A Selection of Interesting Sets

Jeffrey Beyerl

January 25, 2010

Overview

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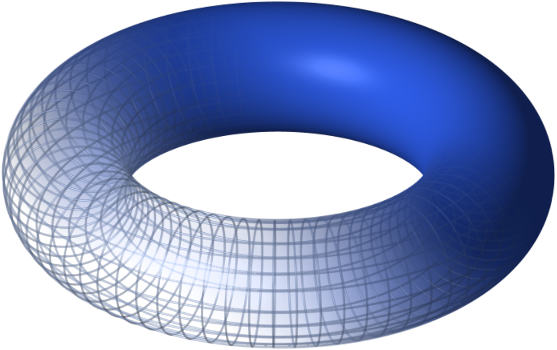
- Some large, some small
- All of them interesting! (to me at least...)

Set #1: A Torus



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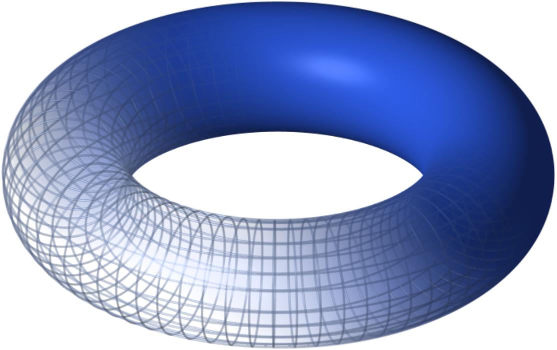

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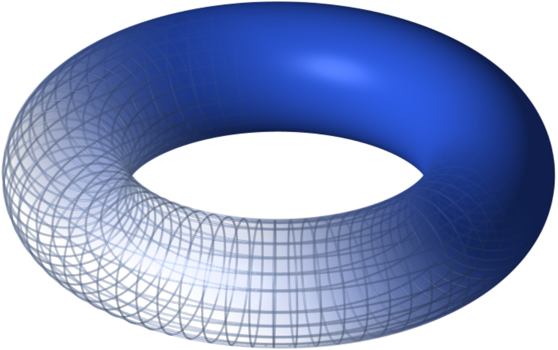

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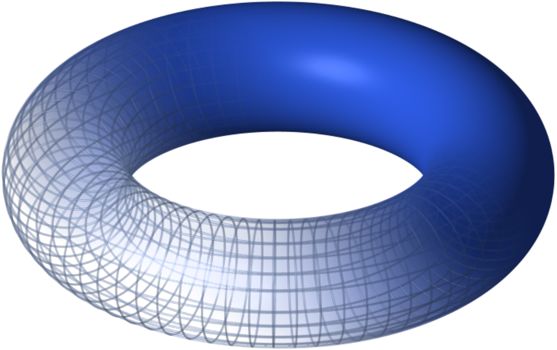
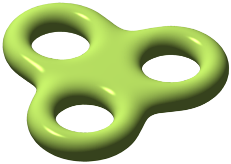
- It is homeomorphic (equivalent) to any smooth surface with one hole or handle (genus 1) such as a coffee mug.
- Formally a torus is $\{ax + by | 0 \leq a, b < 1\}$ for some fixed $x, y \in \mathbb{C}$, equipped with an identification between opposite sides.



Set #2: A 3-torus



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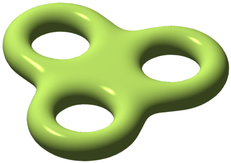

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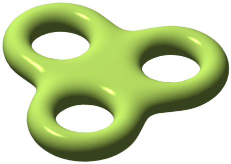

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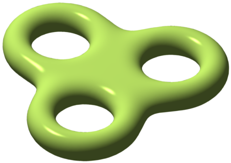

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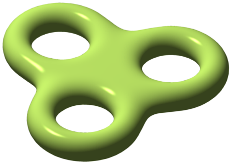
- It is still smooth, connected, compact, and all that good stuff.
- Homeomorphic (equivalent) to the tables in E-7
- And not to be confused with the 3-dimensional torus $S^1 \times S^1 \times S^1$. ($S^1$ being a circle)



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- But yet is uncountable. (and I'd love to know an example of some irrational number in it...)

