MATH 1591 - Final Review

1 Main Topics of Chapter 1

1. Basic rules of limits.
2. How to use the direct substitution to find a limit.
3. How to use the factoring technique to find a limit.
4. How to use the rationalizing technique to find a limit.
5. How to use the Squeeze Theorem to find a limit.
6. How to use the important limit

   $$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin cx}{cx} = \lim_{x\to 0}\frac{\sin(x^n)}{x^n} = 1$$

   to find a limit.
7. How to compute limits of piece-wise defined functions.
8. How to use the one-sided limits to prove that a limit does not exists.
9. The definition of continuity.
10. Properties of continuity.
11. How to find discontinuous points of a function.
12. How to use the intermediate value theorem to prove a function has a zero.
13. How to determine infinite limits.
14. How to find the vertical asymptotes.
15. How to find limits at infinity.
16. how to find the horizontal asymptotes.
17. the $\varepsilon - \delta$ definition of a limit.

2 Main Topics of Chapter 2

1. Definition of derivative of a function?
2. Geometrical meaning of derivative: slope of a tangent line.
3. Tangent line equation.
4. Physical meaning of derivative: velocity.
5. Differentiation rules:
   - Power rule
   - Addition rule
   - Difference rule
   - Product rule
   - Quotient rule
   - Chain rule
6. Two important differentiation techniques:
   - Logarithmic differentiation
   - Implicit differentiation
7. Various derivative formulas

3 Main Topics of Chapter 3

1. Linear approximation of a function
2. L'Höpital's Rule
3. How to find the absolute maximum and minimum
   Follow the following three steps.
   - find critical numbers of a function
   - evaluate the function at critical numbers and the endpoints
   - compare these values
4. How to find a local maximum and minimum
   - Use the First derivative Test
   - Use the Second derivative Test
5. How to find intervals of increase, decrease and concavity?
6. How to find inflection points?
7. How to sketch the graph of a function?
   - find the vertical and horizontal asymptotes;
   - find the intervals of increase and decrease;
   - find the local minimum and maximum;

- find the intervals of concavity and inflection points;
- make a table
- plot critical points like max, min and inflection points and then connect them.

8. How to solve an optimization problem?
   - Draw a diagram;
   - write the primary equation for a quanty Q that is to be maximized or minimized;
   - write the second equation from a given condition;
   - Calculate the derivative of Q and set it equal to zero.
9. How to find a related rate from another given rate?
   - Draw a diagram;
   - Figure out what are given and what is to be found
   - write down an equations relating all of the relevant quantities;
   - differentiate the equations with respect to time t;
   - Solve for the unknown rate.

4 Main Topics of Chapter 4

1. Definitions of antiderivatives, indefinite integrals and definite integrals.
2. Riemann Sum.
3. The integral mean value theorem
4. properties of integrals
5. Fundamental theorem: $\int_a^b f(x)dx = F(b) - F(a)$ and $\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = f(g(x))g'(x)$
6. Calculation of indefinite or definite integrals
   (a) rewriting integrands;
   (b) change of variables $u = g(x)$;
   (c) use trigonometric identities;
   (d) use the long division;
7. how to use definite integrals to find areas

5 Main Topics of Chapter 5

1. Area between two curves
2. Volume of solid of revolution
3. Arc length
4. Surface area of solid revolution
5. Work
6. Center of Mass.

6 Review Exercises

6.1 Evaluation of Limits

1. Use the **direct substitution** to find the following limits.

   (a) $\lim_{x \to 1} (3x^3 - 4x^2 + 3)$.

   (b) $\lim_{x \to 3} \frac{\sqrt{x+1}}{x-4}$.

   (c) $\lim_{x \to 1} \sin\left(\frac{\pi x}{2}\right)$.

   (d) $\lim_{x \to 1} \ln \frac{x}{e^x}$.

2. Use the **dividing out** technique to find the following limits.

   (a) $\lim_{x \to 5} \frac{x-5}{x^2-25}$.

   (b) $\lim_{x \to 3} \frac{x^2-x-6}{x^2-5x+6}$.

   (c) $\lim_{x \to 0} \frac{e^{2x}-1}{e^x-1}$.

   (d) $\lim_{x \to 3} \ln \frac{\sqrt{x+1}-2}{x-3}$.

3. Use the **rationalizing** technique to find the following limits.

   (a) $\lim_{x \to 4} \frac{\sqrt{x+5}-3}{x-4}$.

   (b) $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$.

4. Use the **L'Hôpital's Rule** to find the following limits.

   (a) $\lim_{x \to 0} \frac{1-\cos x}{\sin x}$.

   (b) $\lim_{x \to \infty} \frac{x^2}{e^x}$.

