Limits and Asymptotics

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1 Series, Limits and Asymptotics

Consider the function

$$f(x) = \frac{1+x^2}{1+x},$$

The crudest statement we could make about its behavior is $f(x) \to \infty$ as $x \to \infty$. Often, we would like to know how fast it goes to infinity. The answer is clear here – it grows like x.

What about $f(x) - x$?

$$f(x) - x = \frac{1+x^2}{1+x} - x$$

$$\frac{1+x^2}{1+x} = \frac{x^2+1}{x\left(1+\frac{1}{x}\right)}.$$

If $|x| > 1$,

$$\left(1+\frac{1}{x}\right) = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots$$

so

$$\begin{aligned}
\frac{1+x^2}{1+x} &= \frac{x^2+1}{x}\left(1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots\right) \\
&= \left(1+\frac{1}{x}\right)\left(1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots\right) \\
&= x - 1 - \frac{2}{x} - \frac{2}{x^2} + \ldots
\end{aligned}$$

and thus

$$f(x) - x = -1 + \frac{2}{x} - \frac{2}{x^2} + \ldots.$$

So $f(x) - x$ remains bounded as $x \to \infty$. In fact, as you can see it approaches the limit -1. What about $f(x) - x + 1$? We could say that it tends to zero as $x \to \infty$, or we could say that it behaves like $\frac{2}{x}$ when x is large. We could address many similar questions with the series for $f(x)$. $f(x)$ is the asymptotic expansion when x is large. The series is useless as $x \to 0$ as it is neither convergent nor asymptotic.

Writing

$$\begin{aligned}
f(x) &= \frac{1+x^2}{1+x} = (1+x^2)(1 - x + x^2 - x^3 + \ldots) \\
&= 1 - x + 2x^2 - \ldots
\end{aligned}$$

gives a convergent asymptotic expansion as $x \to 0$.

2 The Symbols O, o, *and* $\sim$

We will now consider some precise definitions of the growth rates of functions.

2.1 O (read as big "oh")

Let $f(x)$ and $g(x)$ be continuous functions with g(x) continuous and x_0 a fixed point.

We write $f(x) = O\left(g(x)\right)$ $(x \to x_0)$ if there is a constant A s.t. $|f(x)| \leq A|g(x)|$ for all x in some neighborhood of x_0. (If $x_0 = \infty$, this means for sufficiently large x.

We can write this as:

$$\frac{|f(x)|}{|g(x)|} \leq A$$

where A is continuous as $x \to x_0$.

Roughly speaking (as a mnemonic) we can say that $f(x) = 0\,(g(x))$ means that $f(x)$ doesn't grow any faster than $g(x)$ as $x \to x_0$.

So $f(x) = 0(1)$ as $x \to \infty$ means that

$$\lim_{x\to\infty} \frac{f(x)}{1} \leq A$$

i.e. $f(x)$ is bounded.

2.2 o (read little "oh")

$$f(x) = o\,(g(x)) \quad (x \to x_0)$$

if

$$\lim_{x\to\ x_0} \frac{|f(x)|}{|g(x)|} \to 0 \ as\ x \to x_0$$

In other words, $f(x) = o\,(g(x))$ can be interpreted as saying that $f(x)$ grows more slowly than $g(x)$ as $x \to x_0$. So if $g(x) \to 0$ as $x \to x_0$, and $f(x) = o\,(g(x))$, $f(x)$ must go to zero more rapidly.

2.3 $\sim$ (is asymptotic to)

$$f(x) \sim g(x) \quad (x \to x_0) \quad \text{if} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$$

i.e. $f(x)$ and $g(x)$ grow at the same rate as $x \to x_0$

2.4 Summary

Thus $o, O,\ and\ \ \sim$ characterize three states of knowledge.
$\sim$ we know the most
o we know less
O we know the least

In some ways $\sim$ is superfluous, since if

$$f(x) \sim g(x) \quad (x \to x_0) \ \ \text{then}$$

$$f(x) = g(x)(1 + o(1)) \ \ \text{as} \ \ (x \to x_0)$$

but $\sim$ is useful.

$O, o,$ and $\sim$ suppress information — be careful

2.5 More complicated forms

$$f(x) = g(x) + O(h(x)) \quad (x \to x_0)$$

means

$$f(x) - g(x) = O(h(x)) \quad (x \to x_0)$$

or

$$\lim x \to x_0 \frac{|f(x) - g(x)|}{|h(x)|} \leq A$$

Similarly,

$$f(x) = g(x) + o(h(x)) \quad (x \to x_0)$$

means

$$f(x) - g(x) = o(h(x)) \quad (x \to x_0)$$

3 Examples to Study

$$\sqrt{n^2+1} \sim n \quad (n \to \infty)$$
$$\frac{n}{n+1} \sim n \quad (n \to \infty)$$
$$\sin x \sim x \quad (x \to 0);\ x \text{ in radians}$$

$$\begin{aligned}
(1+x)^{-1} \quad &= \quad o(1) \quad (x \to \infty) \\
&= \quad o(x^{-1}) \quad (x \to \infty) \\
&= \quad O(x^{-2}) \quad (x \to \infty) \\
&\sim \quad x^{-2} \quad (x \to \infty) \\
&= \quad x^{-2} + o(x^-2) \quad (x \to \infty) \\
&= \quad x^{-2} + O(x^{-4}) \quad (x \to \infty) \\
&= \quad x^{-2} + x^{-4} + O(x^{-6}) \quad (x \to \infty)
\end{aligned}$$

Note that $f(x) \sim x + \sqrt{x} \quad (x \to \infty)$ conveys no more information than $f(x) \sim x \ \ (x \to \infty)$, since $f(x)$ is dominated by x.

Stirling's approximation can be written as

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n; \quad (n \to \infty)$$

$$\sin x = O(1) as x \to \infty$$

$$\frac{\sin x}{1} \leq A \quad \Rightarrow \quad \sin x \text{ is bounded.}$$

$$\frac{1}{1+x^2} = O(1) \qquad (x \to \infty)$$

$$\Rightarrow \quad \frac{1}{1+x^2} \to 0 \text{ as } \quad x \to \infty$$

since

$$\lim_{x \to \infty} \frac{|(1+x^2)^{-1}|}{1} = 0.$$