Math 1592 Solutions of Quiz 5

Problem 1. (10 points, 5 each) Find the radius and interval of convergence of the following power series:

1. $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$.

   Geometric Series. By the geometric series, the series converges if $\frac{|x|}{2} < 1$ and diverges if $\frac{|x|}{2} \geq 1$. From $\frac{|x|}{2} < 1$,we get $|x| < 2$. So the radius is $R = 2$ and interval of convergence is $(-2, 2)$.

2. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n5^n}$

   By Root test. Let $u_n = \frac{(-1)^{n+1}(x-5)^n}{n5^n}$. Then

$$\lim_{n\to\infty}\sqrt[n]{|u_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^{n+1}(x-5)^n}{n5^n}\right|} = \lim_{n\to\infty}\frac{|x-5|}{5\sqrt[n]{n}} = \frac{|x-5|}{5}.$$

   By the root test, the series converges if $\frac{|x-5|}{5} < 1$ and diverges if $\frac{|x-5|}{5} > 1$. From $\frac{|x-5|}{5} < 1$, we get $|x-5| < 5$. So the radius is $R = 5$ and it converges on the interval $(0, 10)$. To determine the interval of convergence, we need to check the endpoints. At $x = 0$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n5^n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(0-5)^n}{n5^n} = \sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{n} = -\sum_{n=0}^{\infty}\frac{1}{n}.$$

   By the p-series test, it diverges. At $x = 6$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n5^n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(10-5)^{n+1}}{n5^n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}.$$

   It is an alternating series and convergent. So the interval of convergence is $(0, 10]$.

Problem 2. (10 points, 5 each) Find a power series for the function:

1. $f(x) = \frac{1}{2-x}, \quad c = 5.$

   By geometric series.

$$\begin{aligned} f(x) &= \frac{1}{2-x} = \frac{1}{-3-(x-5)} = -\frac{1}{3}\frac{1}{1+\frac{(x-5)}{3}} \\ &= -\frac{1}{3}\left(1 - \frac{(x-5)}{3} + \left(\frac{(x-5)}{3}\right)^2 + \cdots + + \left(-\frac{(x-5)}{3}\right)^n + \cdots\right). \end{aligned}$$

2. $f(x) = \ln(1 + x), \quad c = 0.$

Using the geometric series, we get

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots$$

By integration, we get

$$\ln(1+x) + C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + \cdots$$

Set $x = 0$, we get $C = 0$. So

$$\ln(1+x) + C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + \cdots$$