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Existence of positive periodic solutions to nonlinear second order differential equations

Fuyi Li[a,*], Zhanping Liang[a,b]

[a] *Department of Mathematics, Shanxi University, Taiyuan 030006, People's Republic of China*
[b] *Department of Mathematics, Xinzhou Teachers' University, Xinzhou 034000, Shanxi, People's Republic of China*

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Abstract

In this paper, we discuss the existence of positive periodic solutions to the nonlinear differential equation

$$u''(t) + a(t)u(t) = f(t, u(t)), \quad t \in R,$$

where $a : R \to [0, +\infty)$ is an ω-periodic continuous function with $a(t) \not\equiv 0$, $f : R \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(\cdot, u) : R \to [0, +\infty)$ is also an ω-periodic function for each $u \in [0, +\infty)$. Using the fixed point index theory in a cone, we get an essential existence result because of its involving the first positive eigenvalue of the linear equation with regard to the above equation.

Keywords: Positive periodic solution; Fixed point index; Krein–Rutman theorem; The first positive eigenvalue

1. Introduction

In this paper, we are concerned with the existence of positive periodic solutions to the nonlinear differential equation

$$u''(t) + a(t)u(t) = f(t, u(t)), \quad t \in R, \tag{1.1}$$

where we assume that

* Corresponding author.
E-mail address: fyli@sxu.edu.cn (F. Li).

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(H_1) $a : R \to [0, +\infty)$ is an ω-periodic continuous function and $a(t) \not\equiv 0$;
(H_2) $f : R \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(\cdot, u) : R \to [0, +\infty)$ is also an ω-periodic function for each $u \in [0, +\infty)$.

A function u is said to be a positive ω-periodic solution to Eq. (1.1) if and only if

(i) $u \in C^2(R)$, $u(t + \omega) = u(t)$, $u(t) \geq 0$ for all $t \in R$ and $u(t) \not\equiv 0$;
(ii) $u''(t) + a(t)u(t) = f(t, u(t))$, $t \in R$.

It is easy to verify that the positive ω-periodic solution to Eq. (1.1) is equivalent to the positive solution to the following periodic boundary value problems (PBVPs):

$$u''(t) + a(t)u(t) = f(t, u(t)), \quad t \in [0, \omega], \tag{1.2}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega). \tag{1.3}$$

Recently, the fixed point index theory has successfully been used to deal with the existence of positive solutions to two-point boundary value problems, and a great number of satisfactory results have been attained; see [1–4]. Now some authors are trying to apply the fixed point index theory to periodic boundary value problems; see [5,6]. Motivated by the method of [5], we study the existence of positive solutions to periodic boundary value problems (1.2) and (1.3) by using the theory and obtain some further results. These results are essential because they involve the first positive eigenvalue of the linear problem corresponding to the PBVPs (1.2) and (1.3). We also improve the conditions concerning a and M given in [5]. Function a can have a zero point and M can equal $(\pi/\omega)^2$, while in [5] it is assumed that $a(t) > 0$ for all $t \in [0, \omega]$ and $0 < M < (\pi/\omega)^2$.

2. Preliminaries

In this section, we will give some lemmas which are very important for proving the main result of this paper. We assume throughout that a satisfies (H_1) and f satisfies (H_2). Let $M = \max_{t\in[0,\omega]} a(t)$. $C[0, \omega]$ denotes the usual continuous function space with norm $\|u\| = \max_{t\in[0,\omega]} |u(t)|$ for all $u \in C[0, \omega]$ and $C^+[0, \omega] = \{u \in C[0, \omega] : u(t) \geq 0, t \in [0, \omega]\}$ denotes the cone [7–9] in $C[0, \omega]$. We define a partial ordering $\leq$ with respect to $C^+[0, \omega]$ by $u \leq v$ iff $v - u \in C^+[0, \omega]$. Sometimes we shall write $u < v$ to indicate that $u \leq v$ but $u \neq v$.

