Vectors and Index Notation

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1 Basic Vector Review

1.1 Unit Vectors

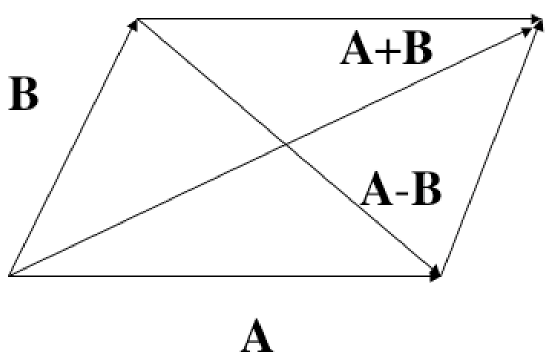
We will denote a unit vector with a superscript caret, thus $\hat{a}$ denotes a unit vector.

$$\hat{a} \Rightarrow |\hat{a}| = 1$$

If $\vec{x}$ is a vector in the x-direction $\hat{x} = \frac{\vec{x}}{|\vec{x}|}$ is a unit vector. We will use $i,\ j,\ and\ k$, or $\hat{x}, \hat{y},\ and \hat{z}$, or $e_1,\ e_2\ and\ e_3$ and a variety of variations without further comment.

1.2 Addition and Subtraction

Addition and subtraction are depicted below



1.3 Scalar Products

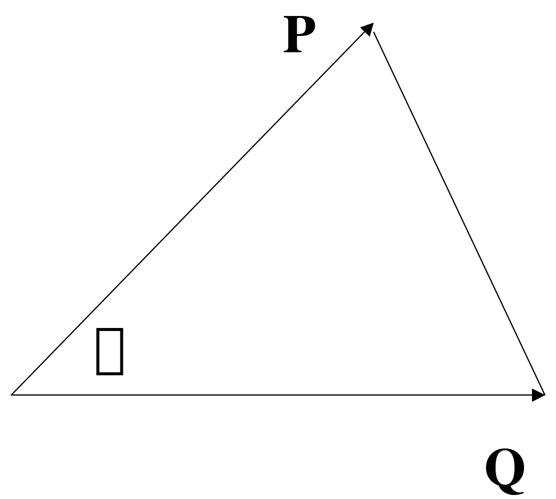
$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a}.\vec{b} = \vec{b}.\vec{a} \text{ since } \cos\theta = \cos(-\theta)$$

1.4 Vector Products

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \text{ Why?}$$
$$\vec{c} = \vec{a} \times \vec{b}, \ \vec{c} \text{ is perpendicular to } \vec{a} \text{ and } \vec{b}$$

1.4.1 Geometric Interpretation



$$\begin{aligned}\text{Area of a triangle} &= \tfrac{1}{2} \text{ base} \times \text{perpendicular height} \\ &= \tfrac{1}{2}|\vec{Q}||\vec{P}| \sin\theta\end{aligned}$$

So

$$A = \frac{1}{2}|\vec{Q} \times \vec{P}|$$

is the area of a triangle and, accordingly,

$$|\vec{Q} \times \vec{P}| = \text{the area of a parallelogram.}$$

1.4.2 Area as a vector

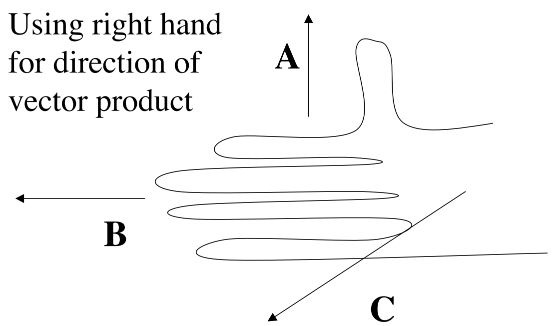
Areas can also be expressed as vector quantities, for the parallelogram considered above, we could have written $\vec{A} = \vec{Q} \times \vec{P}$. If $\hat{n}$ is a unit vector normal to a plane of area A, then $\vec{A} = |\vec{A}|\hat{n} = A\hat{n}$, where A is the numerical value of the area.

