Consider the boundary value problem

\[-y''(x) = f(x), \quad 0 < x < 1,\]
\[y(0) = y(1) = 0,\]

where \(f\) is any given function. The weak form of the problem is

\[
\int_0^1 y'(x)u'(x)dx = \int_0^1 f(x)u(x)dx, \quad \text{for all } u \in H^1_0(0,1).
\]  \hspace{1cm} (1)

Let \(x_1 = 0 < x_2 < \cdots < x_N < x_{N+1} = 1\) be the mesh of \([0,1]\) and \(\phi_1(x), \phi_2(x), \cdots, \phi_N(x), \phi_{N+1}(x)\) be the linear finite element basis functions, and

\[H^{N-2} = \{a_2\phi_2(x) + \cdots + a_N\phi_N(x) | a_2, \cdots, a_N \text{ any real numbers} \} \subset H^1_0(0,1).\]

Let \(y_h\) be the linear finite element approximation of \(y\) in the finite dimensional space \(H^{N-2}\) that satisfies

\[
\int_0^1 y'_h(x)u'(x)dx = \int_0^1 f(x)u(x)dx, \quad \text{for all } u \in H^{N-2}.
\]  \hspace{1cm} (2)

You might have noticed that the linear finite element approximation \(y_h\) is exactly equal to \(y\) at the node points \(x_i, i = 1, 2, \cdots, N, N + 1\). This problem is to ask you to prove this fact.

(i) (5 points) For any \(i = 2, \cdots, N\), construct a function \(G_i(x) \in H^{N-2}\) such that

\[
\int_0^1 u'(x)G_i'(x)dx = u(x_i), \quad \text{for all } u \in H^1_0(0,1).
\]  \hspace{1cm} (3)

**Solution 1.** Since \(G_i(x) \in H^{N-2}\), \(G_i(0)\) and \(G_i(1)\) must be equal to zero and we can also guess that \(G_i(x)\) should be linear on \([0, x_i]\) and \([x_i, 1]\). By condition (3), we can guess that \(G_i(x)\) is not differentiable at \(x_i\). We then conclude that \(G_i(x)\) should have the following form

\[G_i(x) = \begin{cases} \frac{ax}{x_i}, & 0 \leq x \leq x_i, \\ \frac{ax(1-x)}{1-x_i}, & x_i \leq x \leq 1, \end{cases}\]

where the constant \(a\) is to be determined. To determine \(a\), we substitute it into (3) and derive by noting that \(u(0) = u(1) = 0\) that

\[
u(x_i) = \int_0^1 u'(x)G_i'(x)dx
\]

\[
= \int_0^{x_i} u'(x)G_i'(x)dx + \int_{x_i}^1 u'(x)G_i'(x)dx
\]

\[
= a \int_0^{x_i} u'(x)dx - \frac{ax_i}{1-x_i} \int_{x_i}^1 u'(x)dx
\]

\[
= ax_i + \frac{ax_i}{1-x_i}u(x_i)
\]

\[
= \frac{a}{1-x_i}u(x_i).
\]
So \( a = 1 - x_i \) and then
\[
G_i(x) = \begin{cases} 
(1 - x_i)x, & 0 \leq x \leq x_i, \\
x_i(1 - x), & x_i \leq x \leq 1,
\end{cases}
\]

**Solution 2.** For any \( 0 < t < 1 \), we define
\[
u_t(x) = \begin{cases} 
x, & 0 \leq x \leq t, \\
\frac{(1-x_i)t}{1-t}, & t \leq x \leq 1.
\end{cases}
\]
It is clear that \( u_t \in H^1_0(0,1) \). Since \( G_i(0) = G_i(1) = 0 \), it therefore follows from (3) that
\[
u_t(x_i) = \int_0^1 u_t'(x)G_i'(x)dx
\]
\[
= \int_0^t u_t'(x)G_i'(x)dx + \int_t^1 u_t'(x)G_i'(x)dx
\]
\[
= \int_0^t G_i'(x)dx - \frac{t}{1-t} \int_t^1 G_i'(x)dx
\]
\[
= G_i(t) + \frac{t}{1-t}G_i(t)
\]
\[
= \frac{1}{1-t}G_i(t).
\]

We then derive that
\[
G_i(t) = (1-t)u_t(x_i)
\]
\[
= \begin{cases} 
(1-x_i)t, & 0 \leq t \leq x_i, \\
x_i(1-t), & x_i \leq t \leq 1.
\end{cases}
\]

One can readily verify that such \( G_i \in H^{N-2} \) and it satisfies (3). Indeed, we have for all \( u \in H^1_0(0,1) \)
\[
\int_0^1 u'(x)G_i'(x)dx = \int_0^{x_i} u'(x)G_i'(x)dx + \int_{x_i}^1 u'(x)G_i'(x)dx
\]
\[
= (1-x_i) \int_0^{x_i} u'(x)dx - x_i \int_{x_i}^1 u'(x)dx
\]
\[
= u(x_i)(1-x_i) + x_iu(x_i)
\]
\[
= u(x_i).
\]

(ii) (5 points) Show that \( y_h(x_i) = y(x_i), \ i = 2, \cdots, N \).

**Solution.** Since \( u = y - y_h \in H^1_0(0,1) \) and \( G_i \in H^{N-2} \), it follows from (1), (2), and (3) that
\[
y(x_i) - y_h(x_i) = \int_0^1 (y'(x) - y_h'(x))G_i'(x)dx
\]
\[
= \int_0^1 f(x)G_i'(x)dx - \int_0^1 f(x)G_i'(x)dx
\]
\[
= 0.
\]
So \( y_h(x_i) = y(x_i) \).