When is the Rankin-Cohen Bracket Operator an Eigenform?

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Setting up some notation

- $\Gamma = SL_2(\mathbb{Z})$ (Level 1)
- M_k = space of level 1 modular forms
- S_k = space of level 1 cuspidal modular forms
- $T_{n,k}$ = n^{th} Hecke Operator for weight k
- $T_{n,k}(x)$ = Hecke Polynomial related to $T_{n,k}$
- E_k = weight k Eisenstein series
- $\Delta = \sum \tau(n)q^n$ = normalized weight 12 cuspidal form
- $[f, g]_n$ = The n^{th} Rankin-Cohen bracket operator

Eigenforms

Definition

A modular form $f \in M_k$ is said to be a Hecke eigenform if it is an eigenvector for all the Hecke operators $\{T_{n,k}\}_{n\in\mathbb{N}}$.

Fact

M_k has a basis of eigenforms.

Example

$M_k = \langle E_k \rangle \oplus S_k$. Both $\langle E_k \rangle$ and S_k are preserved by $T_{n,k}$.

Example

S_k has dimension 1 for $k \in \{12, 16, 18, 20, 22, 26\}$. In this case let $\Delta_k(z)$ be the unique normalized cusp form in S_k. In particular $\Delta_{12}(z) = \Delta(z)$.

Definitions

Definition

The Rankin-Cohen bracket operator is the unique normalized differential bilinear function from $M_k \times M_l$ to M_{k+l+2n} given by

$$[f(z), g(z)]_n = \frac{1}{(2\pi i)^n} \sum_{a+b=n} (-1)^a \binom{n+k-1}{b} \binom{n+l-1}{a} f^{(a)}(z) g^{(b)}(z)$$

Example

$$[f(z), g(z)]_0 = f(z) \cdot g(z)$$

$$[f(z), g(z)]_1 = \frac{1}{2\pi i} \left(wt(f) \cdot f(z) \cdot g'(z) - wt(g) \cdot f'(z) \cdot g(z) \right)$$

Original results of Ghate and Duke

Theorem

The product of two eigenforms is an eigenform only in the following cases:

- $E_4^2 = E_8$
- $E_4 E_6 = E_{10}$
- $E_4 \Delta_{12} = \Delta_{16}$
- $E_4 \Delta_{16} = E_8 \Delta_{12} = \Delta_{20}$
- $E_4 \Delta_{18} = E_6 \Delta_{16} = E_{10} \Delta_{12} = \Delta_{22}$,
- $E_4 \Delta_{22} = E_6 \Delta_{20} = E_8 \Delta_{18} = E_{10} \Delta_{16} = E_{14} \Delta_{12} = \Delta_{26}$.
- $E_6 E_8 = E_4 E_{10} = E_{14}$
- $E_6 \Delta_{12} = \Delta_{18}$

Results of Lanphier and Takloo-Bighash

Theorem

The Rankin-Cohen bracket of two eigenforms is an eigenform only in the following cases:

- $[E_4, E_6]_0 = E_{10}; [E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}$
- $[E_k, E_l]_n = c\Delta_{k+l+2n}$ *with appropriate weights* $(n \geq 1)$
- $[E_k, \Delta_l]_n = c\Delta_{k+l+2n}$ *with appropriate weights.* $(n > 0)$

Examples with multiplication

Example

E_{10} is an eigenform, and $E_{10} = E_4 E_6$.

Example

Consider S_{28} and M_{12}. Then $S_{28} = E_4 \Delta M_{12}$, so that every $h \in S_{28}$ factors as $h = E_4 \Delta g$ for some $g \in M_{12}$.