1.4.3 Direction of the resultant of a vector product

We have options, in simple cases we often use the right-hand screw rule:

If $\vec{c} = \vec{a} \times \vec{b}$, the direction of $\vec{c}$ is the direction in which a right-handed screw would advance in moving from $\vec{a}$ to $\vec{b}$.

Or we can use the right hand rule, as seen in the diagram.



I prefer this version of the right-hand rule - it doesn't require the contortions of the version typically found in beginning texts.

1.5 Components and unit vectors

We can write vectors in component form, for example:

$$\vec{a} = a_x i + a_y j + a_z k,$$

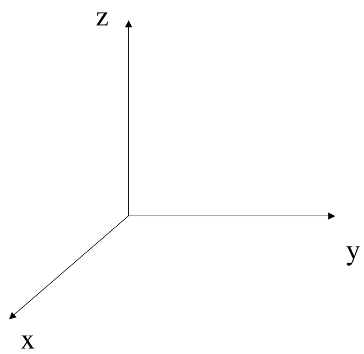
$$\vec{b} = b_x i + b_y j + b_z k,$$

and

$$\vec{c} = c_x i + c_y j + c_z k.$$

In order to calculate in terms of components, we need to be familiar with the scalar and vector products of unit vectors.

Consider a right-handed coordinate system with axes labeled x, y, and z, as shown in the diagram.



If $i, \ j,$ and k are the unit vectors, we find

$$i.i = j.j = k.k = 1$$

and

$$i.j = i.k = j.k = 0$$

for the scalar products, and

$$i \times i = j \times j = k \times k = 0$$

and

$$i \times j = k, \ i \times k = -j, \ j \times k = i$$

for the vector products.

Using these results we can compute

$$\begin{aligned} \vec{a} \times \vec{b} = & \quad (a_x i + a_y j + a_z k) \times (b_x i + b_y j + b_z k) \\ = & \quad a_x b_x (i \times i) + a_x b_y (i \times j) + a_x b_z (i \times k) \\ + & \quad a_y b_x (j \times i) + a_y b_y (j \times j) + a_y b_z (j \times k) \\ + & \quad a_z b_x (k \times i) + a_z b_y (k \times j) + a_z b_z (k \times k) \end{aligned}$$

Simplifying,

$$\vec{a} \times \vec{b} = (a_x b_y - a_y b_x) \underbrace{(i \times j)}_{k} + (a_x b_z - a_z b_x) \underbrace{(i \times k)}_{-j} + (a_y b_z - a_z b_y) \underbrace{(j \times k)}_{i},$$

and finally we have

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y) i + (a_z b_x - a_x b_z) j + (a_x b_y - a_y b_x) k.$$

As a mnemonic, this is often written in the form of a determinant. While the mnemonic is useful, the vector product is not a determinant. (All terms in a determinant must be numbers.)

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Your book goes on to triple products of various types, at this point, I am going to introduce index notation - a far better way of doing vector calculations.

2 Index Notation

You will usually find that index notation for vectors is far more useful than the notation that you have used before. Index notation has the dual advantages of being more concise and more transparent. Proofs are shorter and simpler. It becomes easier to visualize what the different terms in equations mean.

2.1 Index notation and the Einstein summation convention

We begin with a change of notation, instead of writing

$$\vec{A} = A_x i + A_y j + A_z k$$

we write

$$\vec{A} = A_1 e_1 + A_2 e_2 + A_3 e_3 = \sum_{i=1}^{3} A_i e_i.$$

We simplify this further by introducing the Einstein summation convention: if an index appears twice in a term, then it is understood that the indices are to be summed from 1 to 3. Thus we write

$$\vec{A} = A_i e_i$$

In practice, it is even simpler because we omit the basis vectors e_i and just write the Cartesian components A_i. Recall a basis of a vector space is a linearly independent subset that spans (generates) the whole space. The unit vectors i, j, and k are a basis of R_3.

