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Conditional stability for a class of second-order differential equations☆

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Abstract

Using two classes of test functions introduced recently by the authors, a criterion for conditional stability of the trivial solution of a second-order semi-linear differential equation is established.

Keywords: Semi-linear differential equations; Second order; Asymptotic behaviour; Test functions; Conditional stability

1. Preliminaries

Asymptotic integration of ordinary differential equations is an important tool in modern applied mathematics. A variety of practical problems arising, for instance, in astrophysics, materials science and mathematical ecology provoke considerable interest in this topic, reflected in recent publications [1–4]. In particular, Agarwal and O'Regan [2, Section 1.12], [5, Section 13.2] applied a fixed point argument, known as the Furi–Pera theorem [6], to investigate the existence of vanishing-at-infinity solutions of the

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differential equation ($m > 0$)

$$u'' = m^2 u + F(t, u), \qquad t \geq 0,$$

assuming that for each $r > 0$ there exists a continuous non-negative function $\tau_r(t)$ such that $|u| \leq r$ implies $|F(t, u)| \leq \tau_r(t)$ for all $t \geq 0$, and

$$\lim_{t\to+\infty} \mathrm{e}^{-mt} \int_0^t \mathrm{e}^{ms} \tau_r(s) \mathrm{d}s = \lim_{t\to+\infty} \mathrm{e}^{mt} \int_t^\infty \mathrm{e}^{-ms} \tau_r(s) \mathrm{d}s = 0.$$

An extension of these results to a larger class of second-order semi-linear differential equations

$$u'' = a(t)u + F(t, u), \qquad t \geq t_0 \geq 0, \tag{1}$$

has been obtained recently by Agarwal et al. [7] under the assumption that the associated linear equation

$$u'' = a(t)u, \qquad t \geq t_0 \geq 0, \tag{2}$$

possesses a solution $x(t)$ having the Poincaré–Perron property (see [8])

$$\lim_{t\to+\infty} \frac{x'(t)}{x(t)} = -m.$$

It has been established that $a(t)$ can be written in the form

$$a(t) = m^2 + b(t), \tag{3}$$

where $b : [t_0, +\infty) \to \mathbb{R}$ is a continuous function satisfying for all $q > 0$

$$\lim_{t\to+\infty} \mathrm{e}^{-qt} \int_{t_0}^t \mathrm{e}^{qs} b(s) \mathrm{d}s = 0. \tag{4}$$

Remarkably, (4) is equivalent to the celebrated Hartman condition [9, (IV), p. 570],

$$\lim_{t\to+\infty} \left[\sup_{v>0} \frac{1}{1+v} \int_t^{t+v} b(s) \mathrm{d}s \right] = 0,$$

but the proof in [7] relies on completely different tools.

The present work continues the study of the properties of solutions of Eq. (1) undertaken by Agarwal et al. [7] and Avramescu et al. [8] and establishes an interesting link between the results in [2], [7], and a classical theorem due to Wintner [10, p. 67] stating that Eq. (2) has a solution $x(t)$ such that

$$x(t) = \mathrm{e}^{-mt} + o(\mathrm{e}^{-mt}) \qquad \text{and} \qquad x'(t) = -m\mathrm{e}^{-mt} + o(\mathrm{e}^{-mt}) \tag{5}$$

as $t \to +\infty$ provided that

$$\begin{aligned} &\int_{t_0}^{+\infty} \mathrm{e}^{-2mt} \int_{t_0}^t \mathrm{e}^{2ms} |b(s)| \mathrm{d}s \mathrm{d}t < +\infty, \\ &\mathrm{e}^{-2mt} \int_{t_0}^t \mathrm{e}^{2ms} |b(s)| \mathrm{d}s \leq M < +\infty, \end{aligned} \tag{6}$$

for all $t \geq t_0$. For further details the reader is referred to [7] and [8].

In this work, we are concerned with asymptotic behaviour of bounded solutions of Eq. (1). If $b(t)$ in (3) is continuous and absolutely integrable on $[t_0, +\infty)$, Eq. (2) has a solution $y(t)$ such that $y(t) \sim \mathrm{e}^{mt}$ as $t \to +\infty$ [11, p. 126]. Therefore, one cannot expect stability of the trivial solution of Eq. (1), but we can discuss *conditional* stability in the sense that there exists a family of solutions of Eq. (1) which vanish as $t \to +\infty$, cf. [11, pp. 89–90] or [12, p. 76].

Two useful for asymptotic integration classes of test functions $\mathcal{F}$ and $\mathcal{G}$ have been introduced recently by Avramescu et al. [8].

