AM466/562: Finite Element Method

Solutions of Quiz 1

March 8, 2005

Problem 1. (30 points, 3 points each) True or false (Circle your answer).

(1) (True False) The basic idea of Galerkin's approximation is to seek an approximate solution of a weak form in a finite-dimensional subspace $H^{(N)}$ rather than in the whole space like $H^1(0,1)$.

True

(2) (True False) Because the function $\phi(x)$ in the following figure is equal to 1 at the node $x = 0.5$ and zero at all other nodes, it must be a linear finite element basis function.

False

(3) (True False) If $N_1(x), N_2(x)$, and $N_3(x)$ denote the three shape functions on a three-node quadratic element, then $N_1(x) + N_2(x) + N_3(x) = 1$.

True

(4) (True False) In the equation

$$\frac{d}{dx}\left(A(x)\frac{dy}{dx}\right) + B(x)\frac{dy}{dx} + C(x)y + D(x) = 0, \quad 0 \le x \le L,$$

if $C(x) \equiv 0$, then one of the values y generally needs to be specified in the case of Nuemann boundary conditions.

True

(5) (True False) If y is a solution of the weak form of a second order differential equation (strong form), it must be also the solution of the strong form.

False

(6) (True False) Higher order shape functions can increase the accuracy of an approximate solution, but there is a price to pay: the required computational effort increases.

True

(7) (True False) For one-dimensional problems, a stiffness matrix is banded because finite element basis functions are not equal to zero only on at most two elements.

True

(8) (True False) In the following figure, the function $\phi(x)$ defined on the mesh of two three-node quadratic elements is a quadratic basis function.

False

(9) (True False) The popular technique of "blasting" the diagonal is used to handle the known node values, for instance, the Dirichlet boundary conditions.

True

(10) (True False) The functions defined by shape functions, piecewise and element by element, are linearly independent.

True

Problem 2. Consider the boundary value problem

$$\begin{aligned} x^2\frac{d^2y}{dx^2} + y + 1 &= 0, \quad 1 < x < 2, \\ y(1) &= 2, \quad y'(2) = 3. \end{aligned}$$

1. (10 points) Derive the weak form of the problem.

   Multiply the equation by u and integrate from 1 to 2:

$$\int_1^2 (x^2y'' + y + 1)u dx = 0, \quad u \in H^1(1,2), \ and \ u(1) = 0,$$

   Integration by parts gives the weak form

$$\int_1^2 (x^2y'u' + 2xy'u - yu)dx = \int_1^2 u dx + 12u(2), \quad u \in H^1(1,2), \ and \ u(1) = 0.$$

2. (5 points) For the mesh of four linear elements of the same length, write down two linear shape functions $N_1^1(x)$ and $N_2^1(x)$ on the first element $[1, 1.25]$.

$$N_1^1(x) = 4(1.25 - x),$$
$$N_2^1(x) = 4(x - 1).$$

3. (5 points) sketch the linear finite element basis functions $\phi_1(x)$ and $\phi_3(x)$ that represent the nodes $x_1 = 1$ and $x_3 = 1.5$, respectively.

4. (5 points) Use the linear shape functions $N_1^1(x)$ and $N_2^1(x)$ on the first element $[1, 1.25]$ to write out the expressions for the entries k_{ij}^1 of the local element matrix of the first element (You do not need to calculate the integrals).

$$k_{ij}^1 = \int_1^{1.25} \left(x^2 \frac{dN_i^1(x)}{dx} \frac{dN_j^1(x)}{dx} + 2x N_i^1(x) \frac{dN_j^1(x)}{dx} - N_i^1(x) N_j^1(x) \right) dx.$$

The matrix is not symmetric due to the asymmetric term $2xN_i^1(x)\frac{dN_j^1(x)}{dx}$.

5. (5 points) Specify the boundary conditions in the following MESH data file

```
%  MESH data file
%-----------------------------------------------
% NUMNP  + 3 dummy numbers
%-----------------------------------------------
   5       0      0      0
%-----------------------------------------------
% XORD  NPBC      Y      Q
%-----------------------------------------------
   1       1      2      0
   1.25    0      0      0
   1.5     0      0      0
   1.75    0      0      0
   2       0      0      12
```

Problem 3. Let y be the solution of the following variational problem

$$\int_0^1 \frac{dy}{dx}\frac{du}{dx}dx + \int_0^1 yudx = \int_0^1 \sin(x)udx, \quad \text{for all } u \in H_0^1(0,1) \tag{1}$$

Let y_h be Galerkin's approximation of y in the finite dimensional space $H^N \subset H_0^1(0,1)$ that satisfies

$$\int_0^1 \frac{dy_h}{dx}\frac{du}{dx}dx + \int_0^1 y_h udx = \int_0^1 \sin xudx, \quad \text{for all } u \in H^N. \tag{2}$$

(i) (5 points) Show the error $e = y - y_h$ is orthogonal to the finite dimensional space H^{N-2}, that is,

$$\int_0^1 \frac{de}{dx}\frac{du}{dx}dx + \int_0^1 eudx = 0, \quad \text{for all } u \in H^N. \tag{3}$$

Proof. Subtracting (2) from (1) gives (3).

(ii) (5 points) Show that Galerkin's approximation y_h of y in H^N is the best one, that is,

$$\begin{aligned} &\int_0^1 \left(\frac{d(y-y_h)}{dx}\right)^2 dx + \int_0^1 (y-y_h)^2 dx \\ \le\ &\int_0^1 \left(\frac{d(y-u)}{dx}\right)^2 dx + \int_0^1 (y-u)^2 dx, \quad \text{for all } u \in H^N. \end{aligned} \tag{4}$$

Proof. Using (3), we derive that for any $u \in H^N$

$$\begin{aligned}
& \int_0^1 \left(\frac{d(y-u)}{dx}\right)^2 dx + \int_0^1 (y-u)^2 dx \\
= & \int_0^1 \left(\frac{d(y-y_h+y_h-u)}{dx}\right)^2 dx + \int_0^1 (y-y_h+y_h-u)^2 dx \\
= & \int_0^1 \left(\frac{d(y-y_h)}{dx}\right)^2 dx + \int_0^1 (y-y_h)^2 dx \\
& +2\int_0^1 \frac{d(y-y_h)}{dx}\frac{d(y_h-u)}{dx} dx + 2\int_0^1 (y-y_h)(y_h-u) dx \\
& +\int_0^1 \left(\frac{d(y_h-u)}{dx}\right)^2 dx + \int_0^1 (y_h-u)^2 dx \\
= & \int_0^1 \left(\frac{d(y-y_h)}{dx}\right)^2 dx + \int_0^1 (y-y_h)^2 dx \quad \text{(note that } y_h - u \in N^{(N)} \text{ and use (3))} \\
& +\int_0^1 \left(\frac{d(y_h-u)}{dx}\right)^2 dx + \int_0^1 (y_h-u)^2 dx \\
\geq & \int_0^1 \left(\frac{d(y-y_h)}{dx}\right)^2 dx + \int_0^1 (y-y_h)^2 dx.
\end{aligned}$$