Example

Consider $h = E_{16}\Delta - \frac{14903892}{3617} E_4 \Delta^2 - 108\sqrt{18209} E_4 \Delta^2$, which is an eigenform in S_{28}. Then one factorization of h is:

$$h = E_4 \Delta \left(E_{12} - \frac{3075516}{691} \Delta - 108\sqrt{18209} \Delta \right)$$

Examples with $[f, g]_n$

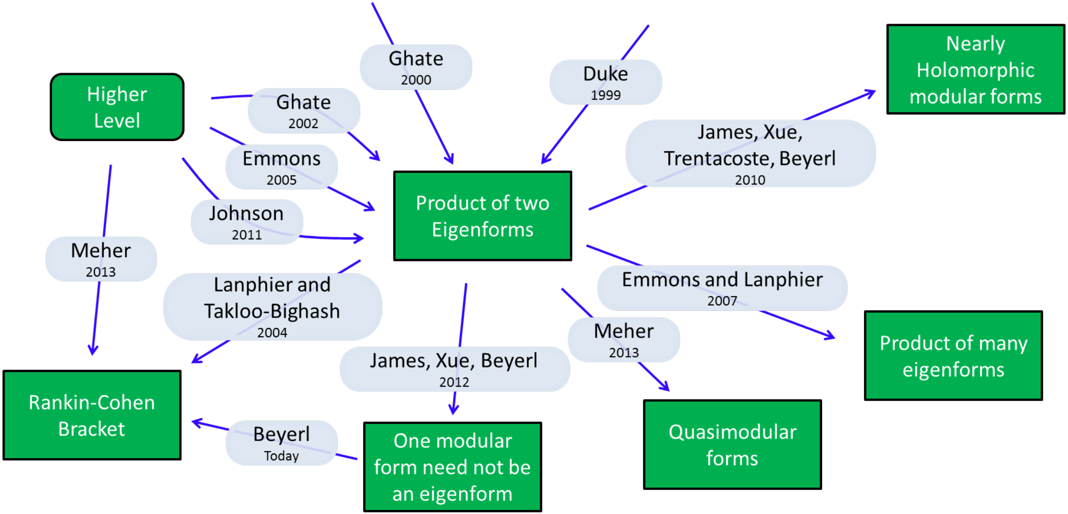
Example

E_4 is an eigenform, and $[E_4, E_8]_4 = c\Delta_{20}$

Example

Consider S_{38} and S_{30}. $[E_6, S_{30}]_1 = S_{38}$, and so for both eigenforms $h \in S_{38}$ there is a modular form $g \in S_{30}$ such that $[E_6, g]_1 = h$. Note that g is not an eigenform.

Generalizations



The Question

Question

Given an eigenform f, for what modular forms g is $[f, g]_n$ also an eigenform? That is, exactly when can $[f, g]_n$ be an eigenform?

Main Tools

Lemma (Rational subspace lemma)

Suppose $S \subseteq S_k$ is a proper $\mathbb{F}$ rational subspace, then (1) implies (2).

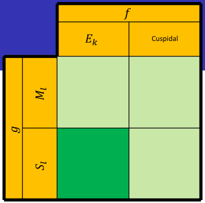
1. *S contains an eigenform.*
2. *For all $m \geq 2$, $T_{m,k}(x)$ is reducible over $\mathbb{F}$.*

Lemma

Let f, g be modular forms and $\sigma \in Gal(\mathbb{C}/\mathbb{F})$. Then $\sigma([f, g]_n) = [\sigma(f), \sigma(g)]_n$

All the possible Cases (i.e. the next 8 slides)

g \ f	E_k	Cuspidal
M_l	"Small < Big" No eigenforms or T reducible "Small = Big" Depends on operator "Small > Big" Eigenforms!	"Small < Big" No eigenforms or T reducible "Small = Big" Eigenforms!
S_l	"Small < Big" No eigenforms or T reducible "Small = Big" Eigenforms!	No Eigenforms. Ever.



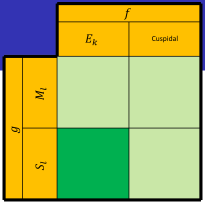
Proposition

Let $f \in M_k$ be an Eisenstein series. If $\dim(S_l) < \dim(S_{k+l+2n})$*, then (1) implies (2):*

1. $[f, S_l]_n$ *contains an eigenform.*
2. *For all* $m \geq 2$, $T_{m,k+l+2n}(x)$ *is reducible.*

Proof.

$[f, S_l]_n$ is a proper rational subspace of S_{k+l+2n}. $\square$

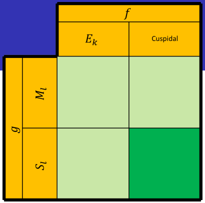


Proposition

Let $f \in M_k$ be an Eisenstein series. If $\dim(S_l) = dim(S_{k+l+2n})$, then $[f, S_l]_n = S_{k+l+2n}$ and hence contains an eigenform.