Definition 1. We say that the functions $f, g : [t_0, +\infty) \to (0, +\infty)$ belong to the classes $\mathcal{F}$ and $\mathcal{G}$, respectively, if $f, g \in C^2([t_0, +\infty), \mathbb{R})$ and for some $\mu_f < 0$ (respectively, for some $\nu_g > 0$)

$$\lim_{t\to+\infty} \frac{f'(t)}{f(t)} = \mu_f, \qquad \left(\lim_{t\to+\infty} \frac{g'(t)}{g(t)} = \nu_g\right).$$

Let $D = \{(t, u) : t \in [t_0, +\infty), u \in \mathbb{R}\}$. Our main assumptions are

(i) Eq. (2) has a solution $f \in \mathcal{F}$;

(ii) $F(t, u)$ satisfies for $(t, u) \in D$ the inequality

$$|F(t, u)| \le h(t)|u|; \tag{7}$$

(iii) $h(t) \ge 0$ is a continuous function such that

$$0 \le \frac{1}{f(t)} \int_t^{+\infty} h(s) f(s)\mathrm{d}s \le C < +\infty, \tag{8}$$

$$\int_{t_0}^{+\infty} G(t)\mathrm{d}t < +\infty, \qquad \text{and} \qquad 0 \le G(t) \le M < +\infty,$$

where $g(t)$ is defined in Lemma 4,

$$G(t) \stackrel{\text{def}}{=} \frac{1}{g^2(t)} \int_{t_0}^{t} h(s)g^2(s)\mathrm{d}s, \qquad t \ge t_0.$$

Clearly, our hypotheses resemble conditions (6).

The following auxiliary lemmas are adopted from [8] to make the exposition self-contained.

Lemma 2. *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then*

$$\lim_{t\to+\infty} \left[f^2(t) \int_{t_0}^{t} \frac{\mathrm{d}s}{f^2(s)} \right] = -\frac{1}{2\mu_f} \qquad \textit{and} \qquad \lim_{t\to+\infty} \left[g^2(t) \int_{t}^{+\infty} \frac{\mathrm{d}s}{g^2(s)} \right] = \frac{1}{2\nu_g}.$$

Furthermore, if $h(t)$ is a continuous, non-negative, integrable on $[t_0, +\infty)$ function, one has

$$\int_{t_0}^{+\infty} \frac{1}{f^2(t)} \int_t^{+\infty} h(s) f^2(s)\mathrm{d}s\mathrm{d}t < +\infty$$

and

$$\int_{t_0}^{+\infty} \frac{1}{g^2(t)} \int_{t_0}^{t} h(s)g^2(s)\mathrm{d}s\mathrm{d}t < +\infty.$$

Note that Lemma 2 accentuates the discussion in [10, p. 67].

Lemma 3. *Let $u(t)$ be a bounded solution of Eq. (1). Then*

$$u'(t) = O(1) \qquad \textit{as } t \to +\infty.$$

Lemma 4. *Let* $g : [t_0, +\infty) \to (0, +\infty)$ *be defined by*

$$g(t) = f(t)\left[1 + \int_{t_0}^{t} \frac{\mathrm{d}s}{f^2(s)}\right], \qquad t \geq t_0.$$

Then (a) $g(t)$ *is a solution of Eq.* (2); (b) $g \in \mathcal{G}$.

2. Conditional stability of the trivial solution

In what follows, we assume that the problem

$$u'' = a(t)u + F(t, u), \qquad t \geq T_0, \tag{9}$$

$$u(T) = u_0 \qquad u'(T) = u_1, \tag{10}$$

has a unique solution for any pair of real numbers (u_0, u_1) and any $T \geq t_0 \geq 0$. It is known (see, for instance, [13]) that (7) implies global existence of solutions of Eq. (1).

Lemma 5. *A solution* $u(t)$ *of Eq.* (1) *satisfies*

$$\lim_{t\to+\infty} u(t) = 0 \tag{11}$$

provided that

$$\lim_{t\to+\infty} \left[u(t) + \frac{1}{\nu_g} u'(t)\right] = 0. \tag{12}$$

Proof. Assume that (12) holds. Then (11) follows from the fact that

$$\lim_{t\to+\infty} \frac{[u(t)\exp(\nu_g t)]'}{[\exp(\nu_g t)]'} = \lim_{t\to+\infty} \left[u(t) + \frac{1}{\nu_g} u'(t)\right]$$

and l'Hôpital's rule. □

The following result is principal in this work and provides a useful criterion for conditional stability of the trivial solution of Eq. (1).