Lemma 1. *If $0 < M \leq (\pi/\omega)^2$, then for each $h \in C[0, \omega]$, there exists a unique solution u satisfying the linear periodic boundary value problems*

$$u''(t) + Mu(t) = h(t), \quad t \in [0, \omega], \tag{2.1}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega). \tag{2.2}$$

The solution u is given by $u := Th$, where T satisfies

(i) $T : C[0, \omega] \to C[0, \omega]$ is a linear completely continuous operator and $\|T\| = 1/M$;
(ii) T is a positive linear operator, i.e. $Th \in C^+[0, \omega]$ for all $h \in C^+[0, \omega]$;
(iii) T is also a strongly positive linear operator, i.e. $(Th)(t) > 0$, $t \in [0, \omega]$ for all $h \in C^+[0, \omega]$ with $h(t) \not\equiv 0$.

Proof. Let

$$r(t) = \alpha_0 \cos \beta(t - \omega/2), \quad t \in [0, \omega],$$

where $\beta = \sqrt{M}$, $\alpha_0 = \frac{1}{2\beta \sin \beta\omega/2}$. It is obvious that $r \in C^2[0,\omega]$, $r(t) \geq 0$, $t \in [0,\omega]$ and

$$r''(t) + Mr(t) = 0, \quad r(0) = r(\omega), \quad r'(0) = r'(\omega) + 1. \tag{2.3}$$

Let $G : [0,\omega] \times [0,\omega] \to [0,+\infty)$ be as follows:

$$G(t,s) = \begin{cases} r(t-s), & 0 \leq s \leq t \leq \omega, \\ r(\omega + t - s), & 0 \leq t \leq s \leq \omega. \end{cases}$$

Then from (2.3) and the maximum principle we can easily show that

$$u(t) := (Th)(t) = \int_0^\omega G(t,s)h(s)\,\mathrm{d}s = \int_0^t r(t-s)h(s)\,\mathrm{d}s + \int_t^\omega r(\omega + t - s)h(s)\,\mathrm{d}s \tag{2.4}$$

is the unique solution to the problems (2.1) and (2.2). Since G is continuous, we can see that $T : C[0,\omega] \to C[0,\omega]$ is a linear completely continuous operator. Obviously, $1/M$ is a solution of the problems (2.1) and (2.2) for $h(t) \equiv 1$, so we get

$$\int_0^\omega G(t,s)\,\mathrm{d}s = \frac{1}{M}, \quad t \in [0,\omega]. \tag{2.5}$$

From (2.5) for all $h \in C[0,\omega]$, we then have that $|(Th)(t)| \leq \int_0^\omega G(t,s)|h(s)|\,\mathrm{d}s \leq \frac{1}{M}\|h\|$, $t \in [0,\omega]$, that is $\|Th\| \leq \frac{1}{M}\|h\|$, so $\|T\| \leq 1/M$. On the other hand, for $h_0 \equiv 1$, $(Th_0)(t) = \int_0^\omega G(t,s)\,\mathrm{d}s = 1/M$, $t \in [0,\omega]$, so $\|Th_0\| = 1/M$. Thus $\|T\| = 1/M$. By the fact that G is nonnegative and (2.4), it is easy to see that $Th \in C^+[0,\omega]$ for all $h \in C^+[0,\omega]$.

For arbitrary given $h \in C^+[0,\omega]$ with $h(t) \not\equiv 0$, now we prove that $(Th)(t) = \int_0^\omega G(t,s)h(s)\,\mathrm{d}s > 0$, $t \in [0,\omega]$. We only need to prove that for each $t \in [0,\omega]$, $G(t,s)h(s) \not\equiv 0$ in $s \in [0,\omega]$. Since $h(s) \not\equiv 0$, there exist $s_0 \in (0,\omega)$ and $\delta > 0$ such that $h(s) > 0$ as $s \in [s_0 - \delta, s_0 + \delta] \subset [0,\omega]$. When $t \in [0,s_0)$, $G(t,s_0)h(s_0) = r(\omega + t - s_0)h(s_0) = \alpha_0 \cos\beta(t - s_0 + \omega/2)h(s_0)$, while $-\omega/2 < \omega/2 - s_0 \leq t - s_0 + \omega/2 < \omega/2$; this implies that $\beta(t - s_0 + \omega/2) \in (-\pi/2, \pi/2)$, so $G(t,s_0)h(s_0) > 0$. Similarly, when $t \in (s_0,\omega]$, $G(t,s_0)h(s_0) = r(t-s_0)h(s_0) = \alpha_0 \cos\beta(t - s_0 - \omega/2)h(s_0) > 0$. When $t = s_0$, $G(t, s_0+\delta)h(s_0+\delta) = G(s_0, s_0+\delta)h(s_0+\delta) = \alpha_0 \cos\beta(\omega/2 - \delta)h(s_0+\delta) > 0$. The proof is completed.

