Introduction to Partial Differential Equations

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1 Opening

- **Welcome to your PDEs class**! My name is Weijiu Liu. I will guide you to navigate through PDEs. We will share the learning task together. You will be the major players and I will be a just facilitator. You will need to read your textbook, attend classes, practice in class, do your homework, and so on. I will just tell you what to read, what to do, and how to do. Whenever you cannot hear me acoustically due to my Chinese accent, please feel free to ask me to repeat. If you cannot hear me mathematically, please read your textbook.
- **Syllabus**.
- **Overview of PDEs**.

2 Lecture 1 – PDE terminology and Derivation of 1D heat equation

Today:

- PDE terminology.
- Classification of second order PDEs.
- Derivation of 1D heat equation.

Next:

- Boundary conditions
- Derivation of higher dimensional heat equations

Review:

- Classification of conic section of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

  where A, B, C are constant. It is
  a **hyperbola** if $B^2 - 4AC > 0$,
  a **parabola** if $B^2 - 4AC = 0$,
  an **ellipse** if $B^2 - 4AC < 0$.

- **Conservation of heat energy**:

  Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- **Fourier's Law**: the heat flux is proportional to the temperature gradient

$$\phi = -K_0 \nabla u. \tag{1}$$

- **Fick's Law of diffusion**: the chemical flux is proportional to the gradient of chemical concentration

$$\phi = -K_0 \nabla u. \tag{2}$$

Teaching procedure:

1. PDE terminology

   - **PDE**: an equation containing an unknown function and its derivatives, e.g.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

- **Order** of a PDE: the highest order of the derivative in the equation.
- **Linear** PDEs: linear in the unknown function and its derivatives, e. g.,

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=G(x,y).$$

- **Nonlinear** PDEs: not linear, e.g., Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=k\frac{\partial^2 u}{\partial x^2}.$$

- **Solution** of a PDE: a function that has all required derivatives and that satisfies the equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

  Solution: $u=a+bx+cy+dz$, where a,b,c,d are constants.

2. Classification of second order equations:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G(x,y),$$

Where A,B,C are constant. It is said to be
hyperbolic if $B^2-4AC>0$,
parabolic if $B^2-4AC=0$,
elliptic if $B^2-4AC<0$.

- The wave equation – hyperbolic

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}.$$

  $A=1,B=0,C=-c^2,B^2-4AC=4c^2>0.$
- The heat equation – parabolic

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}.$$

  $A=0,B=0,C=-k,B^2-4AC=0.$
- The Laplace's equation – elliptic

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

  $A=1,B=0,C=1,B^2-4AC=-4<0.$

3. List of important PDEs.

- The Laplace's equation describing potentials of a physical quantity

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

- The heat equation describing heat conduction (u denoting temperature or concentration of chemical)

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

- The convection-diffusion equation describing heat convection and heat conduction (u denoting temperature or concentration of chemical)

$$\frac{\partial u}{\partial t} + v_1\frac{\partial u}{\partial x} + v_2\frac{\partial u}{\partial y} + v_3\frac{\partial u}{\partial z} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

- The wave equation describing vibration of string or membrane (u denoting displacement)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

- Euler-Bernoulli beam equation describing the buckling of a beam (u denoting displacement)

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = 0.$$

- Korteweg-de Vries equation describing nonlinear shallow water waves (u denoting displacement)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + k\frac{\partial^3 u}{\partial x^3} = 0.$$

- Schrodinger equation describing quantum mechanical behavior (Ψ denoting wavefunctions)

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}\right) + V\Psi.$$

- Maxwell's equations describing electromagnetic fields ($\mathbf{E}$ denoting the intensity of electric field and $\mathbf{H}$ denoting the intensity of magnetic field)

$$\begin{aligned} &\nabla\times\mathbf{H} - \tfrac{\partial(\varepsilon_0\mathbf{E})}{\partial t} = 0, \quad \nabla\times\mathbf{E} + \tfrac{\partial(\mu_0\mathbf{H})}{\partial t} = 0, \\ &\nabla\cdot(\varepsilon_0\mathbf{E}) = 0, \qquad \nabla\cdot(\mu_0\mathbf{H}) = 0, \end{aligned} \tag{3}$$

- Reaction-diffusion system describing reaction diffusion processes of chemicals ($\mathbf{u}$ denoting the concentration vector of chemicals)

$$\frac{\partial(\mathbf{u})}{\partial t} = k\left(\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} + \frac{\partial^2 \mathbf{u}}{\partial z^2}\right) + \mathbf{f}(\mathbf{u}).$$

- Thermoelastic system

$$
\begin{aligned}
\frac{\partial^2 \mathbf{u}}{\partial t^2} &= \mu\left(\frac{\partial^2 \mathbf{u}}{\partial x^2}+\frac{\partial^2 \mathbf{u}}{\partial y^2}+\frac{\partial^2 \mathbf{u}}{\partial z^2}\right)+(\lambda+\mu)\nabla \mathrm{div}\mathbf{u}-\alpha\nabla\theta,\\
\frac{\partial \theta}{\partial t} &= k\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}\right)-\beta \mathrm{div}\left(\frac{\partial \mathbf{u}}{\partial t}\right).
\end{aligned}
$$

- The incompressible Navier-Stokes equations describing dynamics of fluid flows (u, v, w denoting the components of a velocity field of fluid flows)

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+v\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+v\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0.
\end{aligned}
$$

One million dollars are offered by **Clay Mathematics Institute** for solutions of the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof?

4. Derivation of 1D heat equation.

Physical problem: describe the heat conduction in a rod of constant cross section area A.

Physical quantities:

- **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume $= \frac{\text{Energy}}{\text{Volume}}$.
- **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time $= \frac{\text{Energy}}{\text{Area}\cdot\text{Time}}$.
- **Heat sources** $Q(x,t)$ = heat energy per unit volume generated per unit time $= \frac{\text{Energy}}{\text{Volume}\cdot\text{Time}}$.

- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit = $\frac{\text{Energy}}{\text{Mass}\cdot\text{Temperature}}$.
- Mass density $\rho(x)$ = mass per unit volume = $\frac{\text{Mass}}{\text{Volume}}$.

Conservation of heat energy:

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy = Energy density × Volume = $e(x,t)A\Delta x$.
- Heat energy flowing across boundaries per unit time = flow-in flux × Area − flow-out flux × Area = $\phi(x,t)A - \phi(x+\Delta x,t)A$.
- Heat energy generated insider per unit time = $Q(x,t)$ × Volume = $Q(x,t)A\Delta x$.

Then

$$\frac{\partial}{\partial t}[e(x,t)A\Delta x] = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)A\Delta x.$$

Dividing it by $A\Delta x$ and letting Δx go to zero give

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{4}$$

Heat energy = $\frac{\text{Energy}}{\text{Mass}\cdot\text{Temperature}} \cdot \text{Temperature} \cdot \frac{\text{Mass}}{\text{Volume}} \cdot \text{Volume} = c(x)u(x,t)\rho A\Delta x$. So

$$e(x,t)A\Delta x = c(x)u(x,t)\rho A\Delta x,$$

and then

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows from Fourier's law that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q. \tag{5}$$

and then the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho}, \tag{6}$$

where $k = \frac{K_0}{c\rho}$ is called the thermal diffusivity.

