4. Linear Algebra

Vectors and Matrices

Maple has this wonderful linear algebra package. It has to do with vectors and matrices. A vector is simply an array of numbers written as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{1}$$

where a matrix is written as a collection of arrays, i.e.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{2}$$

In (1), we have a single column of 3 numbers, whereas in (5) we have two rows of 3 numbers in each row. In general we could have m rows and n columns in which we would have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{3}$$

where a_{mn} is the element in the m^{th} row and n^{th} column.

In maple, it is first necessary to call in the linear algebra package. This is done by the following maple command

$[> \; with(LinearAlgebra) : \; with(linalg) :$

Next, since we will be entering both vectors and matrices, we will use the palettes in Maple. This is located in the view menu under *palettes* → *Matrix palette*. This should give us a little $4X4$ box with lots of dots. Here we will input the following vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \tag{4}$$

This is done by

$[> \ u := << \ \%? \ \%? \ \%? \ >>;$

$[> \ v := << \ \%? \ \%? \ \%? \ >>;$

noting that after we type $A :=$, we used the first entry on the third line of the palette. We then replace each of the %? with numbers. For example,

$[> \ u := << \ 1, \ 2, \ 3, \ >>;$

$[> \ v := << \ 4, \ 5, \ 6, \ >>;$

would produce

$$u := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v := \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

For matrices, we again use the palette. So, for example if we want a 2×3 matrix as found in (5) we would type

$[> \ A := << \ \%?| \ \%?| \ \%? >, < \ \%?| \ \%?| \ \%? >>;$

and then replace the numbers accordingly

$[> \ B := << \ 1| \ 2| \ 3 >, < \ 4| \ 5| \ 6 >>;$

in which maple would return

$$u \ := \ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Vector Addition

To add vectors or matrices we use the **matadd** command. So if we consider the vectors u and v above, then

$[> \ matadd(u, v, a, b);$

would produce

$$\begin{bmatrix} a + 4b \\ 2a + 5b \\ 3a + 6b \end{bmatrix},$$

where a and b are numbers. If we want $a = 1$ and $b = 1$, we just simply delete the numbers and type

[> $matadd(u, v)$;

Matrix Addition

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B := \begin{bmatrix} -1 & 5 & 2 \\ 4 & -3 & 9 \end{bmatrix}$$

[> $matadd(A, B, a, b)$;

would produce

$$\begin{bmatrix} a-b & 2a+5b & 3a+2b \\ 4a+4b & 5a-3b & 6a+9b \end{bmatrix},$$

again, where a and b are numbers and can be omitted it a and b are both 1.

Scalar Multiplication

In order to multiply a vector or matrix by a number we use scalar multiplication. For example, to multiply the vector

$$u := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

we use the command

[> $scalarmul(u, 2)$;

which would produce

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

Matrix Multiplication

In order to multiply two matrices, it is important that the size of the matrices be of a certain dimension. For example, if

$$A_{mn}\ B_{kl},$$

then we require that $n = k$ and the dimension of the new matrix is $m \times l$. Consider the 2×2 matrix and the 2×3 matrix

$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

As A_{22} and B_{23} we can multiply AB but not BA. Thus, the command

[> $multiply(A, B);$

would produce

$$\begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{bmatrix}$$

Miscellaneous Operations

Determinants

To calculate the determinant of an $n \times n$ matrix we use the command

[> $det(A);$

So, for the A given above, the answer returned would be -2.

Transpose

This command switches the rows and columns, so executing the command

[> $transpose(B);$

would return

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Inverse

Given an $n \times n$ matrix, if a second matrix B exists such that $AB = I$, the identity, the the matrix B is called the inverse. Often this is denoted by A^{-1}. If the inverse exists, then $det(A) \neq 0$. The maple command is

[> $inverse(A);$

So, for the matrix A above, we would obtain

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}.$$

Augment

Often we wish to augment one matrix with another. The maple command is command

[> $<< A|B >>;$

So, for the matrices A and B above, we would obtain

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 3 & 4 & 2 & 4 & 6 \end{bmatrix},$$

Row Operations

Finally,it is possible to have maple perform row operations. Here we will consider three basic operations

1. interchange rows
2. multiple a row by an algebraic expression
3. add a multiple of one row to another

1. $\mathbf{R_i \leftrightarrow R_j}$

The command

$[> \; RowOperation(A,[i,j]);$

would exchange rows i and j in the matrix A

2. $\mathbf{cR_i \rightarrow R_i}$

The command

$[> \; RowOperation(A,i,c);$

would multiple row i by c

3. $\mathbf{R_i + cR_j \rightarrow R_i}$

The command

$[> \; RowOperation(A,[i,j],c);$

would add row i to $c \times$ row j and put the result in row i. We consider the following example, where we will use row reduction to manipulate the matrix A augment with the matrix B.

$$A \; := \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -3 & 1 & 1 \end{bmatrix}, \quad B \; := \begin{bmatrix} 2 \\ 9 \\ 2 \end{bmatrix}$$

We consider the following commands

$[> \; M := << A|B >>;$

[> $M1 := RowOperation(M, [2, 1], -2);$

[> $M2 := RowOperation(M1, [3, 1], 3);$

[> $M3 := RowOperation(M2, 2, -1/5);$

[> $M4 := RowOperation(M3, [3, 2], -7);$

[> $M5 := RowOperation(M4, 3, 1/5);$

[> $M6 := RowOperation(M5, [2, 3], 1);$

[> $M7 := RowOperation(M6, [1, 3], 1);$

[> $M8 := RowOperation(M7, [1, 2], -2);$

which at the end we should end with

$$A \ := \ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

Problems

1. Given the matrices

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 3 & -2 \end{bmatrix}$$

Find (i) $A + 3B$, (ii) $-2A + 4B$

2. Given the matrices

$$A := \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 2 \\ -4 & 5 \\ -2 & -2 \end{bmatrix}$$

Find (i) AB, (ii) BA

3. Given the matrix A and the vectors u and v

$$A := \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, u := \begin{bmatrix} 2 \\ -1 \end{bmatrix}, v := \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Show (i) $Au = 2u$, (ii) $Av = 3v$

4. Solve the system of equations

$$\begin{aligned} x + y + z &= 12, \\ 2x + y + 4z &= 32, \\ 3x - y + 2z &= 14, \end{aligned}$$

by first augmenting the matrices A and B

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}, B := \begin{bmatrix} 12 \\ 32 \\ 14 \end{bmatrix},$$

and row reducing.