AM466/562: Finite Element Method

Quiz 1

March 8, 2005

Name: **ID Number:**

Undergraduate ——— **Graduate** ———

Score:

Problem 1 ————- Problem 2 ————

Problem 3 ————-

Total Score: ————

Note:

1. **Problem 3 is optional for undergraduates. So please indicate whether you are a undergraduate.**
2. No books and notes are allowed.
3. You must show your work in details in order to receive full credits.

Problem 1. (30 points, 3 points each) True or false (Circle your answer).

(1) (True False) The basic idea of Galerkin's approximation is to seek an approximate solution of a weak form in a finite-dimensional subspace $H^{(N)}$ rather than in the whole space like $H^1(0,1)$.

(2) (True False) Because the function $\phi(x)$ in the following figure is equal to 1 at the node $x = 0.5$ and zero at all other nodes, it must be a linear finite element basis function.

(3) (True False) If $N_1(x), N_2(x)$, and $N_3(x)$ denote the three shape functions on a three-node quadratic element, then $N_1(x) + N_2(x) + N_3(x) = 1$.

(4) (True False) In the equation

$$\frac{d}{dx}\left(A(x)\frac{dy}{dx}\right) + B(x)\frac{dy}{dx} + C(x)y + D(x) = 0, \quad 0 \le x \le L,$$

if $C(x) \equiv 0$, then one of the values y generally needs to be specified in the case of Nuemann boundary conditions.

(5) (True False) If y is a solution of the weak form of a second order differential equation (strong form), it must be also the solution of the strong form.

(6) (True False) Higher order shape functions can increase the accuracy of an approximate solution, but there is a price to pay: the required computational effort increases.

(7) (True False) For one-dimensional problems, a stiffness matrix is banded because finite element basis functions are not equal to zero only on at most two elements.

(8) (True False) In the following figure, the function $\phi(x)$ defined on the mesh of two three-node quadratic elements is a quadratic basis function.

(9) (True   False) The popular technique of "blasting" the diagonal is used to handle the known node values, for instance, the Dirichlet boundary conditions.

(10) (True   False) The functions defined by shape functions, piecewise and element by element, are linearly independent.

Problem 2. Consider the boundary value problem

$$\begin{aligned} x^2 \frac{d^2y}{dx^2} + y + 1 &= 0, \quad 1 < x < 2, \\ y(1) &= 2, \quad y'(2) = 3. \end{aligned}$$

1. (10 points) Derive the weak form of the problem.

2. (5 points) For the mesh of four linear elements of the same length, write down two linear shape functions $N_1^1(x)$ and $N_2^1(x)$ on the first element $[1, 1.25]$.

3. (5 points) sketch the linear finite element basis functions $\phi_1(x)$ and $\phi_3(x)$ that represent the nodes $x_1 = 1$ and $x_3 = 1.5$, respectively.

4. (5 points) Use the linear shape functions $N_1^1(x)$ and $N_2^1(x)$ on the first element $[1, 1.25]$ to write out the expressions for entries k_{ij}^1 of the local element matrix of the first element (You do not need to calculate the integrals). Is the local element matrix symmetric?

5. (5 points) Specify the boundary conditions in the following MESH data file

```
%  MESH data file
%----------------------------------------------
% NUMNP  + 3 dummy numbers
%----------------------------------------------
   5       0     0     0
%----------------------------------------------
% XORD  NPBC     Y     Q
%----------------------------------------------
   1
   1.25    0     0     0
   1.5     0     0     0
   1.75    0     0     0
   2
```

Problem 3 (Optional for undergraduates). Let y be the solution of the following variational problem

$$\int_0^1 \frac{dy}{dx}\frac{du}{dx}dx + \int_0^1 yudx = \int_0^1 \sin(x)udx, \quad \text{for all } u \in H_0^1(0,1). \tag{1}$$

Let y_h be Galerkin's approximation of y in the finite dimensional space $H^N \subset H_0^1(0,1)$ that satisfies

$$\int_0^1 \frac{dy_h}{dx}\frac{du}{dx}dx + \int_0^1 y_h udx = \int_0^1 \sin(x)udx, \quad \text{for all } u \in H^N. \tag{2}$$

(i) (5 points) Show the error $e = y - y_h$ is orthogonal to H^N, that is,

$$\int_0^1 \frac{de}{dx}\frac{du}{dx}dx + \int_0^1 eudx = 0, \quad \text{for all } u \in H^N. \tag{3}$$

(ii) (5 points) Show that Galerkin's approximation y_h of y in H^N is the best one, that is,

$$\begin{aligned} &\int_0^1 \left(\frac{d(y-y_h)}{dx}\right)^2 dx + \int_0^1 (y-y_h)^2 dx \\ \le\ &\int_0^1 \left(\frac{d(y-u)}{dx}\right)^2 dx + \int_0^1 (y-u)^2 dx, \quad \text{for all } u \in H^N. \end{aligned} \tag{4}$$