Lemma 2. *Let $(Bh)(t) = (M - a(t))h(t)$, $t \in [0,\omega]$ for all $h \in C[0,\omega]$. Then $B : C[0,\omega] \to C[0,\omega]$ is a positive linear continuous operator, and $\|TB\| < 1$ if $0 < M \leq (\pi/\omega)^2$.*

Proof. It is easy to see that $B : C[0,\omega] \to C[0,\omega]$ is a positive linear continuous operator and $\|B\| \leq M$.

For all $h \in C[0,\omega]$, it follows from (2.4) that

$$\begin{aligned} |(TBh)(t)| &\leq \int_0^\omega G(t,s)(M - a(s))|h(s)|\,\mathrm{d}s \\ &\leq \|h\| \int_0^\omega G(t,s)(M - a(s))\,\mathrm{d}s \\ &= \|h\|(1 - (Ta)(t)) \\ &\leq \|h\| \max_{t\in[0,\omega]} (1 - (Ta)(t)), \quad t \in [0,\omega]. \end{aligned}$$

By assumption (H_1) and Lemma 1, it follows that $(Ta)(t) > 0$ for all $t \in [0,\omega]$, and therefore $\|TB\| \leq \max_{t\in[0,\omega]}(1 - (Ta)(t)) < 1$. The proof is completed.

Remark 1. Lemma 2 improves on the results in [5]. In our Lemma 2, first note that a can have a zero point in $[0, \omega]$; that is, a need not be positive at all points in $[0, \omega]$. Secondly $0 < M \leq (\pi/\omega)^2$; that is to say M can equal $(\pi/\omega)^2$.

Lemma 3. *If $0 < M \leq (\pi/\omega)^2$, then for each $h \in C[0, \omega]$, the following linear PBVPs:*

$$u''(t) + a(t)u(t) = h(t), \quad t \in [0, \omega], \tag{2.6}$$

$$u(0) = u(\omega), \quad u'(0) = u'(\omega) \tag{2.7}$$

have a unique solution u, where $u := Ph$ and

(i) $P : C[0, \omega] \to C[0, \omega]$ *is a positive linear completely continuous operator;*
(ii) $Ph \geq Th$ *for all* $h \in C^+[0, \omega]$*;*
(iii) P *is also a strongly positive operator, i.e.* $(Ph)(t) > 0, t \in [0, \omega]$ *for all* $h \in C^+[0, \omega]$ *with* $h(t) \not\equiv 0$.

Proof. Clearly, problems (2.6) and (2.7) are equivalent to the operator equation $u = TBu + Th$, namely

$$(I - TB)u = Th. \tag{2.8}$$

Since $\|TB\| < 1$ by Lemma 2, $I - TB$ has a bounded inverse $(I - TB)^{-1}$. Therefore, Eq. (2.8) has a unique solution $u = (I - TB)^{-1}Th := Ph$, and $P = (I - TB)^{-1}T$ is linear completely continuous because T is completely continuous. Using the Neumann expansion equation, we have

$$P = (I + TB + (TB)^2 + \cdots + (TB)^n + \cdots)T = T + TBT + (TB)^2T + \cdots + (TB)^nT + \cdots.$$

This implies that P is positive because T and B are both positive. It is obvious that $Ph \geq Th$ for all $h \in C^+[0, \omega]$. This implies also that P is strongly positive since T is strongly positive. The proof is completed.

Lemma 4. *If $0 < M \leq (\pi/\omega)^2$, let $r(P)$ be the spectral radius of operator P; then $r(P) > 0$ and there exists $\varphi > 0$ such that $P\varphi = r(P)\varphi$. Also $\lambda_1 = 1/r(P)$ is the first positive eigenvalue of linear PBVPs corresponding to the problems* (1.2) *and* (1.3)*, and*

$$\int_0^\omega \varphi(t)(Pu)(t)\mathrm{d}t = \frac{1}{\lambda_1}\int_0^\omega \varphi(t)u(t)\mathrm{d}t \quad \text{for all } u \in C[0, \omega]. \tag{2.9}$$

