Math 1191
Mathematica Introduction

Lab 4

Fall, 2005

Mathematica functions related to calculus:

- **Differentiation.** Suppose you have a function f defined in Mathematica. You can find the derivative of f in different ways:

  - `f'[x]` This will spit back the first derivative of f. For multiple derivatives, you can use multiple '', for example `f'''[x]` will give the third derivative of f. Note that this command will only work if f is defined to be a single-variable function.
  - `D[f[x],x]` This will also give the first derivative of f with respect to x. The advantage of the `D[]` command is that you can find *partial* derivatives of multi-variable functions. For example if $g(x, y) = \sin(x) + y^4$ then $g_x(x, y) = \cos(x)$ and $g_y(x, y) = 4y^3$:

    ```
    In[1]:= g[x_,y_] := Sin[x] + y^4
    In[2]:= D[g[x,y],x]
    Out[2]:= Cos[x]
    In[3]:= D[g[x,y],y]
    Out[3]:= 4 y^3
    ```

    The `D[]` command can also be used to find the nth derivative with respect to a variable:

    ```
    In[4]:= D[x^4, {x,4}]
    Out[4]:= 24
    ```

- **Integration.** For integration, the basic command is `Integrate[]` for both indefinite and definite integrals. To compute $\int \sin(x)dx$ and $\int_0^\pi \sin(x)dx$:

  ```
  In[5]:= Integrate[Sin[x],x]
  Out[5]:= -Cos[x]
  In[6]:= Integrate[Sin[x],{x,0,Pi}]
  Out[6]:= 2
  ```

  Note Mathematica does not include a constant of integration for indefinite integrals.

Some integrals do not have closed form solutions, so Mathematica may respond with some strange functions as output. In this example, '`Erf[x]`' is *defined* to be the integral of e^{-x^2}, so Mathematica is not helpful.

```
In[7]:= Integrate[Exp[-x^2],x]
```
`Out[7]:=` $\frac{1}{2}\sqrt{\pi}$ `Erf[x]`

You can also numerically integrate a function (which can be useful if the function is very complicated) with `NIntegrate[]`. This command only works for definite integrals.

```
In[8]:= NIntegrate[Exp[-x^2],{x,-2,2}]
Out[8]:= 1.76416
```

You can also compute improper integrals (limits at infinity and where function becomes infinite):

```
In[9]:= Integrate[1/(1 + x^2),{x,0,Infinity}]
```
`Out[9]:=` $\frac{\pi}{2}$

Note that Mathematica will alert you if it cannot get the integral to converge.

- **Limits.** In addition to the regular command `Limit[]`, there is also a package available, `Calculus‘Limit‘`, that has an enhanced `Limit[]` command. If you get results that you think are a little fishy, you might try loading that package and using the enhanced command.

  Using `Limit[]` is easy. Simply specify the value you are approaching and, if desired, a direction (use `-1` for approaching from the right, `1` for the left):

```
In[10]:= Limit[Exp[2 - x],x -> 2]
Out[10]:= 1
In[11]:= Limit[1/(1 + x), x -> Infinity]
Out[11]:= 0
In[12]:= Limit[(Sin[x])^x, x -> 0, Direction->-1]
Out[12]:= 1
```

Animation. You can animate graphics with the `Animate[]` command, available in the `Graphics‘Animation‘` package. Animations are essentially 'flip book' graphics—Mathematica creates all the individual graphics first (when you use the `Animate[]` command), then when you double-click on one of the cells in the animation, it displays each rapidly to give the appearance of movement.

```
In[13]:= <<Graphics‘Animation‘
In[14]:= Animate[Plot[Sin[a*x],{x,-Pi,Pi}],{a,1,4}]
```

A more involved example:

```
In[15]:= m[x_,a_] := Which[x < a, NIntegrate[Sin[t^2],{t,0,x}]]
In[16]:= Animate[Plot[m[x,a],{x,-2,a},
           PlotRange -> {{-2, 2}, {-1, 1}}], {a,-1.9,1.9}]
```

Assignment

1. Let $f(x) = \cos(3x) + \sin(x)$. Plot f and its first two derivatives on the same pair of axes, using blue for $f(x)$, red for $f'(x)$, and green for $f''(x)$.

2. Recall that the critical points for a differentiable function $f(x)$ are where $f'(x) = 0$. Let $f(x) = (1 - x^2)\arctan(x)$. Find numerical approximations for all critical points of f.

3. Recall that the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is $t(x) = f'(a)(x - a) + f(a)$. Use an animation to show the movement of the tangent line around the curve for $y = \sin(2x)\cos(x)$ as a ranges from $-\pi$ to π. (You should use `PlotRange` to keep the vertical axis constant.)

4. Find $\int x\sin(x)dx$ and $\int_0^\pi x\sin(x)dx$.

5. Numerically integrate $\int_1^\infty 1/(x^2 + \sin(x))$.

6. Find $\lim_{x\to 0}\frac{\sin(x)}{x}$ and $\lim_{x\to 0}\frac{\cos(x)}{x}$.

7. Mathematica does not find $\lim_{x\to 0}\frac{|x|}{x}$ correctly. Show that this limit does not exist by showing that the left-hand and right-hand limits are not equal.

8. Find $\lim_{x\to\infty}(1 + 1/x)^x$.

9. (EXTRA) The Mathematics Problem of the Week last week was to find the line with positive slope which is tangent to the curves $y = x^2 + 4$ and $y = -x^2 + 4x - 8$. Use Mathematica to help you find the equation of the line by following these steps:

   (a) Let $f(x) = x^2 + 4$ and $g(x) = -x^2 + 4x - 8$. The equation of the tangent line passing through $(a, f(a))$ is $s(x) = f'(a)(x - a) + f(a)$ and the equation of the tangent line passing through $(b, g(b))$ is $t(x) = g'(b)(x - b) + g(b)$. (Hint: define s and t as functions of *two* variables, x and a or x and b.)

   (b) We want to find the values of a and b so that $s(x) = t(x)$ for all x. So in particular, $s(0) = t(0)$. This will give us one equation relating a and b. We can get another equation by setting the slopes of the lines equal: $f'(a) = g'(b)$. Solving this pair of equations will give us our values of a and b.

   Plot the curves and the tangent line to show that they do indeed satisfy the requirements of the problem.