5. Practice: Exercise 1.2.3.

3 Lecture 2 – Derivation of higher dimensional heat equations and Initial and boundary conditions

Today:

- Derivation of higher dimensional heat equations
- Initial conditions and boundary conditions

Next:

- Equilibrium

Review:

- **Conservation of heat energy**:

  Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- **Fourier's Law**: the heat flux is proportional to the temperature gradient

$$\phi = -K_0 \nabla u. \tag{7}$$

- **Divergence Theorem**:

$$\int_R \nabla \cdot \mathbf{A} dV = \oint \mathbf{A} \cdot \mathbf{n} dS \tag{8}$$

Teaching procedure:

1. Derivation of 2D or 3D heat equation.

   Physical problem: describe the heat conduction in a region of 2D or 3D space.

   Physical quantities:

   - **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume $= \frac{\text{Energy}}{\text{Volume}}$.
   - **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time $= \frac{\text{Energy}}{\text{Area}\cdot\text{Time}}$.

- **Heat sources** $Q(x,t)$ = heat energy per unit volume generated per unit time $= \frac{\text{Energy}}{\text{Volume}\cdot\text{Time}}$.
- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit $= \frac{\text{Energy}}{\text{Mass}\cdot\text{Temperature}}$.
- Mass density $\rho(x)$ = mass per unit volume $= \frac{\text{Mass}}{\text{Volume}}$.

Conservation of heat energy:

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy $= \int_R e(x,t)dV$.
- Heat energy flowing across boundaries per unit time $= \oint \phi \cdot \mathbf{n} dS$.
- Heat energy generated insider per unit time $= \int_R Q(x,t)dV$.

Then

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\oint \phi \cdot \mathbf{n} dS + \int_R Q(x,t)dV.$$

The divergence theorem give

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\int_R \nabla \cdot \phi dV + \int_R Q(x,t)dV.$$

and then

$$\frac{\partial e}{\partial t} = -\nabla \cdot \phi + Q. \tag{9}$$

Heat energy per unit volume $= c(x)u(x,t)\rho$. So

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows from Fourier's law that

$$c\rho\frac{\partial u}{\partial t} = \nabla \cdot (\mathbf{K_0}\nabla\mathbf{u}) + Q. \tag{10}$$

and then the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{Q}{c\rho}, \tag{11}$$

where $k = \frac{K_0}{c\rho}$ is called the thermal diffusivity.

2. **Laplace**'s equation

$$\nabla^2 u = 0.$$

3. **Poisson**'s equation

$$\nabla^2 u = q.$$

4. Initial conditions
$$u(x, y, z, 0) = f(x, y, z).$$

5. boundary conditions

    - Prescribed temperature boundary condition (**Dirichlet** boundary condition, mathematically)
    $$u(x, y, z, t) = T(x, y, z, t) \quad \text{on} \quad \partial\Omega.$$
    $$u(0, t) = T_1(t), \quad u(L, t) = T_2(t).$$

    - Prescribed heat flux boundary condition (**Neumann** boundary condition, mathematically)
    $$K_0 \nabla u(x, y, z, t) \cdot \mathbf{n} = K_0 \frac{\partial u}{\partial \mathbf{n}} = \phi(x, y, z, t) \quad \text{on} \quad \partial\Omega.$$
    $$-K_0 \frac{\partial u}{\partial x}(0, t) = \phi_1(t), \quad K_0 \frac{\partial u}{\partial x}(L, t) = \phi_2(t).$$

    - Newton's law of cooling boundary condition (**Robin** boundary condition, mathematically)
    $$K_0 \nabla u \cdot \mathbf{n} = K_0 \frac{\partial u}{\partial \mathbf{n}} = -h(u - u_b) \quad \text{on} \quad \partial\Omega.$$
    $$-K_0 \frac{\partial u}{\partial x}(0, t) = -h(u(0, t) - u_1(t)), \quad K_0 \frac{\partial u}{\partial x}(L, t) = -h(u(L, t) - u_2(t)).$$

6. Practice Exercise 1.5.1.

4 Lecture 3 – Equilibrium

Today:

- Equilibrium

Next:

- Eigenvalue problems

Review:

- Integration by Parts: $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$.
- Hölder's inequality:

$$\int_a^b |f(x)g(x)| dx \le \left(\int_a^b |f(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}. \tag{12}$$

- Differential inequality: If

$$\frac{df(t)}{dt} \le af(t),$$

then

$$f(t) \le f(0)e^{at}.$$

Teaching procedure:

1. Diruchlet BC.

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\nabla^2 u, & (13)\\ u(x,y,z,t) &= T(x,y,z) \quad \text{on} \quad \partial\Omega, & (14)\\ u(x,y,z,0) &= f(x,y,z). & (15) \end{aligned}$$

Steady equation

$$\begin{aligned} \nabla^2 w &= 0,\\ w(x,y,z) &= T(x,y,z) \quad \text{on} \quad \partial\Omega, \end{aligned}$$

w is called **equilibrium** or **steady state** solution of (13)-(15).

Example. Consider 1D problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{d^2u}{dx^2}, & (16)\\ u(0,t) &= T_1, \quad u(0,t) = T_2, & (17)\\ u(x,0) &= f(x). & (18) \end{aligned}$$

and its corresponding steady equation

$$
\begin{aligned}
\frac{d^2w}{dx^2} &= 0, \\
w(0) &= T_1, \quad w(L) = T_2.
\end{aligned}
$$

The solution is

$$
w(x) = T_1 + \frac{T_2 - T_1}{L}x.
$$

Theorem 4.1. *For any continuous function $f(x)$, the solution $u(x,t)$ of (16)-(18) converges $w(x)$ as t tends to infinity. More precisely,*

$$
\lim_{t\to\infty} \int_0^L |u(x,t) - w(x)|^2 dx = 0 \tag{19}
$$

Lemma 4.1. *(Poincare's inequality) For any continuous function $f(x)$ with $f(0) = 0$,*

$$
\int_0^L |f(x)|^2 dx \le L^2 \int_0^L \left|\frac{df(x)}{dx}\right|^2 dx. \tag{20}
$$

Proof.

$$
\begin{aligned}
|f(x)| &= \left|\int_0^x \frac{df(s)}{ds} ds\right| \le \int_0^x \left|\frac{df(s)}{ds}\right| ds \le \left(\int_0^x 1^2 ds\right)^{1/2} \left(\int_0^x \left|\frac{df(s)}{ds}\right|^2 ds\right)^{1/2} \\
&\le L^{1/2} \left(\int_0^L \left|\frac{df(s)}{ds}\right|^2 ds\right)^{1/2}.
\end{aligned}
$$

Squaring the inequality and integrating from 0 to L give (20). □

Proof of Theorem 4.1. Let $v = u - w$. Then

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= k\frac{d^2v}{dx^2}, & (21)\\
v(0,t) &= 0, \quad v(0,t) = 0, & (22)\\
v(x,0) &= f(x) - w(x). & (23)
\end{aligned}
$$

Multiplying (21) by v and integrating from 0 to L give

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^L v^2(x,t)dx &= k\int_0^L v(x,t)\frac{d^2v(x,t)}{dx^2}dx \\
&= -k\int_0^L \left(\frac{dv(x,t)}{dx}\right)^2 dx \\
&= -kL^2 \int_0^L |v(x,t)|^2 dx.
\end{aligned}
$$

Solving this inequality gives

$$\int_0^L |v(x,t)|^2 dx \le e^{-2kL^2 t} \int_0^L |v(x,0)|^2 dx = e^{-2kL^2 t} \int_0^L |f(x) - w(x)|^2 dx. \tag{24}$$

$\square$

2. Neumann BC.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\nabla^2 u, && (25)\\
\frac{\partial u}{\partial \mathbf{n}}(x,y,z,t) &= \phi(x,y,z) \quad \text{on} \quad \partial\Omega, && (26)\\
u(x,y,z,0) &= f(x,y,z). && (27)
\end{aligned}$$

Steady equation

$$\begin{aligned}
\nabla^2 w &= 0,\\
\frac{\partial w}{\partial \mathbf{n}}(x,y,z,t) &= \phi(x,y,z) \quad \text{on} \quad \partial\Omega,
\end{aligned}$$

w is called **equilibrium** or **steady state** solution of (25)-(27).