   (c) $\lim_{x \to \infty} \frac{\ln x}{x}$.

   (d) $\lim_{x \to 0} \frac{\sin x}{x^2+3x}$.

   (e) $\lim_{x \to 0^+} (\sin x)^x$.

   (f) $\lim_{x \to \infty} (\sqrt{x^2+1} - x)$.

5. Use the **Squeeze Theorem** to find the following limits.

   (a) $\lim_{x \to 0} x^2 \sin x$.

   (b) $\lim_{x \to 0} x^2 \sin \dfrac{1}{x}$.

6. Determine the infinite limits.

   (a) $\lim_{x \to 1^+} \dfrac{x(1+x)}{1-x}$.

   (b) $\lim_{x \to 2^+} \dfrac{x-3}{x-2}$.

7. Find the vertical asymptotes.

   (a) $f(x) = \frac{1}{x^2}$.

   (b) $f(x) = \frac{2+x}{x^2(1-x)}$.

8. Find limits at infinity.

   (a) $\lim_{x \to \infty} \dfrac{2x+4}{3x^2+1}$.

   (b) $\lim_{x \to \infty} \dfrac{2x^2+1}{3x^2+5}$.

9. Find the horizontal asymptotes.

   (a) $f(x) = \frac{1}{1+e^{-x}}$.

   (b) $f(x) = \frac{3x^2}{x^2+2}$.

10. Use the **one-sided limits** to prove that the limit $\lim_{x \to 0} h(x)$ does not exists, where

$$h(x) = \begin{cases} x^2 + 1 & \text{if } x < 0, \\ x^2 - 1 & \text{if } x \geq 0. \end{cases}$$

11. For the function $f(x) = \frac{x}{x^2 - x}$, find all discontinuous points and all intervals on which it is continuous.

6.2 Calculation of Derivatives

1. Use differentiation rules and formulas to find the derivatives of following functions.

   (a) $y = 5$.
   (b) $y = x^9$.
   (c) $y = \sqrt[6]{x}$.
   (d) $y = 3x^3 + 4x^5 + 7x^{-2}$.
   (e) $y = 5e^x + 6\sin x + 7\cos x$.
   (f) $y = x^3 \cos x$.
   (g) $y = (x^3 + 2)(x^4 - 2x^3 + 1)$.
   (h) $y = \frac{\sin x}{x^2}$.
   (i) $y = \frac{x}{x^2+1}$.

2. Use the **chain rule** to find the derivatives of the following functions.

   (a) $f(x) = (9x + 7)^{2/3}$
   (b) $f(x) = -\frac{8}{(x+3)^3}$
   (c) $f(x) = \left(\frac{3x^2-1}{2x+5}\right)^3$
   (d) $f(x) = \frac{1}{\sqrt{x+2}}$
   (e) $f(x) = \sec(x^3)$
   (f) $f(x) = \sin(\cos x)$
   (g) $f(x) = e^{-3/x^2}$
   (h) $f(x) = \sin^2(2x)$
   (i) $f(x) = (\ln x)^4$
   (j) $f(x) = 5^{-x}$
   (k) $f(x) = \arctan(x^2 - 1$
   (l) $f(x) = \log_5 \sqrt{1 - x^2}$
   (m) $f(x) = x6^{\ln x}$
   (n) $y = 5^{-x/2} \sin(2x) + \frac{10 \log_4 x}{x}$

3. Use the fundamental formula

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

   to find $F'(x)$:

$$(1)\ F(x) = \int_1^x \sqrt[4]{t}\, dt, \quad (2)\ F(x) = \int_0^x \sin t^2\, dt.$$

4. Use the **implicit differentiation** technique to find the derivatives of the following implicit functions.

   (a) $ye^x + xe^y = xy$.

   (b) $x^2 + y^2 = 1$.

5. Use the **logarithmic differentiation** technique to find the derivatives of the following functions.

   (a) $y = \sqrt{(x-1)(x-2)(x-3)}$.

   (b) $y = x^x$.

6. Find the second derivative of the function.

   (a) $y = 4x^{3/2}$.

   (b) $y = x$.

   (c) $y = \sin(2x)$.

6.3 Calculation of Integrals

1. **Rewrite** integrands and then integrate **term by term** to find the integrals.

   (a) $\int_0^1 (4x^3 + 6x^2 - 1)dx$

   (b) $\int \sqrt[3]{x}(x-4)dx$.

   (c) $\int \frac{x^3+x+1}{\sqrt{x}}dx$.

2. Use the method of "**substitution**" to find the following integrals.

   (a) $\int x^2(x^3+5)^4 dx$.