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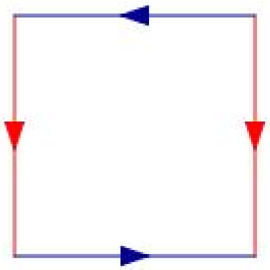

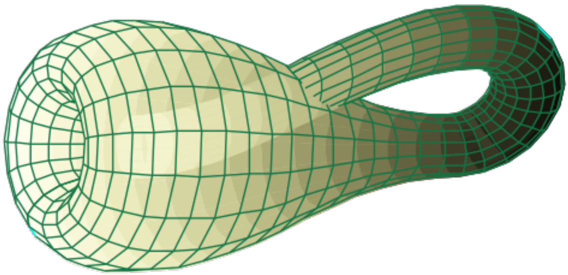

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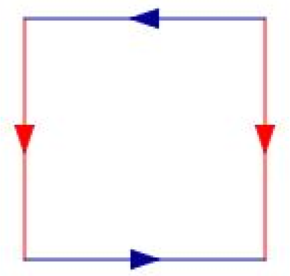

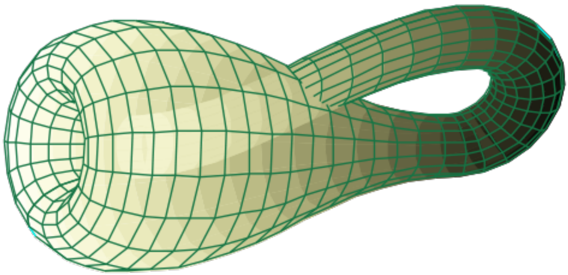

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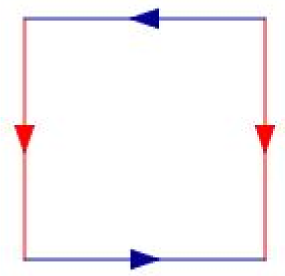

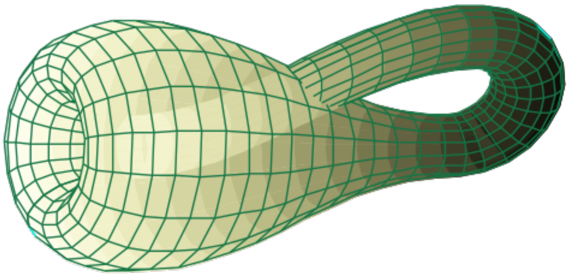

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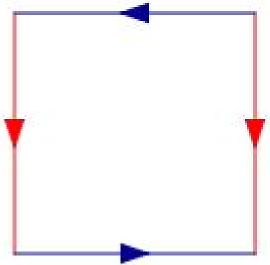

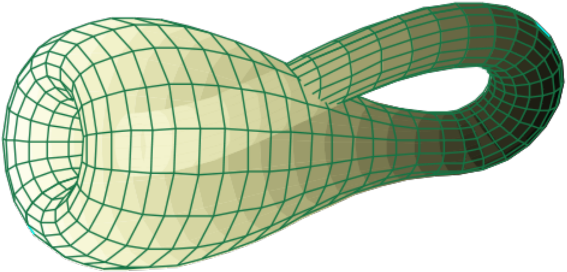

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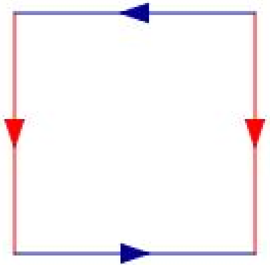

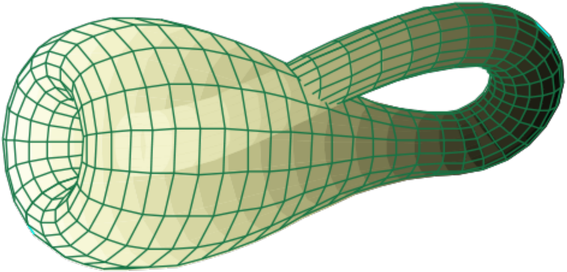

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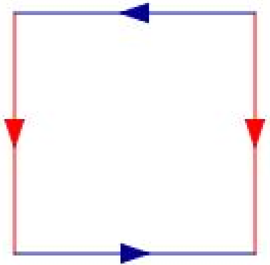
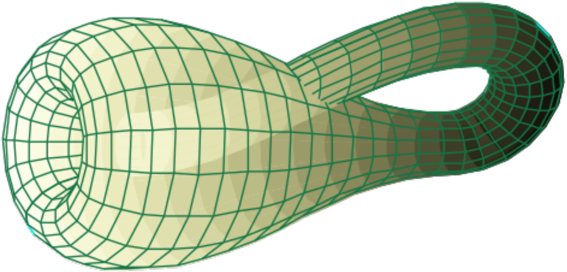
- ...Some of them very big