Example. Consider 1D problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{d^2 u}{dx^2}, && (28)\\
\frac{\partial u}{\partial x}(0,t) &= 0, \quad \frac{\partial u}{\partial x}(0,t) = 0, && (29)\\
u(x,0) &= f(x). && (30)
\end{aligned}$$

and its corresponding steady equation

$$\begin{aligned}
\frac{d^2 w}{dx^2} &= 0,\\
\frac{\partial w}{\partial x}(0) &= 0, \quad \frac{\partial w}{\partial x}(L) = 0.
\end{aligned}$$

The solution is

$$w(x) = C. \quad \text{(any constant)}$$

- Conservation of thermal energy. Integrating (28) from 0 to L gives

  $$\frac{d}{dt}\int_0^L u(x,t)dx = 0.$$

  So

  $$\int_0^L u(x,t)dx = \int_0^L u(x,0)dx = \int_0^L f(x)dx. \tag{31}$$

- Convergence.

  Theorem 4.2. *For any continuous function $f(x)$, the solution $u(x,t)$ of (28)-(30) converges $w(x) = C = \frac{1}{L}\int_0^L f(x)dx$ as t tends to infinity. More precisely,*

$$\lim_{t\to\infty}\int_0^L |u(x,t) - \frac{1}{L}\int_0^L f(s)ds|^2 dx = 0 \tag{32}$$

  Lemma 4.2. *(Poincaré's inequality) For any continuous function $f(x)$,*

$$\int_0^L \left| f(x) - \frac{1}{L}\int_0^L f(s)ds \right|^2 dx \le M \int_0^L \left|\frac{df(x)}{dx}\right|^2 dx, \tag{33}$$

  where M is a positive constant.

  Proof of Theorem 4.2. Let $v = u - \frac{1}{L}\int_0^L f(s)ds$. Then

$$\frac{\partial v}{\partial t} = k\frac{d^2 v}{dx^2}, \tag{34}$$

$$\frac{\partial v}{\partial x}(0,t) = 0, \quad \frac{\partial v}{\partial x}(0,t) = 0, \tag{35}$$

$$v(x,0) = f(x) - \frac{1}{L}\int_0^L f(s)ds. \tag{36}$$

  Multiplying (34) by v and integrating from 0 to L give

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L v^2(x,t)dx &= k\int_0^L v(x,t)\frac{d^2 v(x,t)}{dx^2}dx \\ &= -k\int_0^L \left(\frac{dv(x,t)}{dx}\right)^2 dx \\ &= -kM\int_0^L |v(x,t)|^2 dx.\end{aligned}$$

  Solving this inequality gives

$$\int_0^L |v(x,t)|^2 dx \le e^{-2kMt}\int_0^L |v(x,0)|^2 dx = e^{-2kMt}\int_0^L \left| f(x) - \frac{1}{L}\int_0^L f(s)ds \right|^2 dx.$$

  □

3. Practice. Exercise 1.4.1 (b), (c).

5 Lecture 4 – Eigenvalue problems

Today:

- Eigenvalue problems

Next:

- Linear differential operators
- Separation of variables

Review:

- The method of solving second-order homogeneous linear equations with constant coefficients:

$$ay'' + by' + cy = 0$$
$$am^2 + bm + c = 0$$

  - Distinct real roots m_1 and m_2:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x};$$

  - Repeated real roots $m_1 = m_2$:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x};$$

  - Conjugate complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$:

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x));$$

Teaching procedure:

1. Separation of variables.

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, && (37) \\ u(0,t) &= 0, \quad u(L,t) = 0, && (38) \\ u(x,0) &= f(x). && (39) \end{aligned}$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)G(t), \quad (40)$$

The method of finding such a solution is called the **method of separation of variables**.

Substitute the above expression into the equation (37), we obtain

$$\phi(x)G'(t) = k\phi''(x)G(t),$$

and then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{41}$$

where λ is constant to be determined. The boundary condition yields that

$$\phi(0) = \phi(L) = 0.$$

We then have an eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0, \end{aligned}$$

2. Eigenvalue problems with Dirichlet BC: Find a complex number λ such that the problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0. \end{aligned}$$

has non-zero solutions ϕ. λ is called **eigenvalue** and ϕ called the corresponding **eigenfunction**.

Auxiliary equations:

$$m^2 = -\lambda.$$

- Case 1: $\lambda < 0$. Distinct real roots $m_1 = \sqrt{-\lambda}$ and $m_2 = -\sqrt{-\lambda}$:

$$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.

- Case 2: $\lambda = 0$. Repeated real roots $m_1 = m_2 = 0$:

$$\phi = c_1 + c_2 x.$$

The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.

- Case 3: $\lambda > 0$. Conjugate complex roots $m_1 = i\sqrt{\lambda}$ and $m_2 = -i\sqrt{\lambda}$:

$$\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

$\phi(0) = 0$ implies that $c_1 = 0$. $\phi(L) = 0$ gives

$$\sin(\sqrt{\lambda} L) = 0.$$

So $\sqrt{\lambda} L = n\pi$ $(n = 1, 2, \cdots)$ and then we obtain the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{42}$$

and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{43}$$

3. Eigenvalue problems with Neumann BC:

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{\partial\phi}{\partial x}(0) &= 0, \quad \frac{\partial\phi}{\partial x}(L) = 0. \end{aligned}$$

Auxiliary equations:

$$m^2 = -\lambda.$$

- Case 1: $\lambda < 0$. Distinct real roots $m_1 = \sqrt{-\lambda}$ and $m_2 = -\sqrt{-\lambda}$:

  $$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

  The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.
- Case 2: $\lambda = 0$. Repeated real roots $m_1 = m_2 = 0$:

  $$\phi = c_1 + c_2 x.$$

  The boundary conditions imply that $c_2 = 0$ and c_2 can be any constants. So $\lambda_0 = 0$ is an eigenvalue with the eigenfunction

  $$\phi_0 = 1.$$

- Case 3: $\lambda > 0$. Conjugate complex roots $m_1 = i\sqrt{\lambda}$ and $m_2 = -i\sqrt{\lambda}$:

  $$\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

  $\phi'(0) = 0$ implies that $c_2 = 0$. $\phi'(L) = 0$ gives

  $$\sin(\sqrt{\lambda} L) = 0.$$

  So $\sqrt{\lambda} L = n\pi$ $(n = 1, 2, \cdots)$ and then we obtain the eigenvalues

  $$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{44}$$

  and the corresponding eigenfunctions

  $$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{45}$$

4. Eigenvalue problems with periodic BC:

$$
\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi, \\
\phi(-L) &= \phi(L), \quad \frac{\partial\phi}{\partial x}(-L) = \frac{\partial\phi}{\partial x}(L).
\end{aligned}
$$

Auxiliary equations:

$$m^2 = -\lambda.$$

- Case 1: $\lambda < 0$. Distinct real roots $m_1 = \sqrt{-\lambda}$ and $m_2 = -\sqrt{-\lambda}$:

  $$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

  The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.

