Problem 1. (10 points, 5 each) Find the radius and interval of convergence of the following power series:

1. \[ \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^n \]

Geometric Series. By the geometric series, the series converges if \( |x| < 2 \) and diverges if \( |x| \geq 2 \). From \( |x| < 2 \), we get \( |x| < 2 \). So the radius is \( R = 2 \) and interval of convergence is \((-2, 2)\).

2. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n} \]

By Root test. Let \( u_n = \frac{(-1)^{n+1}(x-5)^n}{n5^n} \). Then

\[
\lim_{n \to \infty} \sqrt[n]{|u_n|} = \lim_{n \to \infty} \sqrt[n]{\left| \frac{(-1)^{n+1}(x-5)^n}{n5^n} \right|} = \lim_{n \to \infty} \frac{|x-5|}{5\sqrt[n]{n}} = \frac{|x-5|}{5}.
\]

By the root test, the series converges if \( \frac{|x-5|}{5} < 1 \) and diverges if \( \frac{|x-5|}{5} > 1 \). From \( \frac{|x-5|}{5} < 1 \), we get \( |x-5| < 5 \). So the radius is \( R = 5 \) and it converges on the interval \((0, 10)\). To determine the interval of convergence, we need to check the endpoints. At \( x = 0 \), we have

\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(0-5)^n}{n5^n} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n} = -\sum_{n=0}^{\infty} \frac{1}{n}.
\]

By the p-series test, it diverges. At \( x = 6 \), we have

\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(10-5)^n+1}{n5^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}.
\]

It is an alternating series and convergent. So the interval of convergence is \((0, 10]\).

Problem 2. (10 points, 5 each) Find a power series for the function:

1. \( f(x) = \frac{1}{2-x}, \quad c = 5. \)

By geometric series.

\[
f(x) = \frac{1}{2-x} = \frac{1}{-3-(x-5)} = -\frac{1}{3} \left( \frac{1}{3} \right) \left( 1- \frac{(x-5)}{3} \right) + \left( \frac{(x-5)}{3} \right)^2 + \cdots + \left( \frac{(x-5)}{3} \right)^n + \cdots.
\]
2. \( f(x) = \ln(1 + x), \quad c = 0. \)

Using the geometric series, we get
\[
\frac{1}{1 + x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots
\]

By integration, we get
\[
\ln(1 + x) + C = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots + \frac{(-1)^n}{n + 1} x^{n + 1} + \cdots
\]

Set \( x = 0 \), we get \( C = 0 \). So
\[
\ln(1 + x) + C = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots + \frac{(-1)^n}{n + 1} x^{n + 1} + \cdots
\]