Theorem 6. *Any bounded solution* $u(t)$ *of Eq.* (1) *satisfies* (11) *and*

$$\lim_{t\to+\infty} u'(t) = 0. \tag{13}$$

Proof. *Step 1*. First we note that, by Lemma 3, the derivative of a bounded solution of Eq. (1) is bounded, i.e., there exists a $\delta = \delta(u) > 0$ such that

$$\max[|u(t)|, |u'(t)|] < \delta. \tag{14}$$

Suppose, for the sake of contradiction, that either the limit

$$\lim_{t\to+\infty} [u(t) + (\nu_g)^{-1} u'(t)]$$

does not exist or differs from zero. Then there exists an $\varepsilon_0 > 0$ and a strictly increasing sequence $(t_n)_{n\geq 1}$ of positive numbers, $t_n \to +\infty$ as $n \to +\infty$, such that

$$\left|u(t_n) + \frac{1}{\nu_g} u'(t_n)\right| > \varepsilon_0,$$

for all $n \geq 1$. Fix an ε satisfying $0 < 2\varepsilon < \min[1, (\nu_g)^{-1}, \varepsilon_0\delta^{-1}]$, and choose an $N = N(\varepsilon) \geq 1$ large enough to ensure that $g'(t) > 0$ for all $t \geq T = t_N$,

$$g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)} \in \left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right),$$

$$g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)} \in \left(\frac{1}{2\nu_g}-\varepsilon, \frac{1}{2\nu_g}+\varepsilon\right),$$

$$g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\left(1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right)^{-1} \in \left(\frac{1}{\nu_g}-\varepsilon, \frac{1}{\nu_g}+\varepsilon\right),$$

and

$$I(T) = \int_T^{+\infty}\frac{1}{g^2(s)}\int_T^s h(\tau)g^2(\tau)\mathrm{d}\tau\mathrm{d}s < \frac{1-2\varepsilon}{2(5+2\varepsilon)(1+(\nu_g)^{-1}+\varepsilon)}\cdot\frac{\varepsilon_0}{\delta}.$$

Step 2. Define the function $p : [T, +\infty) \to \mathbb{R}$ by

$$p(t) \stackrel{\text{def}}{=} \frac{u_0}{g(T)} + [u_1 g(T) - u_0 g'(T)]\int_T^t \frac{\mathrm{d}s}{g^2(s)}.$$

Then

$$|p(t)| \leq \frac{1}{g(T)}\left[|u_0| + \left(\frac{1}{2\nu_g}+\varepsilon\right)|u_1|\right],$$

and $p(t) \to K$ as $t \to +\infty$, where

$$K = \frac{1}{g(T)}\left\{u_0\left[1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right] + u_1\left[g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right]\right\}. \tag{15}$$

Writing (15) in the form

$$K = \frac{1}{g(T)}\left[1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right] \times \left[u_0 + g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\left(1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right)^{-1}u_1\right],$$

we deduce that

$$|K| \leq \frac{1}{g(T)}\left(\frac{1}{2}+\varepsilon\right)\left[|u_0| + \left(\frac{1}{\nu_g}+\varepsilon\right)|u_1|\right]$$

and

$$\left|\left|u_0+\frac{1}{\nu_g}u_1\right| - \left|u_0 + g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\left(1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right)^{-1}u_1\right|\right|$$
$$\leq \left|\frac{1}{\nu_g} - g^2(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\left(1-g(T)g'(T)\int_T^{+\infty}\frac{\mathrm{d}s}{g^2(s)}\right)^{-1}\right||u_1|$$
$$\leq \varepsilon|u_1| < \varepsilon\delta < \frac{\varepsilon_0}{2} < \frac{1}{2}\left|u_0+\frac{1}{\nu_g}u_1\right|.$$

The latter inequality yields an important estimate

$$|K| \geq \frac{1}{g(T)} \frac{1-2\varepsilon}{4} \left| u_0 + \frac{1}{\nu_g} u_1 \right| > 0. \tag{16}$$

Step 3. Let X be a set of all continuous real-valued functions $u(t)$ defined on $[T, +\infty)$ and satisfying

$$\lim_{t \to +\infty} \frac{u(t)}{g(t)} = l \in \mathbb{R}, \qquad l = l(u).$$

It is known (cf. [13]) that X becomes a Banach space under the norm

$$\|u\| = \sup_{t \geq T} \frac{|u(t)|}{g(t)}.$$

Introduce the operator $U : B \to X$ by

$$\frac{(Uu)(t)}{g(t)} = p(t) + \int_T^t \frac{1}{g^2(s)} \int_T^s F(\tau, u(\tau)) g(\tau) \mathrm{d}\tau \mathrm{d}s, \qquad t \geq T, \tag{17}$$

where $B \subset X$ is a closed ball of the radius $2^{-1}|K|$ centred at $p \cdot g$. Then

$$\begin{aligned} \left| \int_T^t \frac{1}{g^2(s)} \int_T^s F(\tau, u(\tau)) g(\tau) \mathrm{d}\tau \mathrm{d}s \right| &\leq \int_T^t \frac{1}{g^2(s)} \int_T^s h(\tau) \left[\frac{|u(\tau)|}{g(\tau)} \right] g^2(\tau) \mathrm{d}\tau \mathrm{d}s \\ &\leq \left[\frac{|K|}{2} + \sup_{v \geq T} |p(v)| \right] I(T), \end{aligned}$$