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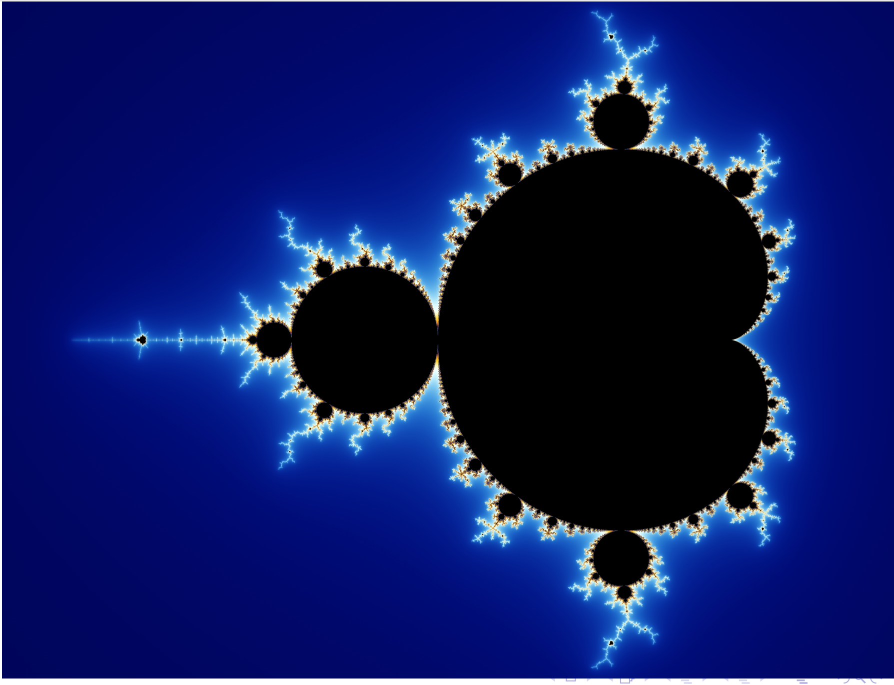
- YouTube video of the Mandelbrot set



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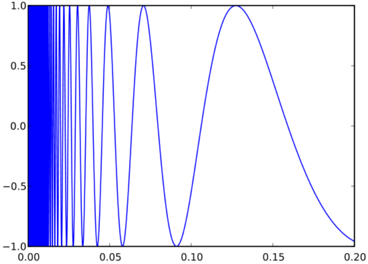

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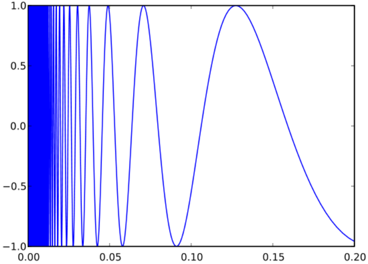

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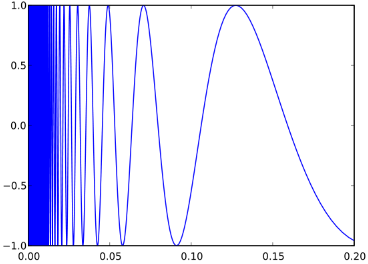

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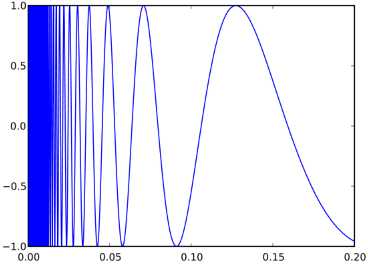

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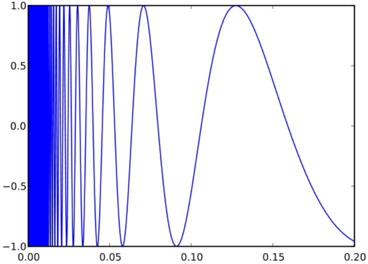

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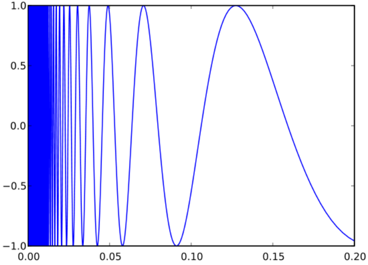
- It is compact, not locally connected, and has Lebesgue measure zero in the plane.



