MATH 3331 - Final Review

1 Solving Equations

1. how to solve separable equations $\frac{dy}{dx} = g(x)h(y)$.

$$\int \frac{dy}{h(y)} = \int g(x)dx.$$

2. how to solve first order linear equations $y' + P(x)y = f(x)$.

   (a) Use integrating factor: $\mu(x) = e^{\int P(x)dx}$

   (b) Multiply the equation by $\mu(x)$ to obtain

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

   (c) Integrate both sides of this last equation.

3. how to solve exact equations $Mdx + Ndy = 0$, $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= M(x,y),\\ f(x,y) &= \int M(x,y)dx + g(y),\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y).\end{aligned}$$

   Solution is

$$f(x,y) = C.$$

4. how to use the substitution $u = y/x$ to solve homogeneous equations of the form $\frac{dy}{dx} = g(y/x)$.

5. how to use the substitution $u = y^{1-n}$ to solve Bernoulli equations $y' + p(x)y = q(x)y^n$.

6. how to solve higher-order homogeneous linear equations with constant coefficients (Section 4.3).

$$ay'' + by' + cy = 0$$

$$am^2 + bm + c = 0$$

- Distinct real roots m_1 and m_2:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x};$$

- Repeated real roots $m_1 = m_2$:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x};$$

- Conjugate complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$:

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x));$$

- Similar formulas for higher-order equations.

7. how to use **the method of undetermined coefficients** to solve higher-order nonhomogeneous linear equations (Section 4.4).

$$ay'' + by' + cy = f(x)$$

$$am^2 + bm + c = 0 \tag{1}$$

$$f(x) = (p_n x^n + \cdots + p_1 x + p_0)e^{\lambda x}\cos(\mu x) + (q_l x^l + \cdots + q_1 x + q_0)e^{\lambda x}\sin(\mu x)$$

The form of a particular solution:

$$y_p(x) = x^r(A_k x^k + \cdots + A_1 x + A_0)e^{\lambda x}\cos(\mu x) + x^r(B_k x^k + \cdots + B_1 x + B_0)e^{\lambda x}\sin(\mu x)$$

where $k = \max(n, l)$ and $r = 0$ if $\lambda + i\mu$ is not a root of (1) or $r = j$ if $\lambda + i\mu$ is a root of (1) with the multiplicity j.

8. how to use **the method of variation of parameters** to solve higher-order nonhomogeneous linear equations (Section 4.6).

$$y'' + P(x)y' + Q(x)y = f(x)$$

- Step 1. Find the general solutions of the associated homogeneous equations

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

- Step 2. Determine a particular solution of the form:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 are solutions of (2 and

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W},$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

- Step 3. The general solutions are

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + u_1(x) y_1(x) + u_2(x) y_2(x).$$

9. how to use the substitution $x = e^t$ to solve Cauchy-Euler equations $ax^2y'' + bxy' + cy = f(x)$ (Section 4.7).

10. how to solve boundary value problems.

    - Step 1. Find general solutions.
    - Step 2. Use boundary conditions to determine the constants in the general solutions.

11. how to solve initial value problems.

    - Step 1. Find general solutions.
    - Step 2. Use initial conditions to determine the constants in the general solutions.

12. how to solve homogeneous linear systems of differential equations.

$$\begin{aligned} \mathbf{X}' &= \mathbf{AX}, \\ \det(\mathbf{A} - \lambda\mathbf{I}) &= 0, \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{K} &= 0. \end{aligned}$$

    (a) Distinct real eigenvalues:

$$\mathbf{X} = c_1 \mathbf{K_1} e^{\lambda_1 t} + c_1 \mathbf{K_2} e^{\lambda_2 t}.$$

    (b) Repeated real eigenvalues:

$$\mathbf{X} = c_1 \mathbf{K_1} e^{\lambda_1 t} + c_2 \left(\mathbf{K_1} t e^{\lambda_1 t} + \mathbf{P} e^{\lambda_1 t} \right).$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}.$$

    (c) Complex eigenvalues: $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$

$$\begin{aligned} \mathbf{X_1} &= e^{\alpha t} \left[\frac{1}{2}(\mathbf{K_1} + \bar{\mathbf{K_1}}) \cos(\beta t) - \frac{i}{2}(-\mathbf{K_1} + \bar{\mathbf{K_1}}) \sin(\beta t) \right], \\ \mathbf{X_2} &= e^{\alpha t} \left[\frac{i}{2}(-\mathbf{K_1} + \bar{\mathbf{K_1}}) \cos(\beta t) + \frac{1}{2}(\mathbf{K_1} + \bar{\mathbf{K_1}}) \sin(\beta t) \right], \end{aligned}$$

13. how to use the method of variation of parameters to solve homogeneous linear systems of differential equations $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$.

$$\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t) dt.$$

2 Mathematical Modeling with Differential Equations

1. how to determine important variable and specify the relationships among them to build a model for real problems.
2. how to use Newtonian mechanics such as Newton second law and analyze forces to build a model.
3. spring-mass systems.
4. the Malthusian model.
5. Logistic model.
6. the first-order reaction model.
7. the second-order reaction model.
8. the falling body model.
9. the mixture model.