- Case 2: $\lambda = 0$. Repeated real roots $m_1 = m_2 = 0$:

  $$\phi = c_1 + c_2 x.$$

  The boundary conditions imply that $c_2 = 0$ and c_2 can be any constants. So $\lambda_0 = 0$ is an eigenvalue with the eigenfunction

  $$\phi_0 = 1.$$

- Case 3: $\lambda > 0$. Conjugate complex roots $m_1 = i\sqrt{\lambda}$ and $m_2 = -i\sqrt{\lambda}$:

  $$\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

  $\phi(-L) = \phi(L)$ gives

  $$c_1 \cos(-\sqrt{\lambda}L) + c_2 \sin(-\sqrt{\lambda}L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L).$$

  $$c_2 \sin(\sqrt{\lambda}L) = 0.$$

  $\frac{\partial\phi}{\partial x}(-L) = \frac{\partial\phi}{\partial x}(L)$ gives

  $$\sqrt{\lambda}[-c_1 \sin(-\sqrt{\lambda}L) + c_2 \cos(-\sqrt{\lambda}L)] = \sqrt{\lambda}[-c_1 \sin(\sqrt{\lambda}L) + c_2 \cos(\sqrt{\lambda}L)].$$

  $$c_1 \sin(\sqrt{\lambda}L) = 0.$$

  Hence we obtain the eigenvalues

  $$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{46}$$

  since, for these λ_n, $c_1 \cos(\sqrt{\lambda_n}x) + c_2 \sin(\sqrt{\lambda_n}x)$ are non-zero solutions. So the corresponding eigenfunctions

  $$\phi_{2n-1} = \cos\left(\frac{n\pi x}{L}\right), \quad \phi_{2n} = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{47}$$

6 Lecture 5 – Separation of variables

Today:

- Linear differential operators
- Dirichlet boundary value problems

Next:

- Continue with Dirichelt BVP.

Review:

- the heat, wave, and Laplace's equations
- solution of linear ode
$$\frac{dG}{dt} = aG$$
is
$$G(t) = Ce^{at}.$$
- The eigenvalue problem
$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0 \end{aligned}$$
has the eigenvalues
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{48}$$
and the corresponding eigenfunctions
$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{49}$$

Teaching procedure:

1. **Linear differential operators**.

   - **Linear operator**. An operator L that maps a function to another function is linear if it satisfies
   $$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$$
   for any two functions u_1, u_2 and any constants c_1, c_2.
   Example. The heat operator
   $$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$$
   is a linear differential operator.

The wave operator
$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$$
is a linear differential operator.
The Laplace's operator
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
is a linear differential operator.

- **Linear differential equation**. If L is a linear differential operator, then

$$L(u) = f$$

is a linear differential equation, where f is a known function.
Example. The heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f$$

is a linear differential equation.
The wave equation
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f$$
is a linear differential equation.
The Poisson's equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$
is a linear differential equation.
The Burgers' equation
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}$$
is a nonlinear equation

- Principle of superposition. If u_1 and u_2 are solutions of a linear equation

$$L(u) = f,$$

then $c_1u_1 + c_2u_2$ is also a solution, where c_1, c_2 are any constants.

Proof.
$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2) = 0.$$
□

2. Dirichlet boundary value problems:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, &(50)\\
u(0,t) &= 0, \quad u(L,t)=0, &(51)\\
u(x,0) &= f(x). &(52)
\end{aligned}$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)G(t), \tag{53}$$

The method of finding such a solution is call the **method of separation of variables**. Substitute the above expression into the equation (50), we obtain

$$\phi(x)G'(t) = k\phi''(x)G(t),$$

and then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{54}$$

where λ is constant to be determined. The boundary condition (51) yields that

$$\phi(0) = \phi(L) = 0.$$

We then have an eigenvalue problem

$$\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi,\\
\phi(0) &= 0, \quad \phi(L)=0,
\end{aligned}$$

which has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{55}$$

and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{56}$$

On the other hand, it follows from (54) that

$$\frac{dG}{dt} = -\lambda k G, \tag{57}$$

which has solutions

$$G(t) = ce^{-\lambda k t} = ce^{-\frac{kn^2\pi^2 t}{L^2}}.$$

We then derive that

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}, \quad n = 1, 2, \cdots, \tag{58}$$

These functions do satisfy the equation (50) and the boundary condition (51), but, unfortunately, they **DO NOT** satisfy the initial condition (52)!

7 Lecture 6 – Infinite Series Solutions

Today:

- Superposition
- Infinite series solutions

Next:

- Neumann and Periodic boundary value problems.

Review:

- Trigonometric identities:

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)].$$

$$\sin^2 u = \frac{1}{2}[1 - \cos(2u)].$$

- Trigonometric integral:

$$\int \cos x dx = \sin x + C.$$

$$\int \sin x dx = -\cos x + C.$$

- Integration by parts:

$$\int_a^b u(x) dv(x) = uv|_a^b - \int_a^b v(x) du(x).$$

- Orthogonal vectors.
- Dirichlet boundary value problems:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, && (59) \\ u(0,t) &= 0, \quad u(0,t) = 0, && (60) \\ u(x,0) &= f(x). && (61) \end{aligned}$$

Functions

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}, \quad n = 1, 2, \cdots, \tag{62}$$

do satisfy the equation and the boundary condition, but **DO NOT** satisfy the initial condition.

Teaching procedure:

1. Superposition:
$$u(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$
If there are N and constants $c_1, c_2, \cdots, c_N$ such that
$$f(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right),$$
then $u(x,t)$ is a solution of (50), (51), and (52).

2. Infinite series solution:
$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$
The initial condition gives
$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right). \tag{63}$$
To determine c_n, we multiply (63) by $\sin\left(\frac{n\pi x}{L}\right)$ and integrate from 0 to L. We then find
$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$
c_n is called **Fourier coefficient**. The trigonometric series (63) with these coefficients is called **Fourier series**.

3. Orthogonality. Two functions $f(x), g(x)$ are said to be **orthogonal** if $\int_0^L f(x)g(x)dx = 0$. A set of functions is called an **orthogonal set of functions** if each member of the set is orthogonal to every other member.

   Example. The set $\{\sin\left(\frac{n\pi x}{L}\right)\}_{n=1}^{\infty}$ is orthogonal.

4. **Example**. Solve the following initial boundary value problem
$$\frac{\partial u}{\partial t} = k\frac{d^2 u}{dx^2}, \tag{64}$$
$$u(0,t) = 0, \quad u(L,t) = 0, \tag{65}$$
with the initial condition

   (a) $u(x,0) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$.

   (b) $u(x,0) = x$.

Solution. (a) The Fourier coefficient are

$$
\begin{aligned}
c_1 &= \frac{2}{L}\int_0^L \left[3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]\sin\left(\frac{1\pi x}{L}\right)dx \\
&= 3, \\
c_2 &= \frac{2}{L}\int_0^L \left[3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]\sin\left(\frac{2\pi x}{L}\right)dx \\
&= 0, \\
c_3 &= \frac{2}{L}\int_0^L \left[3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]\sin\left(\frac{3\pi x}{L}\right)dx \\
&= -1, \\
c_n &= \frac{2}{L}\int_0^L \left[3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)dx \\
&= 0, \quad n = 4, 5, \cdots,
\end{aligned}
$$

So the solution is

$$
u(x,t) = 3\sin\left(\frac{\pi x}{L}\right)e^{-\frac{k\pi^2 t}{L^2}} - \sin\left(\frac{3\pi x}{L}\right)e^{-\frac{9k\pi^2 t}{L^2}}.
$$

(b) The Fourier coefficient are

$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx \\
&= -\frac{2}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2L}{n\pi}\cos(n\pi) + \frac{2L}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L \\
&= \frac{2L}{n\pi}\cos(n\pi), \quad n = 1, 2, \cdots,
\end{aligned}
$$

So the solution is

$$
u(x,t) = \sum_{n=1}^{\infty}\frac{2L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{kn^2\pi^2 t}{L^2}}.
$$

8 Lecture 7 – Neumann and Periodic boundary value problems

Today:

- Neumann boundary value problems
- Periodic boundary value problems

Next:

- Laplace's equation.