So we will often denote $\vec{A}$ as A_i with the understanding that the index can assume the values 1, 2, or 3 independently. A_i stands for the scalar components (A_1, A_2, A_3); we'll refer to the vector A_i even though to get $\vec{A}$ we need to calculate $A_i e_i$.

2.1.1 Examples

$$a_i = b_i \;\Rightarrow\; a_1 = b_1,\; a_2 = b_2,\; a_3 = b_3$$
$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2.2 Summation convention and dummy indices

Consider

$$\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

this indicates that we can write a scalar product as

$$\vec{a}.\vec{b} = a_i b_i$$

In the term a_ib_i, an index like i that is summed over is called a dummy index (or more cavalierly as a dummy variable). The index used is irrelevant - just as the integration variable is irrelevant in an integral (though in this case the term dummy variable is entirely appropriate).

This means

$$a_ib_i = a_jb_j = a_mb_m = a_1b_1 + a_2b_2 + a_3b_3.$$

A term cannot contain an index more than twice, if a compound calculation would lead to such a situation, the dummy index should be changed. An index that appears only once in a term is called a *free* or *floating index*.

For example

$$a_ib_ic_j = (a_1b_1 + a_2b_2 + a_3b_3)c_j$$

In an equation, all terms must contain the same free indices, in particular you should note that

$$a_ib_ic_j \neq a_1b_ic_k$$

If we have $S = A_iB_i$ and we want SC_i, as noted above we change the subscripts on A and B because they are the dummy indices. Do not change the free indices because you risk changing the equation.

Thus,

$$SC_i = A_jB_jC_i.$$

3 The Kronecker delta or the substitution operator

The Kronecker delta,

$$\delta_{ij} = 1 \text{ if } i = j, = 0\ if\ i \neq j.$$

So

$$\delta_{11} = \delta_{22} = \delta_{33}\ and \delta_{12} = \delta_{23} = \delta_{13} = 0$$

We will sometimes find it convenient this result in an array

$$\delta_{ij} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Why is the Kronecker delta slso known as the substitution operator? We can figure this out by making a calculation.

Consider $\delta_{ji}a_i = \delta_{ij}a_i$, let j take on the values 1, 2, and 3. Then we have:

$$j = 1: \ \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = a_1$$

$$j = 2: \ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = a_2$$
$$j = 3: \ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = a_3.$$

From this we can see that

$$\delta_{ij}a_i = a_j.$$

Thus applying Kronecker delta allows us to drop a repeated index and changes one index into another.

3.0.1 Further Examples

$$A_r B_s C_t \delta_{st} = A_r B_s C_s = A_r B_t C_t$$

So we see that if two indices are repeated, only one is dropped. We should note the following obvious results:

$$\delta_{ii} = 1 + 1 + 1 = 3$$

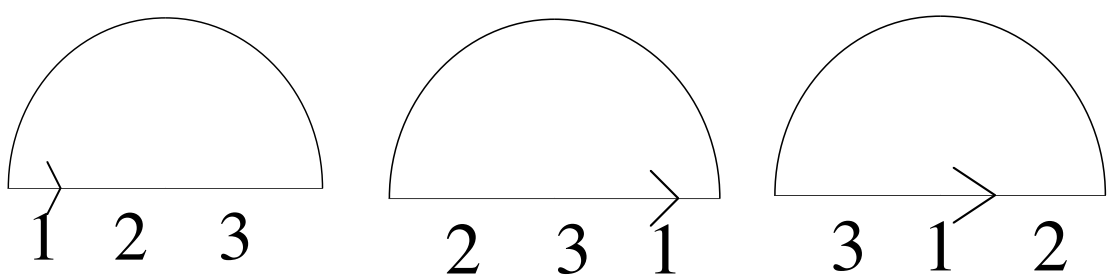
and

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

4 The permutation symbol or the Levi-Civita tensor

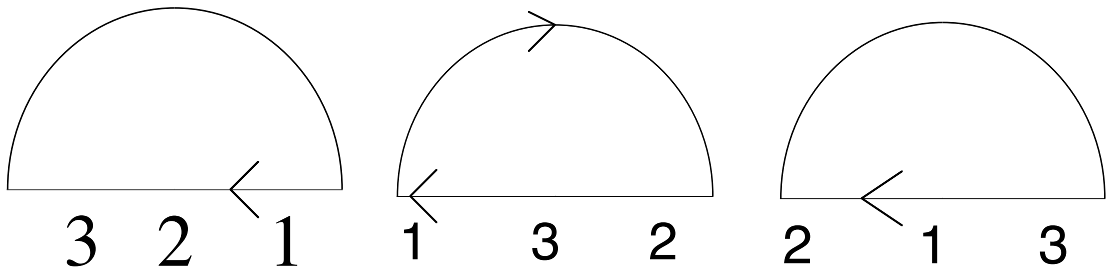
The numbers 1, 2, 3 are in cyclic order if they occur in the order 1,2,3 on a counterclockwise path starting from 1.

Cyclic Permutations



Non-Cyclic Permutations

When the path is clockwise the permutations are non-cyclic.



Cyclic permutations are sometimes called even, non-cyclic permuations are sometimes called odd. This idea can be used in the evaluation of vector products. The idea is introduced through the permutation symbol ε_{ijk}.

$\varepsilon_{ijk} = +1$ if ijk is a cyclic permutation of 1,2,3

$\varepsilon_{ijk} = -1$ if ijk is a non-cyclic permuation of 1,2,3

$\varepsilon_{ijk} = 0$ otherwise, i.e. an index is repeated.

So we find

$$\varepsilon_{123} = \varepsilon_{231} =_{\varepsilon} 312 = +1,$$

and

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1,$$

while

$$\varepsilon_{122} = \varepsilon_{133} = \varepsilon_{112} = 0.$$

We should also note the following properties:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

but when we swap indices

$$\varepsilon_{ijk} = -\varepsilon_{jik} \text{ and } \varepsilon_{ijk} = -\varepsilon_{ikj}.$$

4.1 Vector Product in index notation

Recall

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Now consider

$$c_i = \varepsilon_{ijk} a_j b_k$$

This is a vector characterized by a single free index i. The indices j and k are dummy indices and are summed out. We get the three values of c_i by letting $i = 1,2,3$ independently. This is useful but the method is made more powerful by the methods of the next section.

4.2 The $\varepsilon - \delta$ identity

$$\varepsilon_{ijk}\varepsilon irs = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{rk}$$

This identity can be used to generate all the identities of vector analysis, it has four free indices. To prove it by exhaustion, we would need to show that all 81 cases hold.

Note that the ε's have the repeated index first, and that in the δ's, the free indices are take in this order:

1. both second
2. both third
3. one second, one third
4. the other second, the other third

Let's put this to use by proving what would be a tough identity using ordinary vector methods. We'll prove the bac-cab rule.

Proof that $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$

To prove this, let

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\times\vec{D} = \vec{E}$$

we the convert to index notation as follows: Writing

$$\vec{B}\times\vec{C} = \varepsilon_{ijk} B_j C_k = D_i$$

then

$$\vec{A} \times \vec{D} = \varepsilon_{rsi} A_s D_i = \varepsilon_{rsi} A_s \varepsilon_{ijk} B_j C_k = E_r.$$

Rearranging terms, we have

$$E_r = \varepsilon_{rsi} \varepsilon_{ijk} A_s B_j C_k = \varepsilon_{irs} \varepsilon_{ijk} A_s B_j C_k,$$

and using the $\varepsilon - \delta$ identity

$$E_r = (\delta_{rj} \delta_{sk} - \delta_{rk} \delta_{sj}) A_s B_j C_k,$$

then

$$E_r = \delta_{rj} \delta_{sk} A_s B_j C_k - \delta_{rk} \delta_{sj} A_s B_j C_k$$

then using the substitution properties of the knronecker deltas, this becomes

$$\begin{aligned} E_r = & \quad A_k B_r C_k - A_j B_j C_r \\ = & \quad B_r (A_k C_k) - C_r (A_j B_j) \\ = & \quad \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) \quad \text{Q.E.D.} \end{aligned}$$