Problem Sheet 3

Evaluate the improper integral

$$\begin{aligned}\int_1^\infty \frac{1}{x\ln x}\,dx &= \int_1^2 \frac{1}{x\ln x}\,dx + \int_2^\infty \frac{1}{x\ln x}\,dx\\ &= \lim_{a\to 1^+} \ln|\ln x||_a^2 + \lim_{b\to\infty} \ln(\ln x)|_2^b\\ &= \ln(\ln 2) - \lim_{a\to 1^+} \ln|\ln a| + \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2).\end{aligned}$$

So the integral diverges since both $\int_1^2 \frac{1}{x\ln x}\,dx$ and $\int_2^\infty \frac{1}{x\ln x}\,dx$ diverge.

Evaluate the following limits:

1. $\lim_{n\to\infty}(1+(-1)^n)$. The sequence alternates between 0 and 1. So it does not exist.

2. $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = \frac{1}{1+\lim_{n\to\infty}\frac{1}{\sqrt{n}}} = 1$

3. $\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

4. $\lim_{n\to\infty} \frac{1+(-1)^n}{n^2}$

$$0 \le |\frac{1+(-1)^n}{n^2}| \le \frac{2}{n^2}.$$

Since $\lim_{n\to\infty} \frac{2}{n^2} = 0$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} |\frac{1+(-1)^n}{n^2}|.$$

It then follows from the Absolute Value Theorem that

$$\lim_{n\to\infty} \frac{1+(-1)^n}{n^2} = 0.$$

5. $\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{1}{n(n-1)} = 0.$