Proof. It follows from Lemma 3 that $Ph \geq Th$ for all $h \in C^+[0, \omega]$, especially $P1 \geq T1 = 1/M$, and therefore $P^n1 \geq 1/M^n$. Hence, the spectral radius of operator P satisfies $r(P) = \overline{\lim_{n\to\infty} \|P^n\|^{1/n}} \geq 1/M > 0$. Obviously, $C^+[0, \omega]$ is a total cone of $C[0, \omega]$, i.e. $C[0, \omega] = \overline{C^+[0, \omega] - C^+[0, \omega]}$. According to the Krein–Rutman theorem [10], $r(P)$ is an eigenvalue with a positive eigenvector, i.e. there exists $\varphi > 0$ such that $P\varphi = r(P)\varphi$. Since $P\varphi = r(P)\varphi$ is equivalent to the following PBVPs:

$$\varphi''(t) + a(t)\varphi(t) = \frac{1}{r(P)}\varphi(t), \quad t \in [0, \omega],$$

$$\varphi(0) = \varphi(\omega), \quad \varphi'(0) = \varphi'(\omega),$$

$\lambda_1 = 1/r(P)$ is an eigenvalue value of linear PBVPs corresponding to PBVPs (1.2) and (1.3), and we can easily prove that $\lambda_1 = 1/r(P)$ is the first positive eigenvalue of the linear problems.

Since Pu is the unique solution of the following linear PBVPs:

$$w''(t) + a(t)w(t) = u(t), \quad t \in [0, \omega],$$

$$w(0) = w(\omega), \quad w'(0) = w'(\omega),$$

it follows that

$$\begin{aligned}\lambda_1\int_0^\omega \varphi(t)(Pu)(t)\mathrm{d}t &= \int_0^\omega (\varphi''(t)+a(t)\varphi(t))(Pu)(t)\mathrm{d}t\\ &= \varphi'(t)(Pu)(t)\,|_0^\omega - \int_0^\omega \varphi'(t)(Pu)'(t)\mathrm{d}t + \int_0^\omega a(t)\varphi(t)(Pu)(t)\mathrm{d}t\\ &= -\varphi(t)(Pu)'(t)\,|_0^\omega + \int_0^\omega [\varphi(t)(Pu)''(t)+\varphi(t)a(t)(Pu)(t)]\mathrm{d}t\\ &= \int_0^\omega \varphi(t)u(t)\mathrm{d}t.\end{aligned}$$

Hence (2.9) holds. The proof is completed.

Now, we define operators $F, Q : C^+[0,\omega] \to C^+[0,\omega]$ by

$$\begin{aligned}&(Fu)(t) = f(t,u(t)),\, t\in[0,\omega] \quad \text{for all } u\in C^+[0,\omega],\\ &Q = PF.\end{aligned} \tag{2.10}$$

Noticing that $\varphi(t) = \lambda_1(P\varphi)(t) > 0, t\in[0,\omega]$ from (iii) of Lemma 3, let $b = \min_{t\in[0,\omega]}\varphi(t)$; then $b > 0$, where $\varphi > 0$ and $r(P)\varphi = P\varphi$ with $\int_0^\omega \varphi(t)\mathrm{d}t = \lambda_1$. Choosing the sub-cone K of $C^+[0,\omega]$ given by

$$K = \left\{u\in C^+[0,\omega] : \int_0^\omega u(t)\varphi(t)\mathrm{d}t \ge \delta\|u\|\right\},$$

where $\delta = \frac{b}{\lambda_1\alpha_0\|(I-TB)^{-1}\|}$, we have the following:

Lemma 5. *$Q(C^+[0,\omega]) \subset K$ and $Q : C^+[0,\omega]\to K$ is completely continuous.*