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- Any n-tuple $(a_1, a_2, ..., a_n)$ can be written like this.
- In particular a_k is the element in the set of size k that is not in the set of size $k - 1$.

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- (And by the way - equivalence classes are really sets).
- So yes, 5 is a set.

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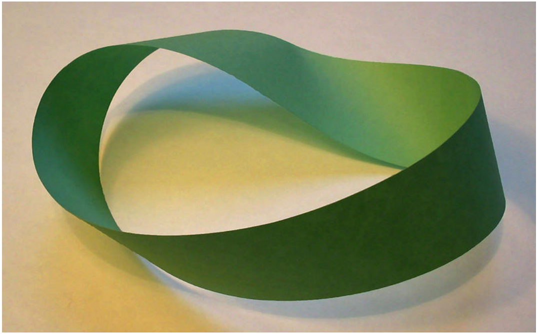
- Locally looks just like $\mathbb{R}$, Globally it does not.
- For instance it is not a metric space (you cannot put a metric on L without destroying its structure).

Set #10: A Mobius Strip



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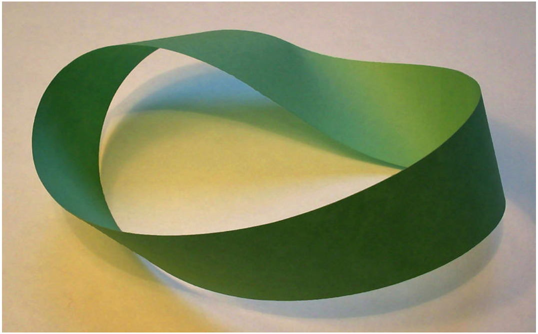

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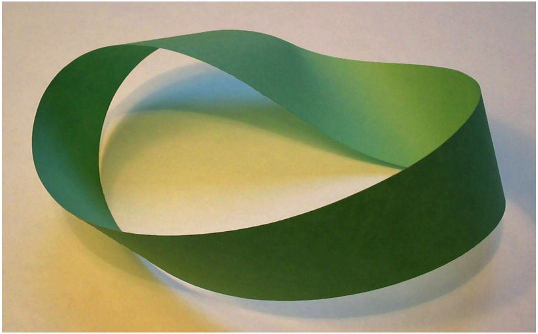

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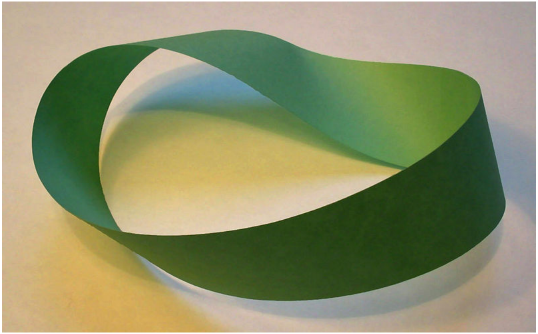

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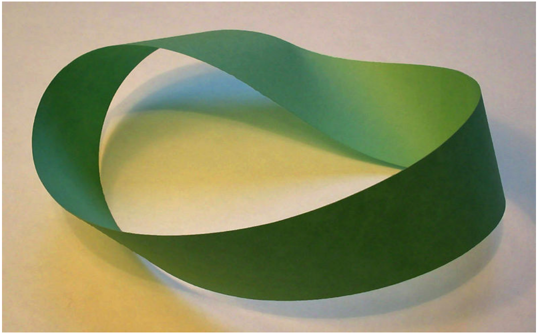
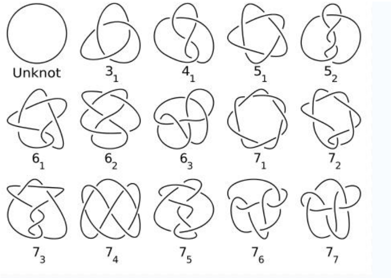
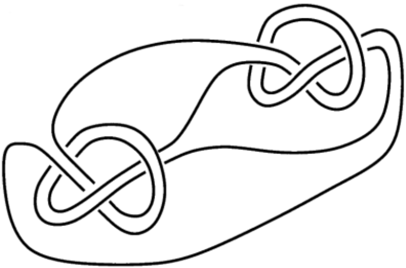
- Dean Kamen's toy country even has currency in the form of Mobius coins (a coin in the shape of a mobius strip)



