ is a group homomorphism.
- Together these give

Theorem

$$Q(D)/ \backsim \cong I(\mathcal{O})/P(\mathcal{O})$$
 as groups via φ .

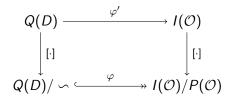
Summary



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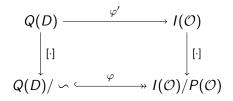
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- $Q(D)/ \sim \cong I(\mathcal{O})/P(\mathcal{O})$ as groups.

Future Ambitions

• Continue deeper this stuff. (60%)

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- Try to generalize this to more general settings (65%)

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The End

Thanks for coming!

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The End

- Thanks for coming!
- ...For references, see my paper.



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