Proof. For arbitrary given $u\in C^+[0,\omega]$, from (2.9) and definitions of Q and F it follows that

$$\begin{aligned}\int_0^\omega \varphi(t)(Qu)(t)\mathrm{d}t &= \int_0^\omega \varphi(t)(PFu)(t)\mathrm{d}t\\ &= \frac{1}{\lambda_1}\int_0^\omega \varphi(t)f(t,u(t))\mathrm{d}t\\ &\ge \frac{b}{\lambda_1}\int_0^\omega f(t,u(t))\mathrm{d}t.\end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned}\|Qu\| &= \|PFu\|\\ &= \|(I-TB)^{-1}TFu\|\\ &\le \|(I-TB)^{-1}\|\|TFu\|\\ &= \|(I-TB)^{-1}\| \max_{t\in[0,\omega]}\int_0^\omega G(t,s)f(s,u(s))\,\mathrm{d}s\\ &\le \|(I-TB)^{-1}\| \max_{(t,s)\in[0,\omega]\times[0,\omega]} G(t,s)\int_0^\omega f(s,u(s))\,\mathrm{d}s\\ &= \alpha_0\|(I-TB)^{-1}\|\int_0^\omega f(s,u(s))\mathrm{d}s.\end{aligned}$$

Using this and (2.11), we have

$$\int_0^\omega \varphi(t)(Qu)(t)\mathrm{d}t \geq \frac{b}{\lambda_1\alpha_0\|(I-TB)^{-1}\|}\|Qu\| = \delta\|Qu\|.$$

Hence, $Qu \in K$; therefore $Q(C^+[0,\omega]) \subset K$. It is obvious that $Q : C^+[0,\omega] \to K$ is completely continuous. The proof is completed.

The proof of the main theorem of this paper is based on fixed point index theory in [7–9]. For $r > 0$, let $K_r = \{u \in K : \|u\| < r\}$ and $\partial K_r = \{u \in K : \|u\| = r\}$, which is the relative boundary of K_r in K. The following two theorems [7–9] are needed in our argument.

Theorem A. *Let $Q : \overline{K_r} \to K$ be a completely continuous operator. If $\mu Qu \neq u$ for all $u \in \partial K_r$ and $\mu \in (0,1]$, then $i(Q, K_r, K) = 1$.*

Theorem B. *Let $Q : \overline{K_r} \to K$ be a completely continuous operator. If there exists $\varphi \in K$ with $\varphi \neq 0$ such that*

$$u \neq Qu + \mu\varphi \quad \textit{for all } u \in \partial K_r \textit{ and } \mu \geq 0,$$

then $i(Q, K_r, K) = 0$.

3. Existence of positive periodic solutions

For convenience, we give some notation:

$$f^0 = \limsup_{u\to 0^+} \max_{t\in[0,\omega]} \frac{f(t,u)}{u}, \qquad f_0 = \liminf_{u\to 0^+} \min_{t\in[0,w]} \frac{f(t,u)}{u},$$
$$f^\infty = \limsup_{u\to +\infty} \max_{t\in[0,\omega]} \frac{f(t,u)}{u}, \qquad f_\infty = \liminf_{u\to +\infty} \min_{t\in[0,\omega]} \frac{f(t,u)}{u}.$$

Theorem 1. *Suppose that* (H_1) *and* (H_2) *hold. If $0 < M \leq (\pi/\omega)^2$, then Eq.* (1.1) *has at least one positive periodic solution in each of the following cases:*

(i) $f^0 < \lambda_1 < f_\infty$;
(ii) $f_0 > \lambda_1 > f^\infty$,

where λ_1 is the first positive eigenvalue of the linear equation corresponding to Eq. (1.1).

Proof. By Lemma 5 and the definition of K, any nonzero fixed point in K of the operator Q defined by (2.10) is a positive solution to PBVPs (1.2) and (1.3). And therefore it is the positive periodic solution to Eq. (1.1). We assume that φ is a positive eigenvector of operator P with respect to the spectral radius $r(P) = 1/\lambda_1$ and $\int_0^\omega \varphi(t)\mathrm{d}t = \lambda_1$. We will show that Q has a nonzero fixed point in K under conditions (i) and (ii), respectively.