for any $u \in B$ and $t \geq T$. Furthermore,

$$\begin{aligned} \frac{|K|}{2} + \sup_{v \geq T} |p(v)| &\leq \frac{1}{g(T)} \left[|u_0| + \left(\frac{1}{2\nu_g} + \varepsilon \right) |u_1| \right] \\ &\quad + \frac{1}{2g(T)} \left(\frac{1}{2} + \varepsilon \right) \left[|u_0| + \left(\frac{1}{\nu_g} + \varepsilon \right) |u_1| \right] \\ &\leq \frac{1}{4g(T)} (5 + 2\varepsilon) \left[|u_0| + \left(\frac{1}{\nu_g} + \varepsilon \right) |u_1| \right] \\ &< \frac{5+2\varepsilon}{4g(T)} \left(1 + \frac{1}{\nu_g} + \varepsilon \right) \delta, \end{aligned}$$

and, finally,

$$\begin{aligned} \left[\frac{|K|}{2} + \sup_{v \geq T} |p(v)| \right] I(T) &< \frac{1}{4g(T)} \left(\frac{1}{2} - \varepsilon \right) \varepsilon_0 \\ &< \frac{1}{4g(T)} \left(\frac{1}{2} - \varepsilon \right) \left| u_0 + \frac{1}{\nu_g} u_1 \right| \leq \frac{|K|}{2}. \end{aligned} \tag{18}$$

Eqs. (17) and (18) yield $UB \subset B$. Thus, the operator U is well defined. Continuity of U can be established as in [13, Section 4], and the estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(Uu)(t)}{g(t)}\right]\right| \le \frac{1}{g^2(t)}\left\{|u_1 g(T) - u_0 g'(T)| + \int_T^t h(\tau)\left[\frac{|u(\tau)|}{g(\tau)}\right] g^2(\tau)\mathrm{d}\tau\right\}$$
$$\le \frac{1}{g^2(T)}|u_1 g(T) - u_0 g'(T)| + \left[\frac{|K|}{2} + \sup_{v\ge T}|p(v)|\right] M < +\infty,$$

where $u \in B$, ensures the equicontinuity of set UB. Next, defining $l(Uu)$ by

$$l(Uu) = K + \int_T^{+\infty} \frac{1}{g^2(s)} \int_T^s F(\tau, u(\tau)) g(\tau)\mathrm{d}\tau \mathrm{d}s,$$

we have

$$\left|\frac{(Uu)(t)}{g(t)} - l(Uu)\right| \le \left[g'(T)|u_0| + g(T)|u_1|\right] \int_t^{+\infty} \frac{\mathrm{d}s}{g^2(s)}$$
$$+ \int_t^{+\infty} \frac{1}{g^2(s)} \int_T^s |F(\tau, u(\tau))| g(\tau)\mathrm{d}\tau \mathrm{d}s$$
$$\le N\left[\int_t^{+\infty} \frac{\mathrm{d}s}{g^2(s)} + \int_t^{+\infty} G(s)\mathrm{d}s\right],$$

where $N = \delta[g(T) + g'(T) + (4g(T))^{-1}(5 + 2\varepsilon)(1 + (\nu_g)^{-1} + \varepsilon)]$, which implies that the set UB is equiconvergent in the sense of [13, p. 349].

Application of the Schauder–Tikhonov theorem yields existence of a fixed point $\zeta \in B$ of the operator U. Since the problem (9), (10) has a unique solution, $u(t) = \zeta(t)$ for all $t \ge T$, this implies

$$p(t) - \frac{|K|}{2} \le \frac{u(t)}{g(t)} \le p(t) + \frac{|K|}{2}, \qquad t \ge T.$$

Passing to the limit as $t \to +\infty$, we conclude that

$$\lim_{t\to+\infty} \frac{u(t)}{g(t)} = l(u) \in \left[K - \frac{|K|}{2}, K + \frac{|K|}{2}\right],$$

which, by virtue of (16) and Definition 1, contradicts (14). □

Remark 7. We conclude the work by commenting on condition (8). Integration by parts and application of l'Hôpital's rule yield that (8) holds for a continuously differentiable function $h(t)$ provided that

$$\int_{t_0}^{+\infty} |h'(s)|\mathrm{d}s < +\infty \qquad \text{and} \qquad 0 \le h(t) \le H < +\infty. \tag{19}$$

These conditions were exploited by Wintner in [10, pp. 64–65] to ensure asymptotic expansions (5) and thus are natural for our discussion. The significant role played by the first condition in (19) in asymptotic integration of Eq. (2) is addressed by Hartman [9, Sections 18, 23].

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