Characteristic Functions in Radar and Sonar

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Outline

- FOUR QUESTIONS
- WHAT ARE CHARCTERISTIC FUNCTIONS?
- SINGLE VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION.
- MULTI-DIMENSIONAL CHARCTERISTIC FUNCTIONS
- RAYLEIGH PROBLEM

FOUR QUESTIONS

Question 1: Given we know the density for $\hat{x}$ is $P(x)$, what is the distribution for $\hat{u} = f(\hat{x})$?

Question 2: Given we know the density for $\hat{x}$ is $P_x(x)$ and the density for $\hat{y}$ is $P_y(y)$, what is the distribution for $\hat{u} = \hat{x} + \hat{y}$?

Question 3: Given we know the density for $\hat{x}$ is $P_x(x)$ and the density for $\hat{y}$ is $P_y(y)$, what is the distribution for $\hat{u} = \hat{x}\hat{y}$?

Question 4: Given we know the density for $\hat{x}$ is $P_x(x)$ and the density for $\hat{y}$ is $P_y(y)$, what is the distribution for $\hat{u} = \frac{\hat{x}}{\hat{y}}$?

DEFINTIONS OF CHARCTERISTIC FUNCTION

DEFINTION: $M_P(\theta) = \langle e^{j\theta x} \rangle = \int_{-\infty}^{\infty} e^{j\theta x} P(x)\ dx$

PROPERTIES

1. $M(0) = 1.$
2. $M^*(-\theta) = M(\theta).$
3. $|M(\theta)| \leq 1.$
4. $|M(\theta)| \leq M(0).$

ALTERNATIVE DEFINITION:

$$M_P(\theta) = \int_{-\infty}^{\infty} e^{j\theta x} P(x)\ dx = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(j\theta x)^n}{n!} P(x) dx = \sum_{n=0}^{\infty} \frac{(j\theta)^n}{n!} \langle x^n \rangle$$

DEFINTIONS OF CHARCTERISTIC FUNCTION

TWO METHODS FOR CALCULATION OF MOMENTS

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x)\, dx$$

$$\langle x^n \rangle = \frac{1}{j^n} \frac{\partial^n M_P(\theta)}{\partial \theta^n} \Big|_{\theta=0}$$

differentiation is easier than integration

Fourier transform pair relationship between the PDF and CF:

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\theta x}\ M_P(\theta) dx$$

SINGLE VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

one would often like to know the density of a new variable, say û, that is a function of the old variable: $\hat{u} = f(\hat{x})$

Probability Density Function: $$P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\theta u} M_u(\theta)\ d\theta$$

Characteristic Function: $$M_u(\theta) = \left\langle e^{j\theta f(x)} \right\rangle = \int_{-\infty}^{\infty} e^{j\theta f(x)} P(x)\ dx$$

Apply CF to PDF: $$P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\theta f(x)} e^{-j\theta x} P(x)\ d\theta\, dx$$

PDF for Transformed variable: $$P(u) = \int_{-\infty}^{\infty} \delta(u - f(x))\, P(x)\, dx$$

This solves the problem for Exercise 1.

Interpretation of Delta Function of Function

$$\delta(g(x)) = \sum_i \delta(x - x_i) \frac{1}{|g'(x_i)|}$$

$$\delta(f(x) - u) = \sum_i \delta(x - x_i) \frac{1}{|f'(x_i)|}$$

$$P(u) = \sum_i \frac{P(x_i)}{|f'(x_i)|}$$

Translation

(Translation): Let $\hat{u} = a\hat{x} + b$, *what is* $P(u)$ *given we know that the PDF of* $\hat{x}$ *is* $P_x(x)$? *Solving for zero of the equation gives*

$$\boxed{x = \frac{u-b}{a}}$$

and note $dx = \frac{1}{a}$, *then the distribution is*

$$\boxed{P(u) = \frac{1}{a} P(x)\Big|_{x=\frac{u-b}{a}} = \frac{1}{a} P_x\left(\frac{u-b}{a}\right)}$$