Review:

- Trigonometric identities:

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)].$$

$$\cos^2 u = \frac{1}{2}[1 + \cos(2u)].$$

- The Neumann eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{d\phi}{dx}(0) &= 0, \quad \frac{d\phi}{dx}(L) = 0 \end{aligned}$$

has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{66}$$

and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{67}$$

- The periodic eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(-L) &= \phi(L), \quad \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L) \end{aligned}$$

has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{68}$$

and the corresponding eigenfunctions

$$\phi_0 = 1, \quad \phi_{2n-1} = \cos\left(\frac{n\pi x}{L}\right), \quad \phi_{2n} = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{69}$$

Teaching procedure:

1. Orthogonality of the sets $\{1, \cos\left(\frac{n\pi x}{L}\right), n = 1, 2, \cdots, \}$ and $\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \cdots, \}$.

2. Neumann boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & (70)\\ \frac{\partial u}{dx}(0,t) &= 0, \quad \frac{\partial u}{dx}(L,t) = 0, & (71)\\ u(x,0) &= f(x). & (72) \end{aligned}$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)G(t), \tag{73}$$

Substitute the above expression into the equation (70), we obtain

$$\phi(x)G'(t) = k\phi''(x)G(t),$$

and then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{74}$$

where λ is constant to be determined. The boundary condition (71) yields that

$$\frac{\phi u}{dx}(0) = \frac{\phi}{dx}(L) = 0.$$

We then have an eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi,\\ \frac{\phi}{dx}(0) &= 0, \quad \frac{\phi}{dx}(L) = 0, \end{aligned}$$

which has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{75}$$

and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{76}$$

On the other hand, it follows from (74) that

$$\frac{dG}{dt} = -\lambda kG, \tag{77}$$

which has solutions

$$G(t) = ce^{-\lambda kt} = ce^{-\frac{kn^2\pi^2 t}{L^2}}.$$

We then derive the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

The initial condition gives

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right). \tag{78}$$

To determine a_n, we multiply (78) by $\cos\left(\frac{n\pi x}{L}\right)$ and integrate from 0 to L. We then find

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx, && (79)\\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. && (80) \end{aligned}$$

Example. Solve the following initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{dx^2}, && (81)\\ \frac{\partial u}{dx}(0,t) &= 0, \quad \frac{\partial u}{dx}(L,t) = 0, && (82)\\ u(x,0) &= x. && (83) \end{aligned}$$

Solution. The Fourier coefficient are

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L xdx = \frac{L}{2},\\ a_n &= \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\\ &= -\frac{2L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\\ &= \frac{2L(1-\cos(n\pi))}{n^2\pi^2}, \quad n = 1, 2, \cdots, \end{aligned}$$

So the solution is

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L(1-\cos(n\pi))}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

3. Periodic boundary value problem:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{d^2u}{dx^2}, &\quad (84)\\
u(-L,t) &= u(L,t), \quad \frac{du}{dx}(-L,t) = \frac{du}{dx}(L,t), &\quad (85)\\
u(x,0) &= f(x). &\quad (86)
\end{aligned}
$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)G(t), \tag{87}$$

Substitute the above expression into the equation (84), we obtain

$$\phi(x)G'(t) = k\phi''(x)G(t),$$

and then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{88}$$

where λ is constant to be determined. The boundary condition (85) yields that

$$\phi(-L) = \phi(L), \quad \frac{\phi}{dx}(-L) = \frac{\phi}{dx}(L).$$

We then have an eigenvalue problem

$$
\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi,\\
\phi(-L) &= \phi(L), \quad \frac{\phi}{dx}(-L) = \frac{\phi}{dx}(L),
\end{aligned}
$$

which has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{89}$$

and the corresponding eigenfunctions

$$\phi_0 = 1, \quad \phi_{2n-1} = \cos\left(\frac{n\pi x}{L}\right), \quad \phi_{2n} = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{90}$$

On the other hand, it follows from (88) that

$$\frac{dG}{dt} = -\lambda kG, \tag{91}$$

which has solutions

$$G(t) = ce^{-\lambda kt} = ce^{-\frac{kn^2\pi^2t}{L^2}}.$$

We then derive the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

The initial condition gives

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \tag{92}$$

To determine the coefficients, we multiply (92) by $\cos\left(\frac{n\pi x}{L}\right)$ or $\cos\left(\frac{n\pi x}{L}\right)$ and integrate from $-L$ to L. We then find

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, && (93)\\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, && (94)\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. && (95) \end{aligned}$$

Example. Solve the following initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{dx^2}, && (96)\\ u(-L,t) &= u(L,t), \quad \frac{\partial u}{dx}(-L,t) = \frac{\partial u}{dx}(L,t), && (97)\\ u(x,0) &= x^2. && (98) \end{aligned}$$

Solution. The Fourier coefficient are

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{L^2}{3},\\ a_n &= \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{1}{n\pi}x^2\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{2}{n\pi}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{2L}{n^2\pi^2}x\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{2L}{n^2\pi^2}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{4L^2\cos(n\pi)}{n^2\pi^2}, \quad n = 1, 2, \cdots,\\ b_n &= 0, \quad n = 1, 2, \cdots. \end{aligned}$$

So the solution is

$$u(x,t) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2\cos(n\pi)}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

4. Exercise 2.4: 2.4.1 (a).

9 Lecture 8 – Laplace's equation

Today:

- Laplace's equation.

Next:

- Review of the heat equation.

Review:

- Trigonometric identities:

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)].$$

$$\cos^2 u = \frac{1}{2}[1 + \cos(2u)].$$

- The Neumann eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{d\phi}{dx}(0) &= 0, \quad \frac{d\phi}{dx}(L) = 0 \end{aligned}$$

has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{99}$$

and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{100}$$

Teaching procedure:

1. Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{101}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = f(x), \tag{102}$$

Look for a solution of the form of separation of variables:

$$u(x, y) = \phi(x)h(y), \tag{103}$$

Substitute the above expression into the equation (101), we obtain

$$\phi''(x)h(y) + \phi(x)h''(y) = 0,$$

and then

$$\frac{h''(y)}{h(y)} = -\frac{\phi''(x)}{\phi(x)} = \lambda, \tag{104}$$

where λ is constant to be determined. The boundary condition yields that

$$\frac{\phi}{dx}(0) = \frac{\phi}{dx}(L) = 0, \quad h(0) = 0.$$

We then have an eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{\phi}{dx}(0) &= 0, \quad \frac{\phi}{dx}(L) = 0, \end{aligned}$$

which has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{105}$$

and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{106}$$

On the other hand, it follows from (104) that

$$h''(y) = \frac{n^2\pi^2}{L^2} h(y), \quad h(0) = 0. \tag{107}$$

The general solutions are

$$h(y) = c_1 e^{\frac{n\pi y}{L}} + c_2 e^{-\frac{n\pi y}{L}}.$$

The boundary condition $h(0) = 0$ gives

$$c_1 + c_2 = 0.$$

So

$$h(y) = c_1 \left(e^{\frac{n\pi y}{L}} - e^{-\frac{n\pi y}{L}}\right).$$

We then derive the infinite series solution:

$$u(x, t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \left(e^{\frac{n\pi y}{L}} - e^{-\frac{n\pi y}{L}}\right).$$

