Solutions of Problem 6

Problem 1. Find the radius and interval of convergence of the following power series:

1. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$.

   Geometric Series. By the geometric series, the series converges if $\frac{|x|}{2} < 1$ and diverges if $\frac{|x|}{2} \geq 1$. From $\frac{|x|}{2} < 1$,we get $|x| < 2$. So the radius is $R = 2$ and interval of convergence is $(-2, 2)$.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   By the ratio test. Let $u_n = \frac{x^n}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0.$$

   By the ratio test, the series converges for any x. So the radius is $R = \infty$ and interval of convergence is $(-\infty, \infty)$.

3. $\sum_{n=0}^{\infty} n! \left(\frac{x}{2}\right)^n$

   By the ratio test. Let $u_n = n! \left(\frac{x}{2}\right)^n$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)! \left(\frac{x}{2}\right)^{n+1}}{n! \left(\frac{x}{2}\right)^n}\right| = \lim_{n\to\infty} \left|(n+1)\frac{x}{2}\right| = \infty$$

   for any $x \neq 0$. By the ratio test, the series diverges for any $x \neq 0$. So the radius is $R = 0$ and interval of convergence is $\{0\}$.

4. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$

   By Root test. Let $u_{n+1} = \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$. Then

$$\lim_{n\to\infty} \sqrt[n+1]{|u_{n+1}|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^{n+1}}{(n+1)4^{n+1}}\right|} = \lim_{n\to\infty} \frac{|x-2|}{4\sqrt[n+1]{n+1}} = \frac{|x-2|}{4}.$$

   By the root test, the series converges if $\frac{|x-2|}{4} < 1$ and diverges if $\frac{|x-2|}{4} > 1$. From $\frac{|x-2|}{4} < 1$, we get $|x-2| < 4$. So the radius is $R = 4$ and it converges on the interval $(-2, 6)$. To determine the interval of convergence, we need to check the endpoints. At $x = -2$, we have

$$\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-2-2)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}.$$

   By the alternating series test, it converges. At $x = 6$, we have

$$\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(6-2)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)}.$$

   It is harmonic series and divergent. So the interval of convergence is $[-2, 6)$.

Problem 2. Find a power series for the function:

1. $f(x) = \frac{1}{2-x}, \quad c = 5.$
   By geometric series.

$$\begin{aligned} f(x) &= \frac{1}{2-x} = \frac{1}{-3-(x-5)} = -\frac{1}{3}\frac{1}{1+\frac{(x-5)}{3}} \\ &= -\frac{1}{3}\left(1 - \frac{(x-5)}{3} + \left(\frac{(x-5)}{3}\right)^2 + \cdots + + \left(-\frac{(x-5)}{3}\right)^n + \cdots\right). \end{aligned}$$

2. $f(x) = \ln(1+x), \quad c = 0.$
   Using the geometric series, we get

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots$$

   By integration, we get

$$\ln(1+x) + C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + \cdots$$

   Set $x = 0$, we get $C = 0$. So

$$\ln(1+x) + C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + \cdots$$

3. $f(x) = \frac{x}{x^2-1}, \quad c = 0.$
   By the partial fractions, we have

$$\begin{aligned} f(x) &= \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} \\ &= \frac{1}{2}(1 - x + x^2 + \cdots + (-1)^n x^n + \cdots) \\ &\quad -\frac{1}{2}(1 + x + x^2 + \cdots + x^n + \cdots) \\ &= -x - \cdots - x^{2n+1} - \cdots \end{aligned}$$

4. $f(x) = \sin x^2, \quad c = 0.$
   By direct substitution.

$$\sin x^2 = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{4n+2} + \cdots$$

5. $f(x) = \cos x, \quad c = \pi/4.$
   By Taylor series.

$$\begin{aligned} f(x) &= \cos x \quad f(\pi/4) = \frac{\sqrt{2}}{2} \\ f'(x) &= -\sin x \quad f'(\pi/4) = -\frac{\sqrt{2}}{2} \\ f''(x) &= -\cos x \quad f''(\pi/4) = -\frac{\sqrt{2}}{2} \\ f'''(x) &= \sin x \quad f'''(\pi/4) = \frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \cos x \quad f^{(4)}(\pi/4) = \frac{\sqrt{2}}{2} \end{aligned}$$

   and so on. Therefore

$$\cos x = \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 - \cdots\right]$$