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Square Law Detector

(Square Law Detector): Let $\hat{u} = \hat{x}^2$, *what is* $P(u)$ *given we know that the PDF of* $\hat{x}$ *is* $P_x(x)$*?*
Answer: Solve for zeros

$$\boxed{x_1 = \sqrt{u}}$$

and

$$\boxed{x_2 = -\sqrt{u}}$$

note $2xdx = du$, *then * becomes*

$$\boxed{P(u) = \frac{1}{2\sqrt{u}} P_x(u)|_{x_1,x_2} = \frac{1}{2\sqrt{u}}\left[P(\sqrt{u}) + P(-\sqrt{u})\right]} \qquad \#$$

If $\hat{x}$ *is* $N(0,\sigma)$, *then the PDF is*

$$\boxed{P_u(u) = \frac{1}{\sigma\sqrt{(2\pi u)}} e^{-\frac{u}{2\sigma^2}} \Theta(u).}$$

where Θ *is the unit step function*

$$\begin{array}{|l|} \hline \Theta(x) = 1 \;\; (x \geq 0) \\ \hline \qquad\quad = 0 \;\; (otherwise) \\ \hline \end{array} \qquad \#$$

SINGLE VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

The coordinate transformation $y = R\sin(\hat{\varphi})$ *a PDF* $f_\varphi(\varphi)$ *is onto but not one-to-one over interval* $[-\infty, \infty]$. *Thus* ref: * *has an infinite number of zeros. It is more connivent to determine the CF directly, so the transformation of the PDF is*

$$M_\theta(\omega) = \int_{-\infty}^{\infty} e^{j\omega R\sin(\varphi)} f_\varphi(\varphi)d\varphi.$$

The exponential can be written as

$$\sum_{n=-\infty}^{\infty} J_n(\omega R)e^{jn\varphi} = e^{j\omega R\sin(\varphi)},$$

so the CF is given by

$$M_\theta(\omega) = \sum_{n=-\infty}^{\infty} J_n(\omega R)\int_{-\infty}^{\infty} e^{jn\varphi} f(\varphi)d\varphi = \sum_{n=-\infty}^{\infty} J_n(\omega R)F(n);$$

which can be rewritten as

$$M_\varphi(\omega) = J_0(\omega R) + \sum_{n=1}^{\infty} J_n(\omega R)[F(n) + (-1)^n F(-n)]$$

where $F(n)$ *is the Fourier transform of the PDF* $f_\varphi(\varphi)$ *evaluated at* $\omega = n$. *Depending on the problem, the CF is sufficient, but sometimes it is still useful to know the PDF. Now if we apply the identity (Abramowitz)*

$$\int_{-\infty}^{\infty} e^{-j\omega t}J_n(t)\ dt = \frac{2(-j)^n T_n(\omega)}{\sqrt{(1-\omega^2)}}\Theta(1-|\omega|),$$

where $T_n(x)$ *is the n-th order Chebyshev polynomials, the PDF of the coordinate transformation is*

$$f_y(y) = \frac{\left[a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})T_n(\frac{y}{R})\right]}{\pi\sqrt{(R^2 - y^2)}}\Theta\left(1 - \left|\frac{y}{R}\right|\right),$$

where $a_n = (-j)^n F(n)$.

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MULTI-VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

Repeating the same procedure for the single dimensional case to the multi-dimensional case yields:

$$P(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \int\int \delta[u - f(x,y)]\delta[v - g(x,y)]P(x,y)\ dxdy$$

This due to Cohen, is sufficient to allow us to solve the problem of determining many of the two dimensional combinations of random variables such as those in Exercises 2-4.

MULTI-VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

Let $\hat{u} = f(\hat{x}, y)$, *the CF is*

$$M_u(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\theta f(x,y)}\; P(x,y)\; dxdy,$$

so the density becomes

$$P(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\theta u} M_u(\theta)\; d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\theta[u-f(x,y)]} P(x,y)\; dydxd\theta$$

integrating out θ *gives delta functions*

$$P(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta[u - f(x,y)] P(x,y)\; dx.$$

$\#(C)$

EXAMPLE:

Translation

Let $\hat{z} = \hat{x} + \hat{y}$, *with distributions* $P_x(x)$ *and* $P_y(y)$ *(assuming they are uncorelated), then the*
ribution of $\hat{u}$ *which is denoted by* $P_u(u)$ *is* *ref: C*

$$P_z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta[z - x - y] P_{xy}(x,y)\ dxdy$$

$$= \int_{-\infty}^{\infty} P_{xy}(x, z - x)\ dx$$

$$= \int_{-\infty}^{\infty} P_x(x) P_y(z - x)\ dx\ (uncorrelated)$$

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EXAMPLE:
Multiplication

Let $\hat{z} = \widehat{xy}$, *with distribution* $P_{xy}(x,y)$ *or distributions* $P_x(x)$ *and* $P_y(y)$ *(assuming they are uncorelated), then the distribution of* $\hat{u}$ *which is denoted by* $P_z(z)$ *is* *ref: C*

$$P_z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta[z - xy] P_{xy}(x,y)\ dxdy.$$