The boundary condition $u(x, H) = f(x)$ gives

$$f(x) = u(x, H) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \left(e^{\frac{n\pi H}{L}} - e^{-\frac{n\pi H}{L}}\right). \tag{108}$$

To determine a_n, we multiply the above by $\cos\left(\frac{n\pi x}{L}\right)$ and integrate from 0 to L. We then find

$$a_n = \frac{2\left(e^{\frac{n\pi H}{L}} - e^{-\frac{n\pi H}{L}}\right)}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{109}$$

2. Exercise 2.5.1 (b).

10 Review of the heat equation and Laplace's equation

1. The heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho}. \tag{110}$$

2. Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{111}$$

3. Equilibrium:

$$k\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho} = 0. \tag{112}$$

4. The eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0 \end{aligned}$$

   has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{113}$$

   and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{114}$$

5. The Neumann eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{d\phi}{dx}(0) &= 0, \quad \frac{d\phi}{dx}(L) = 0 \end{aligned}$$

   has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{115}$$

   and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{116}$$

6. separation of variables:

$$u(x,t) = \phi(x)G(t). \tag{117}$$

7. Dirichlet boundary value problems:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{118}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{119}$$

$$u(x,0) = f(x). \tag{120}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

8. Neumann boundary value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{121}$$

$$\frac{\partial u}{dx}(0,t) = 0, \quad \frac{\partial u}{dx}(L,t) = 0, \tag{122}$$

$$u(x,0) = f(x). \tag{123}$$

has the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \tag{124}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{125}$$

9. Separation of variables

$$u(x,y) = \phi(x)h(y), \tag{126}$$

for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{127}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0, \quad u(x,0) = 0, \quad u(x,H) = f(x), \tag{128}$$

10. Review Exercises: 1.4.1 (d), 2.3.2(b), 2.3.3 (d), 2.5.1 (e)

11 Lecture 9 – Convergence of Fourier Series

Today:

- Piecewise smooth functions.
- Periodic extension.
- Convergence of Fourier Series

Next:

- Since and cosine series.

Review:

- The periodic eigenvalue problem

$$
\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi, \\
\phi(-L) &= \phi(L), \quad \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)
\end{aligned}
$$

  has the eigenvalues

$$
\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{129}
$$

  and the corresponding eigenfunctions

$$
\phi_0 = 1, \quad \phi_{2n-1} = \cos\left(\frac{n\pi x}{L}\right), \quad \phi_{2n} = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{130}
$$

Teaching procedure:

1. Orthogonality of the set of eigenfunctions

$$
\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cdots, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right), \cdots\right\}.
$$

   Proof. Let ϕ_1 and ϕ_2 be two different eigenfunctions corresponding to two different eigenvalues λ_1 and λ_2. Multiplying the equation

$$
-\frac{d^2\phi_1}{dx^2} = \lambda_1\phi_1
$$

   by ϕ_2 and integrating from $-L$ to L yield

$$
-\int_{-L}^{L} \phi_2 \frac{d^2\phi_1}{dx^2} dx = \lambda_1 \int_{-L}^{L} \phi_1\phi_2 dx.
$$

   Integration by parts with the periodic boundary conditions gives

$$
\int_{-L}^{L} \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} dx = \lambda_1 \int_{-L}^{L} \phi_1\phi_2 dx. \tag{131}
$$

Multiplying the equation

$$-\frac{d^2\phi_2}{dx^2} = \lambda_2\phi_2$$

by ϕ_1 and integrating from $-L$ to L yield

$$-\int_{-L}^{L} \phi_1 \frac{d^2\phi_2}{dx^2} dx = \lambda_2 \int_{-L}^{L} \phi_1\phi_2 dx.$$

Integration by parts with the periodic boundary conditions gives

$$\int_{-L}^{L} \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} dx = \lambda_2 \int_{-L}^{L} \phi_1\phi_2 dx. \tag{132}$$

Subtracting (132) from (131) gives

$$(\lambda_1 - \lambda_2) \int_{-L}^{L} \phi_1\phi_2 dx = 0,$$

which implies that

$$\int_{-L}^{L} \phi_1\phi_2 dx = 0.$$

So ϕ_1 is orthogonal to ϕ_2.

$\square$

2. **Fourier series** of $f(x)$ on $[-L, L]$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \tag{133}$$

Fourier coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx, \tag{134}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1, \tag{135}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{136}$$

Example.

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases}$$

Fourier series of $f(x)$ on $[\pi, \pi]$:

$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right].$$

3. **Piecewise smooth function**: $f(x)$ is piecewise function if the interval can be broken up into pieces such that in each piece $f'(x)$ is continuous.

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases}$$

4. **Periodic extension**.

5. **Fourier Theorem**. Let $f(x)$ be a piecewise smooth function on the interval $[-L, L]$ and $F(x)$ denote the periodic extension of f with period $2L$. Then the Fourier series of $f(x)$ converges to

$$\frac{1}{2}[f(x+) + f(x-)] \quad \text{if } -L < x < L$$

or to

$$\frac{1}{2}[F(x+) + F(x-)] \quad \text{if } x \leq -L \text{ or } x \geq L.$$

Example.

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases}$$

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right] = \begin{cases} \frac{\pi}{2}, & x = -\pi \\ f(x), & -\pi < x < \pi \\ \frac{\pi}{2}, & x = \pi \end{cases}$$

Taking $x = \pi$, we have the identity

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} (-1)^n = \frac{\pi}{2},$$

which can be simplified to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This provide a method to compute an approximate value of π.

6. Exercise 3.2.1 (f).

12 Lecture 10 – Fourier Since and cosine series

Today:

- Since and cosine series.

Next:

- Term-by-term differentiation.

Review:

- $\int_{-L}^{L} f(x)dx = 0$ if f is odd.
- $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$ if f is even.
- **Fourier series** of $f(x)$ on $[-L, L]$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{137}$$

  Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, & (138)\\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, & (139)\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. & (140) \end{aligned}$$

Teaching procedure:

1. **Fourier sine series**: If f is odd, that is, $f(-x) = -f(x)$, then

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = 0, & (141)\\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0, \quad n \geq 1, & (142)\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1. & (143) \end{aligned}$$

   So

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right). \tag{144}$$

   Example.

$$x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{145}$$

where

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1. \tag{146}$$

Sketch the series:

2. **Fourier cosine series**: If f is even, that is, $f(-x) = f(x)$, then

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L f(x)dx, && (147)\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, && (148)\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0, \quad n \geq 1. && (149)
\end{aligned}$$

So

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right). \tag{150}$$

Example.

$$x^2 \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{151}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L x^2 dx, && (152)\\
a_n &= \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, && (153)
\end{aligned}$$

Sketch the series:

3. Odd and even extension of a function.

$$\text{odd extension of } f(x) = \begin{cases} f(x), & x > 0 \\ -f(-x) & x < 0 \end{cases} \tag{154}$$

$$\text{even extension of } f(x) = \begin{cases} f(x), & x > 0 \\ f(-x) & x < 0 \end{cases} \tag{155}$$

Example. e^x. Sketch the extensions:

4. Comparison among Fourier series, Fourier since series, and Fourier cosine series.

- Fourier series of e^x on $[-L, L]$:

$$e^x \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \tag{156}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} e^x dx, \tag{157}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1, \tag{158}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} e^x \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{159}$$

Sketch the series:

- Fourier since series of e^x on $[0, L]$:

$$e^x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 \leq x \leq L, \tag{160}$$

$$\text{odd extension of } e^x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq L, \tag{161}$$

where

$$b_n = \frac{2}{L}\int_0^L e^x \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{162}$$

Sketch the series:

