The product of nearly holomorphic eigenforms is rarely an eigenform

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Modular forms for level $\Gamma = SL_2(\mathbb{Z})$

Definition

A modular form of weight k is a holomorhpic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

Modular Forms

Example

The weight k Eisenstein series is a modular form, given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Example

The weight 12 cusp form is $\Delta(z)$ given by

$$\Delta(z) = \left(\frac{1}{2\pi}\right)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Modular Forms

Fact

Let $\mathcal{E}_k$ be the space of weight k Eisenstein series, and S_k the space of weight k cusp forms. Let M_k be the space of all weight k modular forms. Then

$$M_k = E_k \oplus S_k$$

Hecke Operators

Definition

The Hecke operator T_n is a linear operator $M_k \to M_k$ given by

$$(T_n(f))(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

Definition

A modular form $f \in M_k$ is said to be an eigenform if it is an eigenvector for all the Hecke operators $\{T_n\}_{n\in\mathbb{N}}$.

More on Modular Forms

Fact

M_k has a basis of eigenforms.

Example

M_{12} is generated by E_{12} and $\Delta(z)$.

Example

S_k has dimension 1 for $k \in \{12, 16, 18, 20, 22, 26\}$. In this case let $\Delta_k(z)$ be the unique normalized cusp form in S_k. In particular $\Delta_{12}(z) = \Delta(z)$.

More on Modular Forms

Fact

Hecke operators preserve Eisenstein series (resp. Cusp forms)

Remark

In particular, because $\dim(\mathcal{E}_k) = 1$, *every* $f \in \mathcal{E}_k$ *is an eigenform.*
(resp. if $\dim(S_k) = 1$ *then every* $f \in S_k$ *is an eigenform)*

Products of eigenforms

Example

E_4 and E_6 are eigenforms.
What about $E_4 \cdot E_6$?
$E_4 \cdot E_6 \in M_{10}(\Gamma)$, and so is an eigenform (E_{10}) because $\dim(M_{10}) = 1$.

Results of Ghate and Duke

Theorem

The product of two eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
- $E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22},$
- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26}.$
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$

Rankin-Cohen bracket operator

Definition

The Rankin-Cohen bracket operator
$[f,g]_j : M_k(\Gamma) \times M_l(\Gamma) \to M_{k+l+2j}(\Gamma)$ is given by

$$[f,g]_j := \frac{1}{(2\pi i)^j} \sum_{a+b=j} (-1)^a \binom{j+k-1}{b} \binom{j+l-1}{a} f^{(a)}(z) g^{(b)}(z).$$

Remark

$[f,g]_0 = fg$

Remark

$[\cdot,\cdot]_j$ *is the unique (up to a constant) bilinear operator that maps* $M_k \times M_l$ *to* M_{k+l+2n}.

Rankin-Cohen brackets of eigenforms

Example

E_4 and Δ_{12} are eigenforms.
What about $[E_4, \Delta_{12}]_2$?
$[E_4, \Delta_{12}]_2$ is a weight 20 cusp form, and so because $\dim(S_{20}) = 1$ is an eigenform $(78\Delta_{20})$.

Result of Lanphier

Theorem

The Rankin-Cohen bracket operator of eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8, \quad E_4 E_6 = E_{10}, \quad E_6 E_8 = E_4 E_{10} = E_{14}$
- $[E_k, E_l]_n$ *where* $n \geq 1$,
  $k, l \in \{4, 6, 8, 10, 14\}, k + l + 2n \in \{12, 16, 18, 20, 22, 26\}$
- $[E_k, \Delta_l]_n$ *where* $n \geq 0$,
  $k, l \in \{4, 6, 8, 10, 14\}, k + l + 2n \in \{12, 16, 18, 20, 22, 26\}$

The Maass-Shimura operator

Definition

We define the Maass-Shimura operator δ_k on $f \in M_k(\Gamma)$ by

$$\delta_k(f) = \left(\frac{1}{2\pi i} \left(\frac{k}{2i \mathrm{Im}\,(z)} + \frac{\partial}{\partial z} \right) f \right) (z).$$

Write $\delta_k^{(r)} := \delta_k \circ \cdots \circ \delta_k$, with $\delta_k^{(0)} = id$. A function of the form $\delta_k^{(r)}(f)$ is called a nearly holomorphic modular form of weight $k + 2r$. The space of these forms is denoted $\widetilde{M}_k$.