If we note that $\sum_i \delta(x - x_i)\frac{1}{|f'(x_i)|}$, *so*

$$\delta[z - xy] = \frac{\delta(y - \frac{z}{x})}{|x|}$$

so

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} P_{xy}\left(\frac{z}{x}, x\right) dx.$$

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EXAMPLE:

Division or Monopulse Ratio

Let $\hat{z} = \frac{\hat{x}}{\hat{y}}$, *with distribution* $P_{xy}(x,y)$ *or distributions* $P_x(x)$ *and* $P_y(y)$ *(assuming they are uncorelated), then the distribution of* $\hat{z}$ *which is denoted by* $P_z(z)$ *is*

$$P_z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta[z - \frac{x}{y}] P_{xy}(x,y)\ dydx.$$

If we note that

$$\delta[z - \frac{x}{y}] = \delta(y - zx)|x|$$

so

$$f_z(z) = \int_{-\infty}^{\infty} |x| P_{xy}(zx, x) dx.$$

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MULTI-VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

RAYLEIGH PROBLEM

Sum of two dimension random amplitudes and phases:

$$\hat{X} = \sum_{i=1}^{N} \hat{x}_i = \sum_{i=1}^{N} \hat{a}_i \cos(\hat{\theta}_i)$$

Arose in Scattering off Rough Surfaces,

Applications in Radar, Sonar, Acoustics, Communications, Physics, etc.

INSTANCE OF TRACKING PROBLEM

Transformations from spherical to Cartesian coordinates

$$\hat{x} = (\hat{r})\cos\left(\hat{\theta}\right)\sin(\hat{\varphi}),$$

$$y = (\hat{r})\sin\left(\hat{\theta}\right)\sin(\hat{\varphi}),$$

$$\hat{z} = (\hat{r})\cos(\hat{\varphi}).$$

Rewritten as:

$$\hat{x} = \tfrac{\hat{r}}{2}\left[\sin\left(\hat{\theta}+\hat{\varphi}\right)+\sin\left(\hat{\theta}-\hat{\varphi}\right)\right]$$

$$y = \tfrac{\hat{r}}{2}\left[\cos(\hat{\theta}+\hat{\varphi})-\cos(\hat{\theta}+\hat{\varphi})\right]$$

MULTI-VARIABLE APPLICATIONS OF CHARCTERISTIC FUNCTION

Then the distribution for z is: $f_z(z) = \int_{-1}^{1} \frac{P_r(\frac{z}{\varphi})\left[a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})T_n(\varphi)\right]}{\pi|\varphi|\sqrt{(1-\varphi^2)}} d\varphi.$

two sum case $z = z_1 + z_2$

$f_z(z) \overset{\text{(uncorrelated)}}{=} \int_{-\infty}^{\infty} P_{z_1}(z_1) P_{z_2}(z_1 - z)\, dz_1$
$= \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{P_{r_1}(\frac{z_1}{\varphi_1})\left[a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})T_n(\varphi_1)\right]}{\pi\vert\varphi_1\vert\sqrt{(1-\varphi_1^2)}} d\varphi_1 \times$
$\int_{-1}^{1} \frac{P_{r_2}(\frac{z_2+z_1-z}{\varphi_2})\left[a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})T_n(\varphi_2)\right]}{\pi\vert\varphi_2\vert\sqrt{(1-\varphi_2^2)}} d\varphi_2 dz_1$

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Case n is obtained by repeated application of Case 2

Conclusions

- Demonstrated how to use CF to bypass difficulties in computing combinations of random variables using a method based on Cohen
- Using Fourier analysis and application of definition of Dirac delta function, combined PDF is obtained with considerable simplification
- Motivation engineering applications that occur in radar/sonar applications that involve combinations of probability density functions
- Have demonstrated simple method to obtain PDF for different combinations of random variables
- Rayleigh problem for i=2 solved which implies general solution for case i=n