- Fourier cosine series of e^x on $[0, L]$:

$$e^x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad 0 \leq x \leq L, \tag{163}$$

$$\text{even extension of } \; e^x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad -L \leq x \leq L \tag{164}$$

where

$$a_0 = \frac{1}{L}\int_0^L e^x dx, \tag{165}$$

$$a_n = \frac{2}{L}\int_0^L e^x \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{166}$$

Sketch the series:

5. **Physical examples.**

- Fourier sine series for Dirichlet Boundary conditions:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{167}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{168}$$

$$u(x,0) = f(x). \tag{169}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- Fourier cosine series for Neumann boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & (170)\\ \frac{\partial u}{dx}(0,t) &= 0, \quad \frac{\partial u}{dx}(L,t) = 0, & (171)\\ u(x,0) &= f(x). & (172) \end{aligned}$$

has the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx, & (173)\\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. & (174) \end{aligned}$$

- Fourier series for periodic boundary condition:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{d^2 u}{dx^2}, & (175)\\ u(-L,t) &= u(L,t), \quad \frac{du}{dx}(-L,t) = \frac{du}{dx}(L,t), & (176)\\ u(x,0) &= f(x) & (177) \end{aligned}$$

has the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] e^{-\frac{kn^2\pi^2 t}{L^2}},$$

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \qquad (178)$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, & (179)\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n\geq 1, & (180)\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n\geq 1. & (181)
\end{aligned}
$$

6. **Even and odd parts of a function**.

   Even part of $f(x)$:

$$f_e(x) = \frac{1}{2}[f(x)+f(-x)].$$

   Odd part of $f(x)$:

$$f_o(x) = \frac{1}{2}[f(x)-f(-x)].$$

   Example: e^x.

7. **Numerical evidence for convergence of Fourier series**.

$$100 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

   where

$$c_n = \frac{2}{L}\int_0^L 100\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & n \text{ even} \\ \frac{400}{n\pi}, & n \text{ odd.}\end{cases}$$

   Set $L=1$:

$$100 = \frac{400}{\pi}\left(\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots\right).$$

8. Exercise 3.3.2 (b).

13 Lecture 11 – One-dimensional wave equation (Section 4.4)

Today:

- One-dimensional wave equation

Next:

- Two-dimensional wave equation (Section 7.3).

Review:

- The eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0 \end{aligned}$$

has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{182}$$

and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{183}$$

Teaching procedure:

1. Physical background of the wave equation

2. Dirichlet Initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{184}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{185}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial}t(x,0) = g(x). \tag{186}$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)h(t), \tag{187}$$

Substituting the above expression into the equation (184), we obtain

and then

$$\frac{h''(t)}{c^2 G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{188}$$

where λ is constant to be determined. The boundary condition (185) yields that

We then have an eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0, \end{aligned}$$

which has the eigenvalues

and the corresponding eigenfunctions

On the other hand, it follows from (188) that

$$\frac{d^2 h}{dt^2} = -\frac{c^2 n^2 \pi^2}{L^2} h, \tag{189}$$

which has solutions

we then have infinite series solution:

The initial condition gives

$$\begin{aligned} f(x) &= u(x,0) = \\ g(x) &= \frac{\partial u}{\partial t}(x,0) = \end{aligned}$$

where

$$\begin{aligned} a_n &= \\ b_n &= \end{aligned}$$

3. **Example**. Solve the following initial boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= 1, \quad \frac{\partial u}{\partial t}(x,0) = x. \end{aligned}$$

14 Lecture 12 – Higher-dimensional heat and wave equation (Section 7.3)

Today:

- Higher-dimensional heat equation
- Higher-dimensional wave equation

Next:

- Three-dimensional Laplace's equation (Section 7.3).

Review:

- The eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0 \end{aligned}$$

  has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{190}$$

  and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{191}$$

- The eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi'(0) &= 0, \quad \phi'(L) = 0 \end{aligned}$$

  has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{192}$$

  and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{193}$$

Teaching procedure:

1. Solve initial boundary value problem:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), && (194)\\
\frac{\partial u}{\partial x}(0,y,t) &= 0, \quad \frac{\partial u}{\partial x}(L,y,t)=0, \quad u(x,0,t)=0, \quad u(x,H,t)=0, && (195)\\
u(x,y,0) &= f(x,y). && (196)
\end{aligned}
$$

Look for a solution of the form of separation of variables:

$$u(x,y,t)=\phi(x,y)h(t). \tag{197}$$

Substituting the above expression into the equation (194), we obtain

and then

$$\frac{h'(t)}{kh(t)}=\frac{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}}{\phi(x,y)}=-\lambda, \tag{198}$$

where λ is constant to be determined. The boundary condition (196) yields that

We then have an eigenvalue problem

To solve this eigenvalue problem, we have to separate x and y again by letting

$$\phi(x,y)=\alpha(x)\beta(y).$$

Plugging it into the eigenvalue problems yields

and then

which gives two eigenvalue problems

The first one has the eigenvalues

and the corresponding eigenfunctions

The second one has the eigenvalues

and the corresponding eigenfunctions

Thus the original two-dimensional eigenvalue problem has the eigenvalues

and the corresponding eigenfunctions

On the other hand, we have

$$\frac{dh}{dt} = -k\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)h, \tag{199}$$

which has solutions

we then have infinite series solution:

The initial condition gives

$$f(x) \quad = \quad u(x,y,0) =$$

where

$$a_{mn} \quad =$$

2. Example. Solve initial boundary value problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= 0.01\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), && (200)\\
\frac{\partial u}{\partial x}(0,y,t) &= 0, \quad \frac{\partial u}{\partial x}(L,y,t)=0, \quad u(x,0,t)=0, \quad u(x,H,t)=0, && (201)\\
u(x,y,0) &= y(1-y)\sin(\pi x). && (202)
\end{aligned}$$

See Figure 1.

3. Solve Initial boundary value problem:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), && (203)\\
\frac{\partial u}{\partial x}(0,y,t) &= 0, \quad \frac{\partial u}{\partial x}(L,y,t)=0, \quad u(x,0,t)=0, \quad u(x,H,t)=0, && (204)\\
u(x,y,0) &= f(x,y), \quad \frac{\partial u}{\partial t}(x,y,0)=g(x,y). && (205)
\end{aligned}$$

Look for a solution of the form of separation of variables:

$$u(x,y,t)=\phi(x,y)h(t). \tag{206}$$

Substituting the above expression into the equation (203), we obtain

and then

$$\frac{h''(t)}{c^2h(t)}=\frac{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}}{\phi(x,y)}=-\lambda, \tag{207}$$

