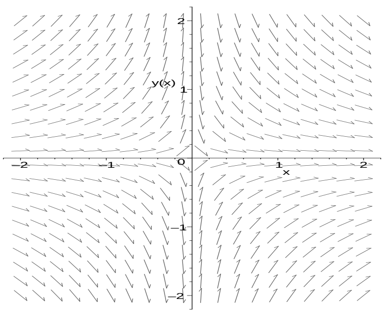
MATH 3331 - Differential Equations

Sample Test 1

Problem 1. Use the direction field of the following equation to roughly sketch its solution.

$$y' = -y/x, \quad y(1) = 1.$$



Problem 2. Find the equilibrium points of the equation

$$\frac{dy}{dt} = (5-y)(2+y).$$

Then sketch the phase portrait of the equation and classify equilibrium points as stable, unstable, and semistable where applicable.

Problem 3. Solve the following first order differential equations

(1) $x\frac{dy}{dx} - y = 2xy^2, y(1) = 1.$

This is a Bernoulli's equation. Dividing the equation by y^2 gives

$$xy^{-2}\frac{dy}{dx} - y^{-1} = 2x.$$

$$-x\frac{d}{dx}(y^{-1}) - y^{-1} = 2x.$$

Set $u = y^{-1}$. Then

$$\frac{du}{dx} + \frac{1}{x}u = -2.$$

Integrating factor is

$$\mu(x) = x$$

Multiplying the equation by x gives

$$x\frac{du}{dx} + u = -2x.$$

$$\frac{dux}{dx} = -2x.$$

$$ux = -x^2 + C.$$

$$\frac{x}{y} = -x^2 + C.$$

By the initial condition, we have

$$1 = -1 + C$$

$$C = 2.$$

So the solution is

$$y = \frac{x}{2 - x^2}.$$

(2) $\frac{dy}{dx} = \frac{xy+2y^2}{x^2+xy}$

This is a homogeneous equation. Set $y = ux$. $\frac{dy}{dx} = u + x\frac{du}{dx}$. Substituting them into the equation gives

$$u + x\frac{du}{dx} = \frac{u + 2u^2}{1 + u}$$

$$x\frac{du}{dx} = \frac{u^2}{1 + u}$$

$$\frac{u + 1}{u^2} du = \frac{dx}{x}$$

$$-\frac{1}{u} + \ln|u| = \ln|x| + C.$$

$$-\frac{x}{y} + \ln|\frac{y}{x}| = \ln|x| + C.$$

(3) $\frac{dy}{dx} = \frac{y}{2x+y^3e^y}$

Rewrite the equation as

$$ydx - (2x + y^3e^y)dy = 0.$$

So we have

$$M = y, \quad N = -(2x + y^3e^y).$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -2.$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{3}{y}.$$

So the integrating factor is

$$\mu(y) = y^{-3}.$$

Multiplying the equation by y^{-3} gives

$$y^{-2}dx - (2xy^{-3} + e^y)dy = 0.$$

$$y^{-2}dx + xd(y^{-2}) - d(e^y) = 0.$$

$$d(y^{-2}x) - d(e^y) = 0.$$

$$y^{-2}x - e^y = C.$$

Problem 4. Blood carries a drug into an organ at a rate of 5 cm^3/sec and leaves at the same rate. The organ has a liquid volume of 125 cm^3. We assume that the mixture in the organ is homogeneous. If the concentration of the drug in the blood entering the organ is 0.2 g/cm^3, what is the concentration of the drug in the organ at time t if there was no trace of the drug initially? When will the concentration of the drug in the organ reach 0.1 g/cm^3?

Solution. Let $x(t)$ denote the amount of the drug in the organ at time t. Then we have

$$\frac{dx}{dt} = 0.2 \times 5 - 5 \times \frac{x}{125}, \tag{0.1}$$

$$x(0) = 0. \tag{0.2}$$

Solving the equation gives

$$x(t) = 25\left(1 - e^{-t/25}\right).$$

So the concentration of the drug in the organ at time t is

$$\frac{x(t)}{125} = \frac{1}{5}\left(1 - e^{-t/25}\right).$$

Let T be the time at which the concentration of the drug in the organ reaches 0.1 g/cm^3. Then we have

$$0.1 = \frac{1}{5}\left(1 - e^{-T/25}\right).$$

So $T = 25 \ln 2 = 17.33$ seconds. □