Suppose condition (i) holds. Since $f^0 < \lambda_1$, by the definition of f^0, there exist $r_0 > 0$ and $\varepsilon > 0$ such that

$$f(t,u) \leq \lambda_1(1-\varepsilon)u \quad \text{for all } t \in [0,\omega] \text{ and } u \in [0,r_0]. \tag{3.1}$$

Let $r \in (0, r_0)$; we now prove that $\mu Qu \neq u$ for all $u \in \partial K_r$ and $\mu \in (0,1]$. In fact, if there exist $u_0 \in \partial K_r$ and $\mu_0 \in (0,1]$ such that $\mu_0 Qu_0 = u_0$, then $u_0 \leq Qu_0$. Multiplying this inequality by φ, integrating on $[0,\omega]$, using (2.9) and (3.1), we have

$$\begin{aligned}\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t &\le \int_0^\omega \varphi(t)(Qu_0)(t)\mathrm{d}t\\ &= \int_0^\omega \varphi(t)(PFu_0)(t)\mathrm{d}t\\ &= \frac{1}{\lambda_1}\int_0^\omega f(t,u_0(t))\varphi(t)\mathrm{d}t\\ &\le \frac{1}{\lambda_1}\cdot\lambda_1(1-\varepsilon)\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t\\ &= (1-\varepsilon)\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t.\end{aligned}$$

Since $\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t \ge \delta\|u_0\| = \delta r > 0$, this implies that $1 \le 1-\varepsilon$, which is a contradiction. Hence Q satisfies the hypotheses of Theorem A. Therefore we have

$$i(Q, K_r, K) = 1. \tag{3.2}$$

On the other hand, since $f_\infty > \lambda_1$, by the definition of f_∞, there exist $R_0 > 0$ and $\varepsilon > 0$ such that

$$f(t,u) \ge \lambda_1(1+\varepsilon)u \quad \text{for all } t\in[0,\omega] \text{ and } u \ge R_0.$$

Since $f(t,u) - \lambda_1(1+\varepsilon)u$ is continuous on $[0,\omega]\times[0,R_0]$, we may choose $C > 0$ such that $f(t,u) - \lambda_1(1+\varepsilon)u \ge -C$ for all $(t,u)\in[0,\omega]\times[0,R_0]$. So altogether we have

$$f(t,u) \ge \lambda_1(1+\varepsilon)u - C \quad \text{for all } t\in[0,\omega] \text{ and } u \ge 0. \tag{3.3}$$

Let $R > \max(C/\varepsilon\delta, R_0, r_0)$. If there exist $u_0 \in \partial K_R$ and $\mu_0 \ge 0$ such that $u_0 = Qu_0 + \mu_0\varphi$, then $u_0 \ge Qu_0$. Multiplying this inequality by φ, integrating on $[0,\omega]$, using (2.9) and (3.3), and noting that $\int_0^\omega \varphi(t)\mathrm{d}t = \lambda_1$, we have

$$\begin{aligned}\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t &\ge \int_0^\omega \varphi(t)(Qu_0)(t)\mathrm{d}t\\ &= \int_0^\omega \varphi(t)(PFu_0)(t)\mathrm{d}t\\ &= \frac{1}{\lambda_1}\int_0^\omega f(t,u_0(t))\varphi(t)\mathrm{d}t\\ &\ge \frac{1}{\lambda_1}\int_0^\omega (\lambda_1(1+\varepsilon)u_0(t) - C)\varphi(t)\mathrm{d}t\\ &= (1+\varepsilon)\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t - C.\end{aligned}$$

By the definition of K, we get $\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t \ge \delta\|u_0\| = \delta R$. Therefore it follows that $R \le C/\varepsilon\delta$, which is a contradiction with the choice of R. Hence hypotheses of Theorem B hold. Therefore we have

$$i(Q, K_R, K) = 0. \tag{3.4}$$

Now by the additivity of fixed point index, (3.2) and (3.4), we have

$$i(Q, K_R\setminus\overline{K_r}, K) = i(Q, K_R, K) - i(Q, K_r, K) = -1.$$

Therefore Q has a fixed point in $K_R \setminus \overline{K}_r$.

Suppose condition (ii) holds. Since $f_0 > \lambda_1$, by the definition of f_0, there exist $r_0 > 0$ and $\varepsilon > 0$ such that

$$f(t,u) \geq \lambda_1(1+\varepsilon)u \quad \text{for all } t \in [0,\omega] \text{ and } u \in [0,r_0]. \tag{3.5}$$

Let $r \in (0,r_0)$. If there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 0$ such that $u_0 = Qu_0 + \mu_0\varphi$, then $u_0 \geq Qu_0$. Multiplying this inequality by φ, integrating on $[0,\omega]$, using (2.9) and (3.5), we have

$$\begin{aligned}\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t &= \int_0^\omega \varphi(t)(PFu_0)(t)\mathrm{d}t\\ &= \frac{1}{\lambda_1}\int_0^\omega f(t,u_0(t))\varphi(t)\mathrm{d}t\\ &\geq \frac{1}{\lambda}\cdot\lambda_1(1+\varepsilon)\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t.\end{aligned}$$

