Math 1191
Mathematica Introduction

Lab 4

Fall, 2005

Mathematica functions related to calculus:

- **Differentiation.** Suppose you have a function \( f \) defined in Mathematica. You can find the derivative of \( f \) in different ways:
  
  - \( f'(x) \) This will spit back the first derivative of \( f \). For multiple derivatives, you can use multiple `'`, for example \( f'''(x) \) will give the third derivative of \( f \). Note that this command will only work if \( f \) is defined to be a single-variable function.
  
  - \( D[f[x],x] \) This will also give the first derivative of \( f \) with respect to \( x \). The advantage of the \( D[] \) command is that you can find partial derivatives of multi-variable functions. For example if \( g(x, y) = \sin(x) + y^4 \) then \( g_x(x, y) = \cos(x) \) and \( g_y(x, y) = 4y^3 \):
    
    \[
    \text{In}[1] := g[x_,y_] := \sin[x] + y^4 \\
    \text{In}[2] := D[g[x,y],x] \\
    \text{Out}[2] := \cos[x] \\
    \text{In}[3] := D[g[x,y],y] \\
    \text{Out}[3] := 4 \cdot y^3
    \]
    
    The \( D[] \) command can also be used to find the \( n \)th derivative with respect to a variable:
    
    \[
    \text{In}[4] := D[x^4, \{x,4\}] \\
    \text{Out}[4] := 24
    \]

- **Integration.** For integration, the basic command is \texttt{Integrate[]} for both indefinite and definite integrals. To compute \( \int \sin(x)dx \) and \( \int_0^\pi \sin(x)dx \):
  
  \[
  \text{In}[5] := \text{Integrate}[\sin[x],x] \\
  \text{Out}[5] := -\cos[x] \\
  \text{In}[6] := \text{Integrate}[\sin[x],\{x,0,\pi\}] \\
  \text{Out}[6] := 2
  \]
  
  Note Mathematica does not include a constant of integration for indefinite integrals.
Some integrals do not have closed form solutions, so Mathematica may respond with some strange functions as output. In this example, \( \text{Erf}[x] \) is defined to be the integral of \( e^{-x^2} \), so Mathematica is not helpful.

\[
\text{In}[7]:= \text{Integrate}\left[\text{Exp}\left[-x^2\right], x\right]
\]
\[
\text{Out}[7]:= \frac{1}{2} \sqrt{\pi} \text{Erf}[x]
\]

You can also numerically integrate a function (which can be useful if the function is very complicated) with \textit{NIntegrate}[]. This command only works for definite integrals.

\[
\text{In}[8]:= \text{NIntegrate}\left[\text{Exp}\left[-x^2\right], \{x, -2, 2\}\right]
\]
\[
\text{Out}[8]:= 1.76416
\]

You can also compute improper integrals (limits at infinity and where function becomes infinite):

\[
\text{In}[9]:= \text{Integrate}\left[\frac{1}{1 + x^2}, \{x, 0, \infty\}\right]
\]
\[
\text{Out}[9]:= \frac{\pi}{2}
\]

Note that Mathematica will alert you if it cannot get the integral to converge.

- **Limits.** In addition to the regular command \textit{Limit}[], there is also a package available, \textit{Calculus'Limit'}, that has an enhanced \textit{Limit}[] command. If you get results that you think are a little fishy, you might try loading that package and using the enhanced command.