Proof.

$[f, g]_n$ is never zero if exactly one of f and g is cuspidal. Hence the operator $[f, \cdot]_n : S_l \to S_{k+l+2n}$ is an injective linear operator. Therefore $[f, S_l]_n = S_{k+l+2n}$ □

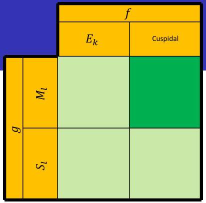


Proposition

Let $f \in S_k$ be an eigenform. Then: $[f, S_l]_n$ contains no eigenforms.

Proof.

The q-coefficient in the Fourier series is zero. $\square$



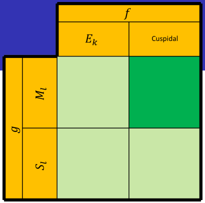
Proposition

Let $f \in S_k$ be an eigenform. If $\dim(M_l) < \dim(S_{k+l+2n})$, then (1) implies (2).

1. *$[f, M_l]_n$ contains an eigenform.*
2. *For all $m \geq 2$, $T_{m,k+l+2n}(x)$ is reducible.*

Proof.

$[f, M_l]_n$ is a proper rational subspace of S_{k+l+2n}. □

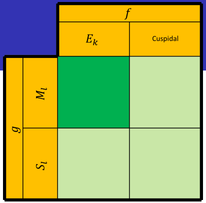


Proposition

Let $f \in S_k$ be an eigenform. If $\dim(M_l) = \dim(S_{k+l+2n})$, then $[f, M_l]_n = S_{k+l+2n}$ and hence contains eigenforms.

Proof.

The only case where this happens is when $f = \Delta$, $n = 1$, and $l \equiv 0 \mod 12$. In this case $dim(M_l) = dim(S_{k+l+2n})$ and the operator $[\Delta, \cdot]_1 : M_l \to S_{k+l+2n}$ is injective. Hence $[\Delta, M_l]_n = S_{k+l+2n}$. $\square$



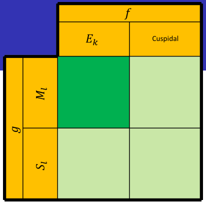
Proposition

Let f be an Eisenstein series and $n \geq 1$. If $\dim(M_l) < \dim(S_{k+l+2n})$, then (1) implies (2):

1. $[f, M_l]_n$ *contains an eigenform.*
2. *For all* $m \geq 2$, $T_{m,k+l+2n}(x)$ *is reducible.*

Proof.

$[f, M_l]_n$ is a rational subspace of S_{k+l+2n} $\square$

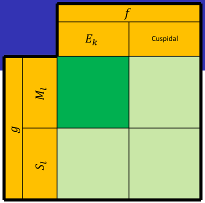


Proposition

Let f be an Eisenstein series and $n \geq 1$. If $\dim(M_l) > dim(S_{k+l+2n})$, then: $[f, M_l]_n = S_{k+l+2n}$ and hence contains an eigenform.

Proof.

The function $[f, \cdot]_n : M_l \to S_{k+l+2n}$ reduces the dimension by at most one, so $[f, M_l]_n = S_{k+l+2n}$. □



Proposition

Let f be an Eisenstein series and $n \geq 1$. If $\dim(M_l) = dim(S_{k+l+2n})$ and $[f, \cdot]_n : M_l \to S_{k+l=2n}$ is not injective, then (1) implies (2):

1. $[f, M_l]_n$ *contains an eigenform.*
2. *For all $m \geq 2$, $T_{m,k+l+2n}(x)$ is reducible.*

Proof.

$[f, M_l]_n$ is a rational subspace of S_{k+l+2n} □

Summary

- Compare the "small space" and the "big space"
- With f an eigenform, there is a modular form g such that $[f, g]_n$ is an eigenform only when $[f, \cdot]_n$ maps the "small space" onto the "big space" or all Hecke Polynomials $T_{\cdot,k+l+2n}(x)$ are reducible.
- The first condition happens quite a few times (a finite set of infinite classes)
- The second condition is conjectured to never happen (Maeda's conjecture)

Thank You!