Set #11: Knots





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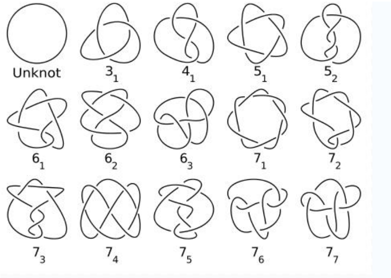

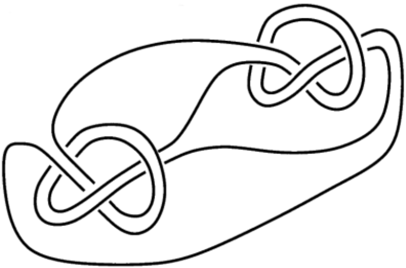

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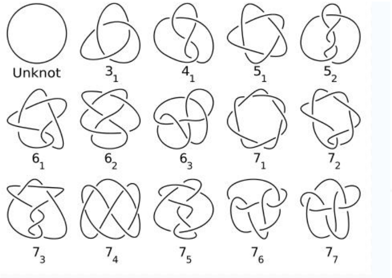

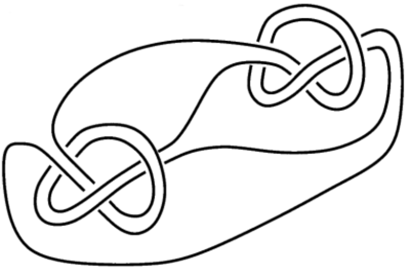

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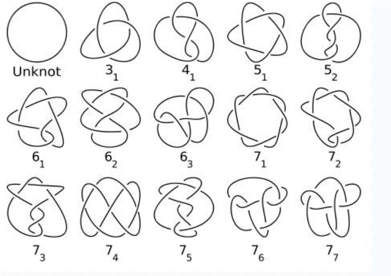
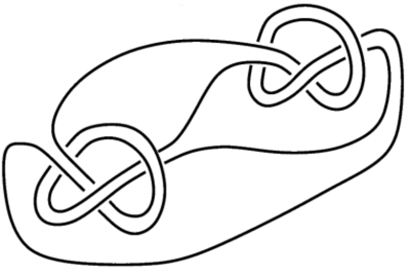
- It seems like there is a plethora of cool stuff in knot theory.

(That I know nothing about...)





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- $f(a) = b$ iff $(a, b) \in R$
- However, often people leave off the codomain and/or the domain and just think of a function as a relation on $A \times B$ satisfying each $a \in A$ appearing exactly once.

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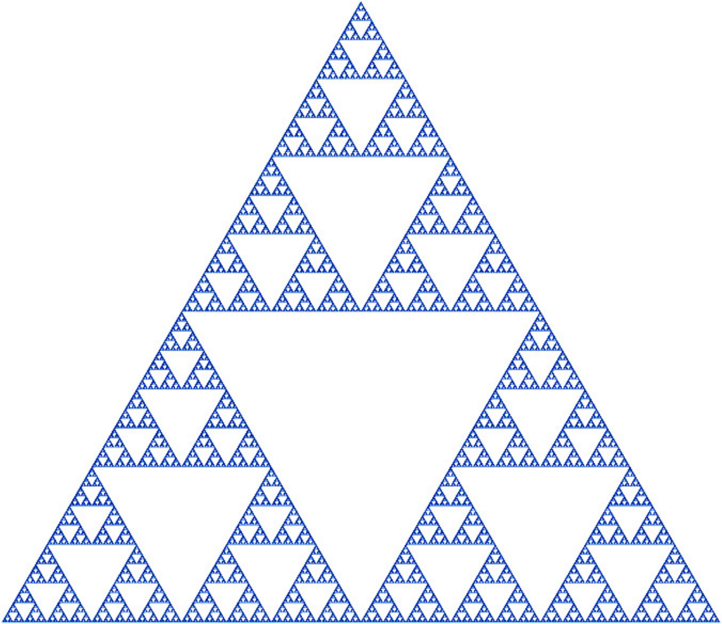
- There are other ways of generating a Sierpinski triangle as well including the Chaos Game.



References

References

- Wikipedia

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- Wikipedia
- Topology By John Gilbert Hocking, Gail S. Young

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- Wikipedia
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- (And the ones that look bad are because I changed them from .svg to .jpg because I don't know how to get LaTeXto display a .svg)

Set #17:

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