Calculus

Weijiu Liu
Department of Mathematics
University of Central Arkansas
201 Donaghey Avenue, Conway, AR 72035, USA

1 Opening

- **Welcome to your Calculus I class**! My name is Weijiu Liu. I will guide you to navigate through Calculus. We will share the learning task together. You will be the major players and I will be a just facilitator. You will need to read your textbook, attend classes, practice in class, do your homework, and so on. I will just tell you what to read, what to do, and how to do. Whenever you cannot hear me acoustically due to my Chinese accent, please feel free to ask me to repeat. If you cannot hear me mathematically, please read your textbook.
- **Syllabus**.
- **Overview of Calculus**.

2 Lecture 1 – Concept of Limit and how to find limits graphically and numerically

Today:

- Problem of limit.
- Find limits graphically.
- Find limits numerically.

Next:

- Basic rules of limit
- Techniques of finding limits

Review:

- Slope of a line: $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- Absolute value of a number:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0, \end{cases}$$

- Trigonometric functions:

$$\begin{aligned} \sin\theta &= \frac{y}{r}, \\ \cos\theta &= \frac{x}{r}, \\ \tan\theta &= \frac{\sin\theta}{\cos\theta}, \\ \cot\theta &= \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}, \\ \sec\theta &= \frac{1}{\cos\theta}, \\ \csc\theta &= \frac{1}{\sin\theta} \end{aligned}$$

Teaching procedure:

1. Formulation of problem of limit – tangent line problems.
2. Find limits graphically and numerically, see slides.
3. Exercises 1.2: 2 (a), (b), (c), (f), (g), (h), 4.

3 Lecture 2 – Techniques of finding limits

Today:

- Basic rules of limit
- Techniques of finding limits

Next:

- Continuation of techniques of finding limits
- Limits of trigonometric, exponential and logarithm functions.

Review:

- $x^2 - y^2 = (x-y)(x+y)$, $x^2 \pm 2xy + y^2 = (x \pm y)^2$, $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.
- Polynomials $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.
- Rational functions $R(x) = \frac{p(x)}{q(x)}$, where p, q are polynomials.
- Rationalization: $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$.
- Factoring trinomials: $ax^2+bx+c = (p_1x+q_1)(p_2x+q_2) = p_1p_2x^2+(p_1q_2+p_2q_1)x+q_1q_2$. So $a = p_1p_2, b = p_1q_2 + p_2q_1, c = q_1q_2$.

$$\begin{matrix} p_1 & q_1 \\ p_2 & q_2 \end{matrix}$$

Teaching procedure:

1. Basic rules of limit (see page 88).
2. Factoring polynomials:
$$x^2 + x - 2, x^2 - 3x + 2, x^2 + 2x - 3, x^3 - 1.$$
3. By **direct substitution**.
   Example. Evaluate the limits:
$$\lim_{x\to 2} \sqrt[3]{2x+1}, \quad \lim_{x\to 2} \frac{x-5}{x^2+4}.$$
4. By **factoring**.
   Example. Evaluate the limits:
$$\lim_{x\to 1} \frac{x^3-1}{x^2+2x-3}, \quad \lim_{x\to 1} \frac{x^2+x-2}{x^2-3x+2}.$$
5. By **rationalizing**.
   Example. Evaluate the limits:
$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}.$$
6. Exercises 1.3: 1, 2, 7, 13, 14.

4 Lecture 3 – Techniques of finding limits

Today:

- Limits of trigonometric, exponential and logarithm functions.
- Continuation of techniques of finding limits.

Next:

- Squeeze Theorem.
- limits of piecewise-defined functions.

Review:

- Inverse functions: $y = x^3$, $x = \sqrt[3]{y} = y^{1/3}$. Inverse function: $y = \sqrt[3]{x}$.
- Exponential function: $y = b^x$, $b > 0$.
- Logarithm function: $y = \log_b x$ if and only if $x = b^y$. Natural logarithm: $y = \log_e x = \ln x$.

$$\log_b b = 1, \quad \ln e = 1.$$

- Inverse Trigonometric functions:
  $y = \sin^{-1} x$ if and only if $\sin y = x$.
  $y = \cos^{-1} x$ if and only if $\cos y = x$.
  $y = \tan^{-1} x$ if and only if $\tan y = x$.

Teaching procedure:

1. General strategy: Always use **Direction Substitution** first. If it does not work, then use other methods.

$$\lim_{x\to 2} \sqrt[3]{2x+1}, \quad \lim_{x\to 0} \frac{\tan x}{x}.$$

2. Limit of trigonometric, exponential, and logarithm functions (see page 91).

   Example. Evaluate the limits:

$$\lim_{x\to 0} \sin(x^2+1), \quad \lim_{x\to 0} \cos^{-1}(x^2+1), \quad \lim_{x\to 0} \ln(e^{x^2+1}).$$

3. Use the important limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

   Example. Evaluate the limits:

$$\lim_{x\to 0} \frac{\tan x}{x}.$$

4. By **Youself**.

   Example. Evaluate the limits:

$$\lim_{x \to 0} \left(\frac{2}{x} - \frac{2}{|x|} \right).$$

5. Exercises 1.3: 3, 9, 11, 12, 21, 22.

5 Lecture 4 – Squeeze Theorem and Limits of Piecewise-defined functions

Today:

- Squeeze Theorem.
- limits of piecewise-defined functions.

