4. Linear Algebra

Vectors and Matrices
Maple has this wonderful linear algebra package. It has to do with vectors and matrices. A vector is simply an array of numbers written as

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

(1)

where a matrix is written as a collection of arrays, i.e.

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

(2)

In (1), we have a single column of 3 numbers, whereas in (5) we have two rows of 3 numbers in each row. In general we could have \( m \) rows and \( n \) columns in which we would have

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\]

(3)

where \( a_{mn} \) is the element in the \( m^{th} \) row and \( n^{th} \) column.

In maple, it is first necessary to call in the linear algebra package. This is done by the following maple command

\[
> \text{with(LinearAlgebra)} : \text{with(linalg)} : 
\]

Next, since we will be entering both vectors and matrices, we will use the palettes in Maple. This is located in the view menu under \textit{palettes} \( \rightarrow \) \textit{Matrix palette}. This should give us a little 4X4 box with lots of dots. Here we will input the following vectors

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad \begin{bmatrix}
4 \\
5 \\
2
\end{bmatrix}
\]

(4)
This is done by

\[
> u := \langle\langle \, \%? \, \%? \, \%? \, \%? \, \rangle; \\
> v := \langle\langle \, \%? \, \%? \, \%? \, \%? \, \rangle; \\
\]

noting that after we type \( A := \), we used the first entry on the third line of the palette. We then replace each of the \( \%? \) with numbers. For example,

\[
> u := \langle\langle 1, 2, 3 \, \rangle; \\
> v := \langle\langle 4, 5, 6 \, \rangle; \\
\]

would produce

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
\]

For matrices, we again use the palette. So, for example if we want a \( 2 \times 3 \) matrix as found in (5) we would type

\[
> A := \langle\langle \, \%? \| ? \| ? \, \rangle; \\
\]

and then replace the numbers accordingly

\[
> B := \langle\langle 1 \| 2 \| 3 \, \rangle; \\
\]

in which maple would return

\[
\begin{bmatrix} 
1 & 2 & 3 \\
4 & 5 & 6 
\end{bmatrix}
\]

**Vector Addition**

To add vectors or matrices we use the \texttt{matadd} command. So if we consider the vectors \( u \) and \( v \) above, then

\[
> \text{matadd}(u, v, a, b); \\
\]

would produce

\[
\begin{bmatrix}
a + 4b \\
2a + 5b \\
3a + 6b
\end{bmatrix},
\]

where \( a \) and \( b \) are numbers. If we want \( a = 1 \) and \( b = 1 \), we just simply delete the numbers and type
Matrix Addition

\[ A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B := \begin{bmatrix} -1 & 5 & 2 \\ 4 & -3 & 9 \end{bmatrix} \]

\[ \text{matadd}(A, B, a, b) \]

would produce

\[ \begin{bmatrix} a - b & 2a + 5b & 3a + 2b \\ 4a + 4b & 5a - 3b & 6a + 9b \end{bmatrix}, \]

again, where \( a \) and \( b \) are numbers and can be omitted if \( a \) and \( b \) are both 1.

Scalar Multiplication

In order to multiply a vector or matrix by a number we use scalar multiplication. For example, to multiply the vector

\[ u := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \]

we use the command

\[ \text{scalarmul}(u, 2) \]

which would produce

\[ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}. \]

Matrix Multiplication

In order to multiply two matrices, it is important that the size of the matrices be of a certain dimension. For example, if

\[ A_{mn} B_{kl}, \]

then we require that \( n = k \) and the dimension of the new matrix is \( m \times l \). Consider the \( 2 \times 2 \) matrix and the \( 2 \times 3 \) matrix

\[ A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \]
As $A_{22}$ and $B_{23}$ we can multiply $AB$ but not $BA$. Thus, the command

```
> multiply(A, B);
```

would produce

```
[ 5  11  17 ]
[  11  25  39 ]
```

**Miscellaneous Operations**

**Determinants**

To calculate the determinant of an $n \times n$ matrix we use the command

```
> det(A);
```

So, for the $A$ given above, the answer returned would be $-2$.

**Transpose**

This command switches the rows and columns, so executing the command

```
> transpose(B);
```

would return

```
[ 1  2 ]
[  3  4 ]
[  5  6 ]
```

**Inverse**

Given an $n \times n$ matrix, if a second matrix $B$ exists such that $AB = I$, the identity, the the matrix $B$ is called the inverse. Often this is denoted by $A^{-1}$. If the inverse exists, then $det(A) \neq 0$. The maple command is

```
> inverse(A);
```

So, for the matrix $A$ above, we would obtain

```
[ -2  1 ]
[  3 -\frac{1}{2} ]
```

**Augment**

Often we wish to augment one matrix with another. The maple command is command

```
> << A | B >>;
```
So, for the matrices $A$ and $B$ above, we would obtain
\[
\begin{bmatrix}
1 & 2 & 1 & 3 & 5 \\
3 & 4 & 2 & 4 & 6
\end{bmatrix},
\]

**Row Operations**

Finally, it is possible to have maple perform row operations. Here we will consider three basic operations

1. interchange rows

2. multiple a row by an algebraic expression

3. add a multiple of one row to another

1. $R_i \leftrightarrow R_j$

The command
\[
\text{RowOperation}(A, [i,j]);
\]
would exchange rows $i$ and $j$ in the matrix $A$

2. $cR_i \rightarrow R_i$

The command
\[
\text{RowOperation}(A, i, c);
\]
would multiple row $i$ by $c$

3. $R_i + cR_j \rightarrow R_i$

The command
\[
\text{RowOperation}(A, [i,j], c);
\]
would add row $i$ to $c \times$ row $j$ and put the result in row $i$. We consider the following example, where we will use row reduction to manipulate the matrix $A$ augment with the matrix $B$.

\[
A := \begin{bmatrix}
1 & 2 & -1 \\
2 & -1 & 3 \\
-3 & 1 & 1
\end{bmatrix}, \quad B := \begin{bmatrix}
2 \\
9 \\
2
\end{bmatrix}
\]

We consider the following commands
\[
M := << A|B >>;
\]
\begin{verbatim}
> M1 := RowOperation(M, [2, 1], -2);
> M2 := RowOperation(M1, [3, 1], 3);
> M3 := RowOperation(M2, 2, -1/5);
> M4 := RowOperation(M3, [3, 2], -7);
> M5 := RowOperation(M4, 3, 1/5);
> M6 := RowOperation(M5, [2, 3], 1);
> M7 := RowOperation(M6, [1, 3], 1);
> M8 := RowOperation(M7, [1, 2], -2);

which at the end we should end with

\[ A := \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}, \]
\end{verbatim}
Problems

1. Given the matrices

\[
A := \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 3 & -2 \end{bmatrix}
\]

Find (i) \( A + 3B \), (ii) \(-2A + 4B\)

2. Given the matrices

\[
A := \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 2 \\ -4 & 5 \\ -2 & -2 \end{bmatrix}
\]

Find (i) \( AB \), (ii) \( BA \)

3. Given the matrix \( A \) and the vectors \( u \) and \( v \)

\[
A := \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \quad u := \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v := \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

Show (i) \( Au = 2u \), (ii) \( Av = 3v \)

4. Solve the system of equations

\[
\begin{align*}
x + y + z &= 12, \\
2x + y + 4z &= 32, \\
3x - y + 2z &= 14,
\end{align*}
\]

by first augmenting the matrices \( A \) and \( B \)

\[
A := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}, \quad B := \begin{bmatrix} 12 \\ 32 \\ 14 \end{bmatrix}
\]

and row reducing.