Since $\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t > 0$, it follows that $1 \geq 1+\varepsilon$, which is a contradiction. On the basis of Theorem B, we have

$$i(Q,K_r,K) = 0. \tag{3.6}$$

On the other hand, since $f^\infty < \lambda_1$, there exist $R_0 > 0$ and $\varepsilon > 0$ such that

$$f(t,u) \leq \lambda_1(1-\varepsilon)u \quad \text{for all } t \in [0,\omega] \text{ and } u \geq R_0.$$

Because $f(t,u) - \lambda_1(1-\varepsilon)u$ is continuous on $[0,\omega]\times[0,R_0]$, we can choose $C > 0$ such that

$$f(t,u) \leq \lambda_1(1-\varepsilon)u + C \quad \text{for all } t \in [0,\omega] \text{ and } u \in [0,R_0].$$

So altogether we have

$$f(t,u) \leq \lambda_1(1-\varepsilon)u + C \quad \text{for all } t \in [0,\omega] \text{ and } u \geq 0. \tag{3.7}$$

Let $R > \max(C/\varepsilon\delta, R_0, r_0)$. If there exist $u_0 \in \partial K_R$ and $\mu_0 \in (0,1]$ such that $u_0 = \mu_0 Qu_0$, then $u_0 \leq Qu_0$. Therefore

$$\begin{aligned}\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t &\leq \int_0^\omega \varphi(t)(PFu_0)(t)\mathrm{d}t\\ &= \frac{1}{\lambda_1}\int_0^\omega f(t,u_0(t))\varphi(t)\mathrm{d}t\\ &\leq \frac{1}{\lambda_1}\int_0^\omega (\lambda_1(1-\varepsilon)u_0(t)+C)\varphi(t)\mathrm{d}t\\ &= (1-\varepsilon)\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t + C.\end{aligned}$$

In addition, $\int_0^\omega \varphi(t)u_0(t)\mathrm{d}t \geq \delta\|u_0\| = \delta R$. We have $R \leq C/\varepsilon\delta$, which is a contradiction. So

$$i(Q,K_R,K) = 1. \tag{3.8}$$

From (3.6) and (3.8) it follows that

$$i(Q,K_R\setminus\overline{K}_r,K) = i(Q,K_R,K) - i(Q,K_r,K) = 1.$$

Therefore Q has a fixed point in $K_R \setminus \overline{K}_r$, which is nonzero. The proof is completed.

Corollary 1. *Suppose that* (H_1) *and* (H_2) *hold. If* $0 < M \leq (\pi/\omega)^2$ *and one of the following cases:*

(i) $f^0 = 0,\ f_\infty = \infty$ *(superlinear case);*
(ii) $f_0 = \infty,\ f^\infty = 0$ *(sublinear case),*

is satisfied, then the Eq. (1.1) *has at least one positive periodic solution.*

Remark 2. Conditions (i) and (ii) of Theorem 1 are given by the first positive eigenvalue of the linear differential equation corresponding to the Eq. (1.1), so the existence results are essential, and therefore we have improved on the results in [5].

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References

[1] L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.
[2] L.H. Erbe, S. Hu, H. Wang, Multiple positive solutions of some boundary value problems, J. Math. Anal. Appl. 184 (1994) 640–648.
[3] Z. Liu, F. Li, Multiple positive solutions of nonlinear two-point boundary value problems, J. Math. Anal. Appl. 203 (1996) 610–625.
[4] J. Henderson, H. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259.
[5] Y. Li, Positive periodic solutions of nonlinear second order ordinary differential equations, Acta Math. Sinica 45 (2002) 481–488 (in Chinese).
[6] Z. Zhang, J. Wang, On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equation, J. Math. Anal. Appl. 281 (2003) 99–107.
[7] D. Guo, Nonlinear Functional Analysis, Shangdong Science and Technique Publishing House, Jinan, 1985 (in Chinese).
[8] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Inc, London, 1988.
[9] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin Heidelberg, 1985.
[10] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Trans. Amer. Math. Soc. 10 (1962) 199–325.