   (b) $\int \sin(3x)dx$.

   (c) $\int_0^{\sqrt{\pi}} x \sin x^2 dx$.

   (d) $\int (x+1)\sqrt{2-x}dx$.

   (e) $\int_1^5 \frac{x}{\sqrt{2x-1}}dx$.

3. Use the **log integration** rule: $\int \frac{1}{x}dx = \ln|x| + C$ to find the integrals.

   (a) $\int \frac{(\ln x)^2}{x}\, dx$.

   (b) $\int \frac{\cos x}{1+\sin x}\, dx$.

4. Use **trigonometric identities** to find the integrals.

   (a) $\int \tan x\ dx$.

   (b) $\int \sec x\ dx$.

5. Use the **long division** method to find the integrals.

   (a) $\int \frac{x^3-6x-20}{x+5}\,dx$.

6. Use the **inverse trigonometric integration** rules to find the integrals.

   (a) $\int \frac{3}{\sqrt{1-4x^2}}\,dx$.

7. **Complete a square** to find the integrals.

   (a) $\int_{-2}^{2} \frac{dx}{x^2+4x+13}$.

   (b) $\int \frac{2dx}{\sqrt{-x^2+4x}}$.

6.4 Applications of Differentiation

1. Find the equation of the tangent line to the graph of $f(x) = \frac{x}{x+1}$ at the point (0,0).

2. Calvin and Hobbes are at their dads office and spot Susie standing on the sidewalk 256 ft below. They decide it would be really funny to drop a water balloon on her head. If the distance the balloon falls in t seconds is given by $s(t) = 16t^2$, what is the balloons velocity when it hits Susie? What is its acceleration? (Hint: In order to hit Susie, the balloon must travel 256 ft).

3. Find the local and absolute extreme values of $f(x) = x - \sqrt{x}$ on [0,4].

4. For $f(x) = x^4 - 4x^3$,

   (i) find the intervals of increase or decrease;

   (ii) find the local maximum and minimum values;

   (iii) find the intervals of concavity and the inflection points;

   (iv) sketch the graph of f.

5. A rectangular page is to contain 36 square inches of print. The margins on each side are to be 1.5 inches. Find the dimensions of the page such that the least amount of paper is used.

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

7. The radius r of a circle is increasing at a rate of 3 centimeters per minute. Find the rate of change of the area when $r = 6$ centimeters.

6.5 Applications of Integration

1. Find the area of the region bounded by the graphs of the equations

$$y = 1 + \sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0.$$

2. Find the area of the region bounded by the graphs of two functions $f(x) = x^2$ and $f(x) = x^3$.

3. Use the Disk Method:

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx$$

or

$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right) dy$$

to find the volumes of the following solids of revolution about a line.

   (a) Find the volume of the solid formed by revolving the region bounded by $y = 4 - x^2, y = 0, x = 0$ about the x-axis.
   (b) Find the volume of the solid formed by revolving the region bounded by $x = -y^2 + 4y, y = 1, x = 0$ about the y-axis.
   (c) Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}, y = 0, x = 4$ about the y-axis.
   (d) Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}, y = 0, x = 4$ about the line $x = 6$.

4. Use the arc length formula:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

or

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy$$

to find the arc length of the graph of the function over the indicated interval.

   (a) $y = \frac{2}{3}x^{3/2} + 1, \quad [0, 1].$
   (b) $y = \ln(\sin x), \quad [\frac{\pi}{4}, \frac{3\pi}{4}].$

5. Use the area formula:

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2} dx$$

or

$$s = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2} dy$$

to find the area of a surface generated by revolving the curve about the x-axis.

(a) $y = \frac{1}{3}x^3, \quad [0,3]$.

(b) $y = \frac{x^3}{6} + \frac{1}{2x}, \quad [1,2]$.

6. Use the formula

$$\bar{x} \quad = \quad \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx}$$

to find the center of mass of an object with density

$$\rho(x) = 3 - \frac{x}{6}, \quad 0 \le x \le 6.$$

6.6 Fundamental Theorems in Calculus

The problems in this section are optional.

1. Use the Intermediate Value Theorem to show that there is a root of the equation

$$x^3 - 10x^2 + 1 = 0$$

in the interval $(0,1)$.

2. Use Rolle's Theorem (on page 206) to prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

3. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a,b]$, the value c guaranteed by the differential Mean Value Theorem (on page 208) is the midpoint of the interval.

4. Let $G(x) = \int_0^x \left[s \int_0^s f(t)dt\right] ds$, where f is continuous for all real t. Find

(a) $G(0)$.

(b) $G'(0)$.

(c) $G''(0)$.

(d) $G'''(0)$.