Next:

- Definition of continuity
- Continuous functions.

Review:

- Area of a sector: $A = \frac{1}{2}r^2\theta$
- Area of a triangle: $A = \frac{1}{2}hb$.
- If $a \leq b$ and $c > 0$, then $ac \leq bc$.
- Limit rules:

$$\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)].$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \quad (\lim_{x\to a} g(x) \neq 0).$$

- Important limit

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin cx}{cx} = \lim_{x\to 0}\frac{\sin(x^n)}{x^n} = 1.$$

Teaching procedure:

1. Evaluate limit

$$\lim_{x\to 0}\frac{\sin 7x}{3x}.$$

2. Squeeze Theorem (page 92).

3. Evaluate the limit

$$\lim_{x\to 0} x^4 \cos\left(\frac{1}{x^2}\right).$$

4. Let f be defined by

$$f(x) = \begin{cases} 2x+2, & \text{if } x < 2, \\ x^2+1, & \text{if } x \geq 2, \end{cases}$$

Evaluate $\lim\limits_{x\to 2} f(x)$, $\lim\limits_{x\to 1} f(x)$, and $\lim\limits_{x\to 3} f(x)$.

5. Exercises 1.3: 24, 32, 33, 34, 35, 38.

6 Lecture 5 – Continuity

Today:

- Definition of continuity
- Continuous functions.

Next:

- Continuity of composite functions
- Intermediate Value Theorem.

Review:

- graphs of polynomials, sine, cosine, exponential and log functions

Teaching procedure:

1. Definition of continuity: If
$$\lim_{x \to a} f(x) = f(a),$$
then f is said to be continuous at a. Otherwise, it is said to be discontinuous at a.

2. Continuous functions (theorems 4.1 and 4.2, page 100): All polynomials, $\sin x, \cos x, \tan^{-1} x$, and e^x are continuous everywhere; $\ln x$ is continuous for $x > 0$;

3. Discontinuity.

   - Case 1. $f(a)$ is not well defined:
   $$f(x) = \frac{\sin x}{x}.$$

   - Case 2. $\lim_{x \to a} f(x) \neq f(a)$:
   $$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

   - Case 3. $\lim_{x \to a} f(x)$ does not exit:
   $$f(x) = \begin{cases} 2x + 2, & \text{if } x < 2, \\ x^2 + 1, & \text{if } x \geq 2, \end{cases}$$
   $$f(x) = \frac{1}{x}.$$

4. Removable discontinuity:
$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

5. Find discontinuous points of the following functions:

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

$$f(x) = x^2 \tan x.$$

6. Exercises 1.4: 2, 13, 21

7 Lecture 6 – Continuity and Intermediate Value Theorem

Today:

- Continuity of composite functions
- Intermediate Value Theorem.

Next:

- Infinity limits
- Asymptotes

Review:

- the domain of $\ln x$: $(0, \infty)$.
- Solving the quadratic inequality:
$$x^2 - a \geq 0.$$
- definition of continuity: If
$$\lim_{x \to a} f(x) = f(a),$$
then f is said to be continuous at a. Otherwise, it is said to be discontinuous at a.
- Continuous functions (Theorem 4.1, page 100)
- Removable Discontinuity: Find discontinuous points of the following function:
$$f(x) = \frac{x-2}{x^2-4}.$$

Teaching procedure:

1. Continuity of composite function (page 101, theorem 4.3)
$$f(x) = \sin(x^3 + x^2 + x + 1).$$
2. Continuous interval (definition 4.2, page 101): find continuous intervals of the following functions
$$f(x) = \sin(x^3 + x^2 + x + 1).$$
$$f(x) = \sqrt{x^2 - 4}.$$
$$f(x) = \cos\left(\frac{1}{x}\right).$$
$$f(x) = \ln(4 - x^2).$$
3. Intermediate Value Theorem (page 104): Show $f(x) = x^3 - 4x - 2$ has a zero on $[2, 3]$.
4. The method of bisections
5. Exercises 1.4: 25, 29, 31, 43.

8 Lecture 7 – Infinity limits and Asymptotes

Today:

- Infinity limits
- Asymptotes

Next:

- Slant Asymptotes
- Limits of some important functions

Review:

- $\tan x$
- Important limit: $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ $(n > 0)$
- quotient rule of limits

Teaching procedure:

1. Infinite Limits and vertical asymptotes: evaluate

$$\lim_{x \to 1^-} \frac{1-2x}{x^2-1}, \quad \lim_{x \to 1^+} \frac{1-2x}{x^2-1}, \quad \lim_{x \to 1} \frac{1-2x}{x^2-1}.$$

2. Limits at infinity and horizontal asymptotes: evaluate

$$\lim_{x \to \infty} \frac{x^3-x+1}{4x^3-3x-1}, \quad \lim_{x \to \infty} \frac{1-2x}{x^2-1}, \quad \lim_{x \to -\infty} \frac{1-2x}{x^2-1}.$$

3. Exercises 1.5: 2, 8, 28.

9 Lecture 8 – Slant Asymptotes

Today:

- Slant Asymptotes
- Limits of some important functions

Next:

- $\varepsilon - \delta$ definition of limits

Review:

- long division
- $\tan^{-1} x$
- e^x.

