Solutions of Problem 5

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n - 1}$.

   Direct Comparison Test. Since

   $$\frac{4^n}{3^n - 1} \geq \frac{4^n}{3^n}$$

   and the geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n}$ diverges, $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^n - 1}$ **diverges**.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3n^2 - 2n - 15}$.

   Limit Comparison Test. Since

   $$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{3n^2-2n-15}} = \lim_{n\to\infty} \frac{3n^2 - 2n - 15}{n^2} = 3$$

   and the p-series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3n^2 - 2n - 15}$ **converges.**

3. $\displaystyle\sum_{n=1}^{\infty} \tan \frac{1}{n}$

   Limit Comparison Test. Since

   $$\lim_{n\to\infty} \frac{\frac{1}{n}}{\tan \frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\sin \frac{1}{n}} \cos \frac{1}{n} = 1$$

   and the p-series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=0}^{\infty} \tan \frac{1}{n}$ **diverges.**

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

   Ratio Test. Since

   $$\lim_{n\to\infty} \frac{\frac{(2(n+1))!}{(n+1)^5}}{\frac{(2n)!}{n^5}} = \lim_{n\to\infty} \frac{(2(n+1))!}{(2n)!} \frac{n^5}{(n+1)^5} = \lim_{n\to\infty} (2n+1)(2n+2) \frac{n^5}{(n+1)^5} = \infty,$$

   $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$ **diverges**.

5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{3n+1}\right)^{3n}$

Root Test. Since

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{-2n}{3n+1}\right|^{3n}}=\lim_{n\to\infty}\left(\frac{2n}{3n+1}\right)^3=\left(\frac{2}{3}\right)^3<1,$$

$\sum_{n=1}^{\infty}\left(\frac{-2n}{3n+1}\right)^{3n}$ **diverges**.

Determine whether the series converges conditionally or absoluately,or diverges.

1. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\sqrt{n}}$

   Since $\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{1.5}}$ converges (p-series), $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\sqrt{n}}$ absolutely converges.

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+4}}$

   This is an **alternating series**. Since

$$\lim_{n\to\infty}\frac{1}{\sqrt{n+4}}=0$$

   and

$$\frac{1}{\sqrt{n+4}}\le\frac{1}{\sqrt{n+1+4}},$$

   $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+4}}$ converges. But $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{\sqrt{n+4}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+4}}$ diverges.
   So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+4}}$ converges conditionally.