Hecke Operators applied to $\widetilde{M}_k$

Definition

Let $\delta_k^{(r)} f$ be a nearly holomorhpic modular form. The Hecke operator T_n is defined with the same formula:

$$\left(T_n\left(\delta_k^{(r)} f\right)\right)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \delta_k^{(r)} f\left(\frac{nz+bd}{d^2}\right).$$

Definition

A modular form $f \in \widetilde{M}_k$ is said to be an eigenform if it is an eigenvector for all the Hecke operators $\{T_n\}_{n\in\mathbb{N}}$.

Theorem

The product of two nearly holomorphic eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
- $E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22},$
- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26}.$
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$
- $\delta_4 E_4 \cdot E_4 = \frac{1}{2} \delta_8 E_8$

Theorem

The product of two nearly holomorphic eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
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- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{12} = E_{14} \Delta_{12} = \Delta_{26}$.
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$
- $\delta_4 E_4 \cdot E_4 = \frac{1}{2} \delta_8 E_8$

The Rankin-Cohen bracket operator meets the Maass-Shimura operator

Fact (Lanphier)

$$[f,g]_n(z) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} \delta_k^{(r)} f(z) \times \delta_l^{(s)} g(z)$$

Fact

$$\delta_k^{(r)}(f)\delta_l^{(s)}(g) = \sum_{j=0}^{r+s} \frac{\delta_{k+l+2j}^{(r+s-j)}[f,g]_j(z)}{\binom{k+l+2j-2}{j}} \left(\sum_{m=\max(j-r,0)}^{s} (-1)^{j+m} \frac{\binom{s}{m}\binom{r+m}{j}\binom{k+r+m-1}{r+m-j}}{\binom{k+l+r+m+j-1}{r+m-j}} \right)$$

Lemma

Suppose that $\{f_i\}_i$ is a collection of modular forms with distinct weights k_i. Then $\sum_{i=1}^{t} a_i \delta_{k_i}^{\left(n-\frac{k_i}{2}\right)}(f_i)$ $(a_i \in \mathbb{C}^*)$ is an eigenform if and only if every $\delta_{k_i}^{\left(n-\frac{k_i}{2}\right)} f_i$ is an eigenform and they have the same eigenvalue.

Lemma

Let $l < k$ and $f \in M_k(\Gamma), g \in M_l(\Gamma)$ both be eigenforms. Then $\delta_l^{\left(\frac{k-l}{2}\right)} g$ and f do not have the same eigenvalues.

1: Write the product as a sum of Rankin-Cohen brackets

Fact

$$\delta_k^{(r)}(f)\delta_l^{(s)}(g) = \sum_{j=0}^{r+s} \frac{\delta_{k+l+2j}^{(r+s-j)}[f,g]_j(z)}{\binom{k+l+2j-2}{j}} \left(\sum_{m=\max(j-r,0)}^{s} (-1)^{j+m} \frac{\binom{s}{m}\binom{r+m}{j}\binom{k+r+m-1}{r+m-j}}{\binom{k+l+r+m+j-1}{r+m-j}} \right)$$

2: One of the first two terms is nonzero

Fact

For eigenforms f, g, the top term, $(-1)^r \frac{[f,g]_{r+s}}{\binom{k+l+2r+2s-2}{r+s}}$, is nonzero unless if $f = g = E_k$, in which case the second term is nonzero.

Fact

$[f, g]_{r+s}$ *(resp. $[f, g]_{r+s-1}$) is an eigenform finitely many times.*

3: If the summation has more than one term, it is not an eigenform

Example

$$\delta_6 E_6 \cdot E_8 = \frac{-1}{14}[E_6, E_8]_1 + \frac{3}{7}\delta_{14}[E_6, E_8]_0$$

3: If the summation has more than one term, it is not an eigenform

Example

$$\delta_6 E_6 \cdot E_8 = \underbrace{\frac{-1}{14}[E_6, E_8]_1 + \frac{3}{7}\delta_{14}[E_6, E_8]_0}_{\text{Bad!}}$$

4: Compute the last finitely many (110) cases and check for eigenforms.

Fact

- *16 cases are the holomorphic cases*
- *1 case is* $\delta_4 E_4 \cdot E_4 = \frac{1}{2}\delta_8 E_8$
- *The 93 other cases have multiple terms and so are not eigenforms.*

Summary of proof

1. Write the product as a sum of Rankin-Cohen brackets
2. One of the first two terms is nonzero
3. If the summation has more than one term, it is not an eigenform
4. Compute the last finitely (110) many and check for eigenforms