Teaching procedure:

1. Review problem: evaluate
$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}}.$$

2. Important limits: evaluate
$$\lim_{x\to\infty} \tan^{-1} x, \quad \lim_{x\to-\infty} \tan^{-1} x.$$
$$\lim_{x\to\infty} e^x, \quad \lim_{x\to-\infty} e^x.$$
$$\lim_{x\to\infty} \ln x, \quad \lim_{x\to 0^+} \ln x.$$

3. Use of important limits: evaluate
$$\lim_{x\to\infty} \sin(\tan^{-1} x), \quad \lim_{x\to 0^+} \tan^{-1}(\ln x).$$

4. Slant Asymptotes:
$$y = \frac{x^2+1}{x-2}.$$

5. Exercises 1.5: 7, 11, 28, 35.

10 Lecture 9 – $\varepsilon - \delta$ definition of limits

Today:

- $\varepsilon - \delta$ definition of limits

Next:

- Review of Chapter 1

Review:

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Teaching procedure:

1. Introductory example
$$\lim_{x \to 2} 2x = 4.$$

2. $\varepsilon - \delta$ definition of limits

3. Use $\varepsilon - \delta$ definition to prove
$$\lim_{x \to 2} 2x = 4.$$

4. Exercises 1.6: 11.

11 Lecture 10 – Definition of derivative

Today:

- definition of derivative

Next:

- continuity and differentiability

Review:

- Slope

Teaching procedure:

1. Motivation of derivative
2. Definition of derivative
3. Use definition to find the derivative of
$$f(x) = \sqrt{2x+1}.$$
4. Sketch the graph $f'(x)$ from the graph $f(x)$
5. Exercises 2.2: 5, 13.

12 Lecture 11 – Continuity and Differentiability

Today:

- continuity and differentiability

Next:

- Power rule
- difference rule
- higher order derivatives

Review:

- Continuity
- differentiability

Teaching procedure:

1. Discuss the continuity and differentiability of $f(x) = |x|$ at 0.
2. Theorem 2.1: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
3. Not differentiable cases:
$$f(x) = \frac{1}{x-1}, \quad f(x) = |x|, \quad f(x) = \sqrt{x}.$$
4. Numerical differentiation: Find the derivative of $\sqrt{x}$ at 1:
$$\begin{array}{rlll} h = & 0.1 & 0.01 & 0.001 \\ \sqrt{1+h} \approx & 1.0488 & 1.00499 & 1.0004999 \\ \sqrt{1} = & 1 & 1 & 1 \\ \frac{\sqrt{1+h}-\sqrt{1}}{h} \approx & 0.488 & 0.499 & 0.4999 \end{array}$$
5. Exercises 2.2: 39.

13 Lecture 12 – Power Rules

Today:

- Power rule
- General rule
- Higher order derivatives
- Acceleration

Next:

- Product rule
- Quotient rule

Review:

- Exponent properties

Teaching procedure:

1. List Power rule and general rules on board.
2. Examples.
   - Normal case: $f(x) = x^{100}$.
   - x is in the bottom: $f(x) = \frac{1}{x^{10}}$.
   - Radicals: $f(x) = \sqrt[3]{x^2}$.
   - Fractions: $f(x) = \frac{4x^2 - x + 3}{\sqrt{x}}$.
3. Higher order derivatives: $f^{(n)}(x) = \frac{d^n f}{dx^n}$.

$$f(x) = x^4 + 3x^2 - 2, \quad f^{(4)}(x).$$

4. Acceleration: $a(t) = s''(t)$.
5. Exercises 2.3: 7, 11, 15, 20, 25

14 Lecture 13 – Product rule and Quotient rule

Today:

- Product rule
- Quotient rule

Next:

- Chain rule

Review:

- Point-slope form of a line: $y = m(x - x_0) + y_0$.
- Exponent properties

Teaching procedure:

1. List Product rule and Quotient rule on board.
2. Examples.
   - Product rule: $f(x) = (\sqrt[3]{x^2} - 2x)\left(\frac{3}{x^2} - 4\right)$.
   - Quotient rule: $f(x) = \frac{x^2-2}{x^2+5x}$.
   - Power rule: $f(x) = x\sqrt[3]{x} + \frac{5}{x^2}$.
3. Tangent line: Find the equation of tangent line to $f(x) = x\sqrt[3]{x} + \frac{5}{x^2}$ at $x = 1$.
4. Exercises 2.3: 1, 5, 11, 19.