3 Qualitative Analysis

1. how to sketch a phase line, classify equilibrium points as stable, unstable, semistable, attractor, and repeller.
2. how to use direction fields to roughly describe the behavior of solutions as $x \to \infty$.

4 Theoretical Problems

1. how to show that a solution set is fundamental.
2. how to show that a group of functions are linearly independent.

5 Demonstration Questions

1. Solve the following separable equation

$$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}.$$

2. Solve the first order linear equation

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x \cos x.$$

3. Solve the exact equation

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$$

4. Solve the homogeneous equation

$$(xy + y^2 + x^2)dx + x^2 dy = 0.$$

5. Solve the Bernoulli's equation

$$y' - 5y = -\frac{5}{2}xy^3.$$

6. Solve the higher-order equations

$$y'' - y' - 12y = 0.$$
$$y'' - 2y' + y = 0.$$

$$y'' + y = 0.$$

$$y''' + y'' + y' + y = 0.$$
$$y'' - y' - 12y = e^{4x}.$$
$$y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x).$$
$$y'' + y = 2x\sin x.$$

$$y'' + y = \ln x.$$

7. Use the substitution $x = e^t$ to solve Cauchy-Euler equations

$$3x^2y'' + 11xy' - 3y = 0, \quad x > 0.$$

8. Solve the boundary value problem

$$y'' + y = x^2 + 1, \quad y(0) = 5, \quad y(1) = 0.$$

9. Solve the initial value problem

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x}, \quad y(0) = 1, \quad y'(0) = 0.$$

10. Solve the linear system

$$\begin{aligned} x' &= 2y + e^t, \\ y' &= -x + 3y - e^t. \end{aligned}$$

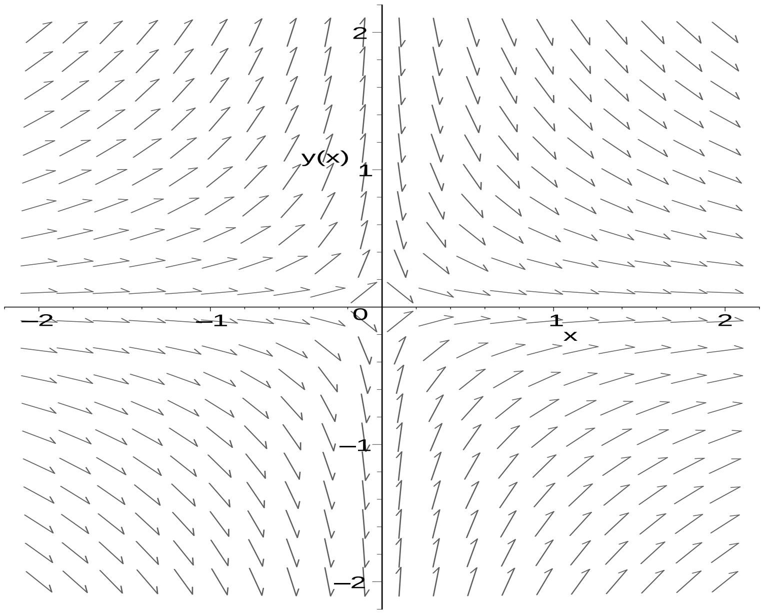
11. For modeling problems, you can practice the following problems in the textbook: Exercises 1.3: 2, 9, 19; Exercises 3.1: 1, 3, 19; Exercises 3.2: 1, 9; Exercises 5.1: 29;

12. For the following equation, sketch a phase line, classify equilibrim points as stable, unstable, semistable, attractor, and repeller

$$y' = (y-1)^2(y-2)(y-3).$$

13. Use the direction field of the following equation to roughly sketch its solutions?

$$y' = -y/x.$$



14. Show that $\{\cos x, \sin x\}$ is a fundamental solution set of

$$y'' + y = 0.$$

15. Show that $1, x, x^2, x^3$ are linearly independent on $(-\infty, \infty)$.