Differentials and Partial Derivatives

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The Chain Rule

Consider $y = f(x)$ and $x = g(t)$ so $y = f(g(t))$. Using the chain rule we can find dy/dt,

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

Now consider a function $w = f(x, y, x)$. Suppose we want to explore the behavior of f along some curve C, if the curve is parameterized by $x = x(t)$, $y = y(t)$, and $z = z(t)$. If we want to calculate $\frac{dw}{dt}$ we need the chain rule for partial derivatives:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Using this, we define the total differential of w as

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

In this expression the $\frac{\partial w}{\partial x_i}dx_i$ are called partial differentials.

Relationships between Partial Derivatives

We can use the definition of the total differential to develop relationships between partial derivatives. In thermal physics, we will usually want to explicitly denote which variables are being held constant. We do this by placing subscripts on our partial derivatives. Thus we can rewrite our expression for the differential of w as

$$dw = \left(\frac{\partial w}{\partial x}\right)_{y,z} dx + \left(\frac{\partial w}{\partial y}\right)_{x,z} dy + \left(\frac{\partial w}{\partial z}\right)_{x,y} dz.$$

Now suppose there is a relationship between x, y, and z such that $f(x, y, z) = 0$. We can then write $x = x(y, z)$, $y = y(x, z)$, and $z = z(x, y)$. We can write the total differential of x as

$$dx = \left(\frac{\partial x}{\partial y}\right)_{z} dy + \left(\frac{\partial x}{\partial z}\right)_{y} dz. \tag{1}$$

Similarly, from $y = y(x, z)$, we write the total differential of y as

$$dy = \left(\frac{\partial y}{\partial x}\right)_{z} dx + \left(\frac{\partial y}{\partial z}\right)_{x} dz. \tag{2}$$

Then, inserting (1) in (2) we have,

$$dx = \left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial x}\right)_{z} dx + \left[\left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial z}\right)_{x} + \left(\frac{\partial x}{\partial z}\right)_{y}\right] dz.$$

Only two of the variables x, y, z are independent. The above equation is true for all values of dx and dz. If we choose x and z as the independent variables and $dz = 0$ and $dx \neq 0$. Then

$$dx = \left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial x}\right)_{z} dx$$

or,

$$\left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial x}\right)_{z} = 1$$

which we rewrite as

$$\left(\frac{\partial x}{\partial y}\right)_{z} = \frac{1}{\left(\frac{\partial y}{\partial x}\right)_{z}}.$$

Next we choose $dx = 0$ and $dz \neq 0$, so

$$\left[\left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial z}\right)_{x} + \left(\frac{\partial x}{\partial z}\right)_{y}\right] dz = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -\left(\frac{\partial x}{\partial z}\right)_y$$

and finally we write

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

Now let's consider f as a function of x, y, and z where x, y and z are related. This means that we can write $f = f(x, y)$ or $f = f(x, z)$, etc. Or, we could equally well write one of the x_i as a function of f and the other x_i. If we write $x = x(f, y)$ and $y = y(f, z)$ then we can immediately write their total differentials as

$$dx = \left(\frac{\partial x}{\partial f}\right)_y df + \left(\frac{\partial x}{\partial y}\right)_f dy$$

and

$$dy = \left(\frac{\partial y}{\partial f}\right)_z df + \left(\frac{\partial y}{\partial z}\right)_f dz$$

leading to

$$dx = \left[\left(\frac{\partial x}{\partial f}\right)_y + \left(\frac{\partial x}{\partial y}\right)_f \left(\frac{\partial y}{\partial f}\right)_z\right] df + \left(\frac{\partial x}{\partial y}\right)_f \left(\frac{\partial y}{\partial z}\right)_f dz.$$

We can also write $x = x(f, z)$ and write its total differential as

$$dx = \left(\frac{\partial x}{\partial f}\right)_z df + \left(\frac{\partial x}{\partial z}\right)_f dz.$$

Then equating the coefficients of df and dz in the expressions for dx, the coefficients of dz result in

$$\left(\frac{\partial x}{\partial y}\right)_f \left(\frac{\partial y}{\partial z}\right)_f = \left(\frac{\partial x}{\partial z}\right)_f$$

which we rewrite as

$$\left(\frac{\partial x}{\partial y}\right)_f \left(\frac{\partial y}{\partial z}\right)_f \left(\frac{\partial z}{\partial x}\right)_f = 1.$$

We will find many of these results useful in our study of thermal physics. You will find that you will learn some of them. However, the important thing is to be able to derive them when they are needed.