Solutions of Problem 4

Determine the convergence of the following series. If convergent and possible, find their sums:

1. \( \sum_{n=0}^{\infty} 2(-1.03)^n \).
   This is a geometric series. It diverges since \(-1.03 < -1\).

2. \( \sum_{n=0}^{\infty} 5 \left( -\frac{1}{3} \right)^n \).
   This is a geometric series. It converges and its sum is
   \[ \sum_{n=0}^{\infty} 5 \left( -\frac{1}{3} \right)^n = \frac{5}{1 + \frac{1}{3}} = \frac{15}{4}. \]

3. \( \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \).
   This is a telescoping series.
   \[ \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \]
   \[ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{1}{2}. \]

4. \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}} \).
   Since
   \[ \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{1 + \frac{1}{n^2}}}}{\frac{1}{n^2}} = \frac{1}{\sqrt{1 + \lim_{n \to \infty} \frac{1}{n^2}}} = 1. \]
   By the n-th term test, it diverges.

5. \( \sum_{n=1}^{\infty} \frac{1}{n} \).
   This is the harmonic series and it diverges.

6. \( \sum_{n=1}^{\infty} \frac{1}{n^{1.00001}} \).
   This is \( p \)-series. It converges since \( p = 1.00001 > 1 \).

7. \( \ln \frac{2}{2} + \ln \frac{3}{3} + \ln \frac{4}{4} + \ln \frac{5}{5} + \ln \frac{6}{6} + \cdot \).
   Since
   \[ \int_{2}^{\infty} \frac{\ln x}{x} \, dx = \int_{2}^{\infty} \ln x \, dx \ln x = \frac{1}{2} (\ln x)^2 \bigg|_{2}^{\infty} = \infty, \]
   by the integral test, the series diverges.