Using \textit{Limit}[] is easy. Simply specify the value you are approaching and, if desired, a direction (use -1 for approaching from the right, 1 for the left):

\[
\text{In}[10]:= \text{Limit}\left[\text{Exp}\left[2 - x\right], x \to 2\right]
\]
\[
\text{Out}[10]:= 1
\]

\[
\text{In}[11]:= \text{Limit}\left[\frac{1}{1 + x}, x \to \infty\right]
\]
\[
\text{Out}[11]:= 0
\]

\[
\text{In}[12]:= \text{Limit}\left[\left(\sin[x]\right)^x, x \to 0, \text{Direction} \to -1\right]
\]
\[
\text{Out}[12]:= 1
\]

**Animation.** You can animate graphics with the \textit{Animate}[] command, available in the \textit{Graphics'Animation'} package. Animations are essentially ‘flip book’ graphics—Mathematica creates all the individual graphics first (when you use the \textit{Animate}[] command), then when you double-click on one of the cells in the animation, it displays each rapidly to give the appearance of movement.

\[
\text{In}[13]:= \ll \text{Graphics'Animation}'
\]
\[
\text{In}[14]:= \text{Animate}\left[\text{Plot}\left[\sin[a x], \{x, -\pi, \pi\}\right], \{a, 1, 4\}\right]
\]

A more involved example:

\[
\text{In}[15]:= m[x_, a_] := \text{Which}[x < a, \text{NIntegrate}[\sin[t^2], \{t, 0, x\}]]
\]
\[
\text{In}[16]:= \text{Animate}\left[\text{Plot}\left[m[x, a], \{x, -2, a\}, \text{PlotRange} \to \{\{-2, 2\}, \{-1, 1\}\}\right], \{a, -1.9, 1.9\}\right]
\]
Assignment

1. Let \( f(x) = \cos(3x) + \sin(x) \). Plot \( f \) and its first two derivatives on the same pair of axes, using blue for \( f(x) \), red for \( f'(x) \), and green for \( f''(x) \).

2. Recall that the critical points for a differentiable function \( f(x) \) are where \( f'(x) = 0 \). Let \( f(x) = (1 - x^2) \arctan(x) \). Find numerical approximations for all critical points of \( f \).

3. Recall that the equation of the tangent line to the graph of \( y = f(x) \) at the point \((a, f(a))\) is \( t(x) = f'(a)(x - a) + f(a) \). Use an animation to show the movement of the tangent line around the curve for \( y = \sin(2x) \cos(x) \) as \( a \) ranges from \(-\pi\) to \( \pi \). (You should use \texttt{PlotRange} to keep the vertical axis constant.)

4. Find \( \int x \sin(x) \, dx \) and \( \int_0^\pi x \sin(x) \, dx \).

5. Numerically integrate \( \int_1^\infty \frac{1}{x^2 + \sin(x)} \, dx \).

6. Find \( \lim_{x \to 0} \frac{\sin(x)}{x} \) and \( \lim_{x \to 0} \frac{\cos(x)}{x} \).

7. Mathematica does not find \( \lim_{x \to 0} \frac{|x|}{x} \) correctly. Show that this limit does not exist by showing that the left-hand and right-hand limits are not equal.

8. Find \( \lim_{x \to \infty} (1 + \frac{1}{x})^x \).

9. (EXTRA) The Mathematics Problem of the Week last week was to find the line with positive slope which is tangent to the curves \( y = x^2 + 4 \) and \( y = -x^2 + 4x - 8 \). Use Mathematica to help you find the equation of the line by following these steps:

   (a) Let \( f(x) = x^2 + 4 \) and \( g(x) = -x^2 + 4x - 8 \). The equation of the tangent line passing through \((a, f(a))\) is \( s(x) = f'(a)(x - a) + f(a) \) and the equation of the tangent line passing through \((b, g(b))\) is \( t(x) = g'(b)(x - b) + g(b) \). (Hint: define \( s \) and \( t \) as functions of \( x \) and a or \( x \) and \( b \).)

   (b) We want to find the values of \( a \) and \( b \) so that \( s(x) = t(x) \) for all \( x \). So in particular, \( s(0) = t(0) \). This will give us one equation relating \( a \) and \( b \). We can get another equation by setting the slopes of the lines equal: \( f'(a) = g'(b) \). Solving this pair of equations will give us our values of \( a \) and \( b \). Plot the curves and the tangent line to show that they do indeed satisfy the requirements of the problem.