1. The finite element interpolant $f_h$ of the function $f(x) = \sin(\pi x)$ is

$$f_h(x) = \sin(\pi/4)\phi_2(x) + \sin(\pi/2)\phi_3(x) + \sin(3\pi/4)\phi_4(x),$$

where $\phi_2, \phi_3, \phi_4$ are the piece-wise linear finite element basis functions that represent the nodes $x = 1/4, 1/2, 3/4$, respectively. Their graphs are as follows

![Graph of the function and interpolant](image)

Figure 1: The finite element interpolant $f_h$ of the function $f(x) = \sin(\pi x)$.

2. (a) Integrating the equation, we can easily find the exact solution

$$y = -\frac{1}{6}x^3 + c_1x + c_2.$$  

The boundary conditions $y(0) = y(1) = 0$ implies $c_1 = 1/6$ and $c_2 = 0$. Then

$$y = \frac{1}{6}x(1 - x^2).$$

(b) To derive the weak form of the problem, we multiply the equation by a sufficiently smooth test function $u(x)$:

$$-y''u = xu, \quad u \in H^1_0(0,1).$$
We then obtain the weak form by integrating over [0, 1]:

\[-y'u|_0^1 + \int_0^1 y'u' \, dx = \int_0^1 xu \, dx,\]
\[\int_0^1 y'u' \, dx = \int_0^1 xu \, dx, \quad u \in H^1_0(0, 1).\]  \hspace{1cm} (1)

(c) We seek the Galerkin approximation

\[y_h = \sum_{j=1}^3 a_j \sin(j\pi x).\]

Then the stiffness matrix and load vector are

\[K_{ij} = \begin{cases} \int_0^1 \frac{d}{dx} \sin(i\pi x) \frac{d}{dx} \sin(j\pi x) \, dx = 0, & i \neq j, \\ \int_0^1 \frac{d}{dx} \sin(i\pi x) \frac{d}{dx} \sin(j\pi x) \, dx = \frac{i^2 \pi^2}{2}, & i = j, \end{cases}\]
\[F_i = \int_0^1 x \sin(i\pi x) \, dx = -\frac{\cos(i\pi)}{i\pi}.\]

Solving the linear system

\[\sum_{j=1}^3 K_{ij} a_j = F_i, \quad i = 1, 2, 3\]

we obtain the approximation

\[a_j = -\frac{2 \cos(j\pi)}{j^3 \pi^3},\]
\[y_h = \sum_{j=1}^N -\frac{2 \cos(j\pi)}{j^3 \pi^3} \sin(j\pi x).\]

Figure 2 shows that \(y_h\) is almost equal to \(y\) with \(N = 10\).

(d) Let the nodes \(x_e = (e - 1)0.25, \ e = 1, 2, 3, 4, 5\) and \(\phi_e\) be the basis function that represents the node \(x_e\). Because the basis functions \(\phi_1\) and \(\phi_5\) do not satisfy the boundary condition, we can use only \(\phi_2, \phi_3\) and \(\phi_4\) in our approximation. Therefore the finite element approximation has the following form

\[y_h(x) = a_2 \phi_2(x) + a_3 \phi_3(x) + a_4 \phi_4(x).\]
Figure 2: Fourier series approximation of the exact solution $y = \frac{1}{6}x(1 - x^2)$.

Since

$$\frac{dN_i^e(x)}{dx} = -\frac{1}{h} = -4, \quad \frac{dN_j^e(x)}{dx} = \frac{1}{h} = 4, \quad e = 1, 2, 3, 4,$$

we have

**Element e=1:**

$$K^1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^1 = \begin{pmatrix} \frac{1}{18} \\ 0 \\ 0 \end{pmatrix}$$

**Element e=2:**

$$K^2 = \begin{pmatrix} 4 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^2 = \begin{pmatrix} \frac{1}{24} \\ \frac{25}{12} \\ 0 \end{pmatrix}$$

**Element e=3:**

$$K^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix}, \quad F^3 = \begin{pmatrix} 0 \\ \frac{175}{24} \\ \frac{7}{12} \end{pmatrix}$$
Element $e=4$:

\[
K^4 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 4
\end{pmatrix}, \quad F^4 = \begin{pmatrix}
0 \\
0 \\
\frac{25}{24}
\end{pmatrix}
\]

\[
K = K^1 + K^2 + K^3 + K^4 = \begin{pmatrix}
8 & -4 & 0 \\
-4 & 8 & -4 \\
0 & -4 & 8
\end{pmatrix},
\]

\[
F = F^1 + F^2 + F^3 + F^4 = \begin{pmatrix}
\frac{1}{16} \\
\frac{1}{8} \\
\frac{1}{16}
\end{pmatrix}
\]

and then the system of equation is

\[
\begin{pmatrix}
8 & -4 & 0 \\
-4 & 8 & -4 \\
0 & -4 & 8
\end{pmatrix}
\begin{pmatrix}
a_2 \\
a_3 \\
a_4
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{16} \\
\frac{1}{8} \\
\frac{1}{16}
\end{pmatrix}
.
\]

Solving it, we obtain

\[
a_2 = \frac{5}{128}, \quad a_3 = \frac{1}{16}, \quad a_4 = \frac{7}{128},
\]

and then the finite element approximation $y_h$ (see figure 3)

\[
y_h(x) = \frac{5}{128} \phi_2(x) + \frac{1}{16} \phi_3(x) + \frac{7}{128} \phi_4(x)
\]

\[
= \begin{cases}
\frac{20}{128} x, & 0 \leq x \leq 1/4, \\
\frac{1}{28} (0.5 - x) + \frac{1}{32} (x - 0.25), & 1/4 \leq x \leq 1/2, \\
\frac{1}{32} (0.75 - x) + \frac{25}{128} (x - 0.5), & 1/2 \leq x \leq 3/4, \\
\frac{28}{128} (1 - x), & 3/4 \leq x \leq 1.
\end{cases}
\]

We note that in this particular problem $a_i = y(x_i)$, $i = 1, 2, 3, 4, 5$, where $y(x)$ is the exact solution.
Figure 3: Finite element approximation of the exact solution $y = \frac{1}{6}x(1 - x^2)$. 