1 Series, Limits and Asymptotics

Consider the function

\[ f(x) = \frac{1 + x^2}{1 + x} \]

The crudest statement we could make about its behavior is \( f(x) \to 1 \) as \( x \to \infty \). Often, we would like to know how fast it goes to infinity. The answer is clear here—it grows like \( x \).

What about \( f(x) - x \)?

\[ f(x) - x = \frac{1 + x^2}{1 + x} - x \]

\[ \frac{1 + x^2}{1 + x} = \frac{x^2 + x}{x(1 + x)} \]

If \( |x| > 1 \),

\[ \left(1 + \frac{1}{x}\right) = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots \]

so

\[ \frac{1 + x^2}{1 + x} = \frac{x^2 + 1}{x} \left(1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots\right) \]

\[ = \left(1 + \frac{1}{x}\right) \left(1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \ldots\right) \]

\[ = x - 1 - \frac{2}{x} + \frac{2}{x^2} + \ldots \]

and thus

\[ f(x) - x = -1 + \frac{2}{x} - \frac{2}{x^2} + \ldots \]

So \( f(x) - x \) remains bounded as \( x \to \infty \). In fact, as you can see it approaches the limit \(-1\). What about \( f(x) - x + 1 \)? We could say that it tends to zero as \( x \to \infty \), or we could say that it behaves like \( \frac{2}{x} \) when \( x \) is large. We could address many similar questions with the series for \( f(x) \). \( f(x) \) is the asymptotic expansion when \( x \) is large. The series is useless as \( x \to 0 \) as it is neither convergent nor asymptotic.

Writing

\[ f(x) = \frac{1 + x^2}{1 + x} = (1 + x^2)(1 - x + x^2 - x^3 + \ldots) \]

\[ = 1 - x + 2x^2 - \ldots \]

gives a convergent asymptotic expansion as \( x \to 0 \).

2 The Symbols \( O \), \( o \), and \( \sim \)

We will now consider some precise definitions of the growth rates of functions.

2.1 \( O \) (read as big “oh”)

Let \( f(x) \) and \( g(x) \) be continuous functions with \( g(x) \) continuous and \( x_0 \) a fixed point.

We write \( f(x) = O(g(x)) \) \( (x \to x_0) \) if there is a constant \( A \) s.t. \( |f(x)| \leq A|g(x)| \) for all \( x \) in some neighborhood of \( x_0 \). (If \( x_0 = \infty \), this means for sufficiently large \( x \).

We can write this as:

\[ \frac{|f(x)|}{|g(x)|} \leq A \]

where \( A \) is continuous as \( x \to x_0 \).
Roughly speaking (as a mnemonic) we can say that $f(x) = 0(g(x))$ means that $f(x)$ doesn’t grow any faster than $g(x)$ as $x \to x_0$.

So $f(x) = O(1)$ as $x \to \infty$ means that
\[
\lim_{x \to \infty} \frac{f(x)}{1} \leq A
\]
i.e. $f(x)$ is bounded.

### 2.2 $o$ (read little “oh”)

\[f(x) = o(g(x)) \quad (x \to x_0)\]

if
\[
\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} \to 0 \text{ as } x \to x_0
\]

In other words, $f(x) = o(g(x))$ can be interpreted as saying that $f(x)$ grows more slowly than $g(x)$ as $x \to x_0$. So if $g(x) \to 0$ as $x \to x_0$, and $f(x) = o(g(x))$, $f(x)$ must go to zero more rapidly.

### 2.3 $\sim$ (is asymptotic to)

\[f(x) \sim g(x) \quad (x \to x_0) \quad \text{if} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1\]
i.e. $f(x)$ and $g(x)$ grow at the same rate as $x \to x_0$

### 2.4 Summary

Thus $o, O,$ and $\sim$ characterize three states of knowledge.

$\sim$ we know the most

$o$ we know less

$O$ we know the least

In some ways $\sim$ is superfluous, since if
\[f(x) \sim g(x) \quad (x \to x_0)\]
then
\[f(x) = g(x)(1 + o(1)) \quad (x \to x_0)\]
but $\sim$ is useful.

$O, o,$ and $\sim$ suppress information — be careful

### 2.5 More complicated forms

\[f(x) = g(x) + O(h(x)) \quad (x \to x_0)\]

means
\[f(x) - g(x) = O(h(x)) \quad (x \to x_0)\]
or
\[
\lim_{x \to x_0} \frac{|f(x) - g(x)|}{|h(x)|} \leq A
\]

Similarly,
\[f(x) = g(x) + o(h(x)) \quad (x \to x_0)\]
means
\[f(x) - g(x) = o(h(x)) \quad (x \to x_0)\]

### 3 Examples to Study

\[
\sqrt{n^2 + 1} \sim n \quad (n \to \infty)
\]
\[
\frac{n}{n+1} \sim n \quad (n \to \infty)
\]
\[
\sin x \sim x \quad (x \to 0); \quad x \text{ in radians}
\]
\[
(1 + x)^{-1} = o(1) \quad (x \to \infty)
\]
\[
= o(x^{-1}) \quad (x \to \infty)
\]
\[
= O(x^{-2}) \quad (x \to \infty)
\]
\[
\sim x^{-2} \quad (x \to \infty)
\]
\[
= x^{-2} + o(x^{-2}) \quad (x \to \infty)
\]
\[
= x^{-2} + O(x^{-4}) \quad (x \to \infty)
\]
\[
= x^{-2} + x^{-4} + O(x^{-6}) \quad (x \to \infty)
\]

Note that $f(x) \sim x + \sqrt{x} \quad (x \to \infty)$ conveys no more information than $f(x) \sim x \quad (x \to \infty)$, since $f(x)$ is dominated by $x$.

Stirling’s approximation can be written as
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad (n \to \infty)
\]
\[
\sin x = O(1) \text{ as } x \to \infty
\]

\[
\frac{\sin x}{1} \leq A \quad \Rightarrow \quad \sin x \text{ is bounded.}
\]

\[
\frac{1}{1 + x^2} = O(1) \quad (x \to \infty)
\]

\[
\Rightarrow \quad \frac{1}{1 + x^2} \to 0 \text{ as } x \to \infty
\]

since

\[
\lim_{x \to \infty} \frac{|(1 + x^2)^{-1}|}{1} = 0.
\]