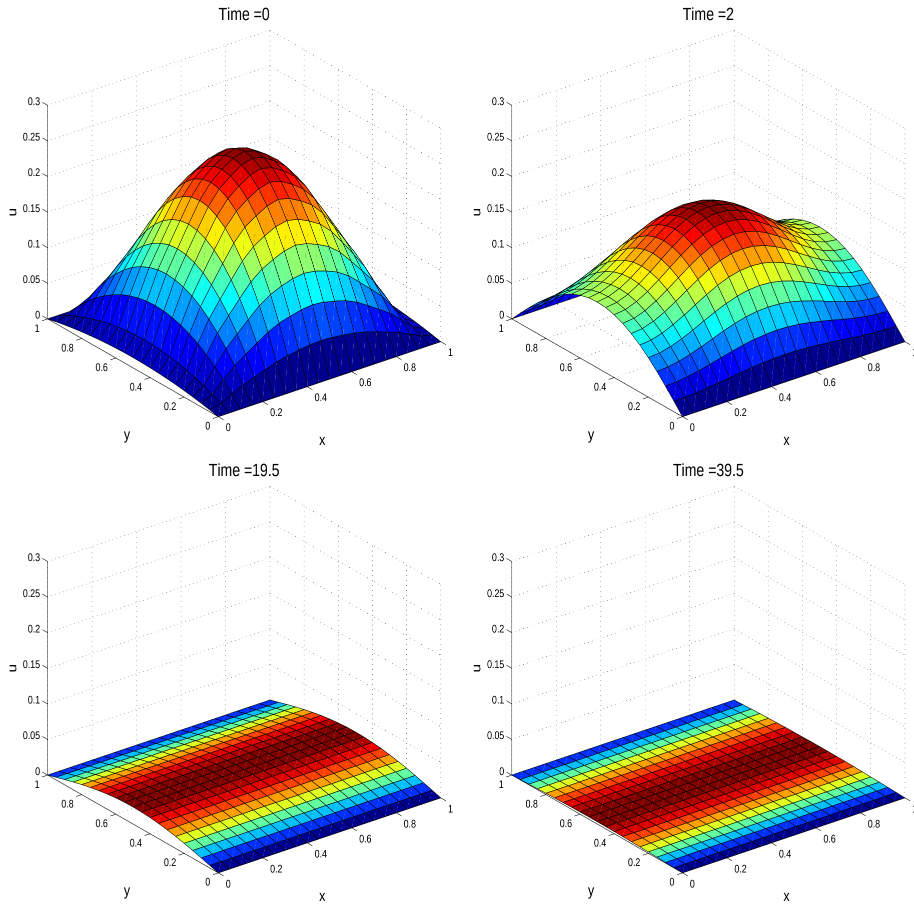

Figure 1: Solution of 2D heat equation (200).

where λ is constant to be determined. The boundary condition (205) yields that

We then have an eigenvalue problem

To solve this eigenvalue problem, we have to separate x and y again by letting

$$\phi(x, y) = \alpha(x)\beta(y).$$

Plugging it into the eigenvalue problems yields

and then

which gives two eigenvalue problems

The first one has the eigenvalues

and the corresponding eigenfunctions

The second one has the eigenvalues

and the corresponding eigenfunctions

Thus the original two-dimensional eigenvalue problem has the eigenvalues

and the corresponding eigenfunctions

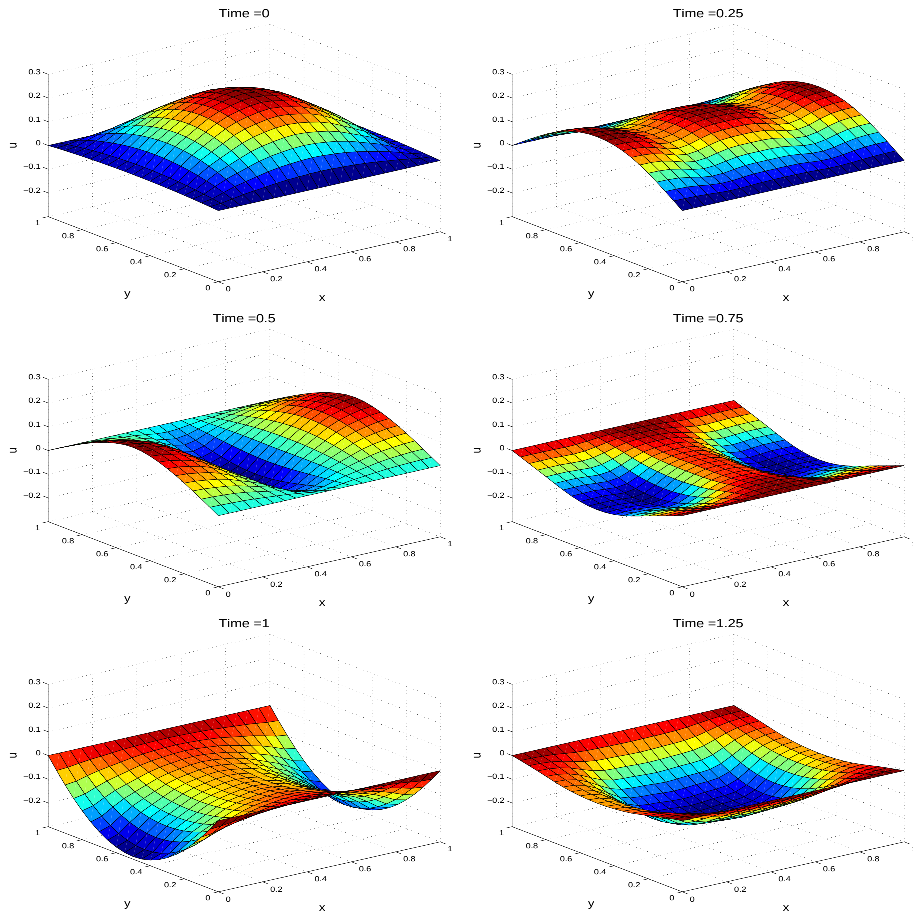

Figure 2: Solution of 2D wave equation (209).

On the other hand, we have

$$\frac{d^2h}{dt^2} = -c^2\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)h, \tag{208}$$

which has solutions

we then have infinite series solution:

The initial condition gives

$$\begin{aligned} f(x) &= u(x,y,0) = \\ g(x) &= \frac{\partial u}{\partial t}(x,y,0) = \end{aligned}$$

where

$$\begin{aligned} a_{mn} &= \\ b_{mn} &= \end{aligned}$$

4. Example. Solve the initial boundary valume problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{209}$$

$$\frac{\partial u}{\partial x}(0,y,t) = 0, \quad \frac{\partial u}{\partial x}(L,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,H,t) = 0, \tag{210}$$

$$u(x,y,0) = y(1-y)\sin(\pi x), \quad \frac{\partial u}{\partial t}(x,y,0) = y(1-y)\sin(\pi x). \tag{211}$$

See Figure 2.

15 Lecture 13 – Higher-dimensional eigenvalue problems (Section 7.4)

Today:

- Higher-dimensional eigenvalue problems

Next:

- Problem-solving (Assignment 6).

Teaching procedure:

1. Eigenvalue problem in rectanglar domain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi(x, y), \tag{212}$$

$$\frac{\partial \phi}{\partial x}(0, y) = \frac{\partial \phi}{\partial x}(L, y) = \phi(x, 0) = \phi(x, H) = 0, \tag{213}$$

has the eigenvalues

$$\lambda_{m,n} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}$$

and the corresponding eigenfunctions

$$\phi_{m,n} = \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{H}\right), m = 1, 2, \cdots, \quad n = 0, 1, 2, \cdots .$$

2. Eigenvalue problem in any domain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi(x, y), \tag{214}$$

$$\phi = 0 \quad \text{on the boundary} \tag{215}$$

See the result on page 290.

16 Main Technique Practice

1. Useful trig identies

$$\begin{aligned}
\sin u \sin v &= \frac{1}{2}[\cos(u-v) - \cos(u+v)], \\
\cos u \cos v &= \frac{1}{2}[\cos(u-v) + \cos(u+v)], \\
\sin u \cos v &= \frac{1}{2}[\sin(u-v) + \sin(u+v)].
\end{aligned}$$

2. Integrals:

$$\begin{aligned}
\int \sin ax dx &= \frac{-\cos ax}{a} + C, \\
\int \cos ax dx &= \frac{\sin ax}{a} + C.
\end{aligned}$$

3. The solution of $h' = ah$ is

4. The solution of $h'' = -a^2 h$ is

5. Compute

   - $\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx =$
   - $\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx =$
   - If $m \neq n$, then $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx =$
   - If $m \neq n$, then $\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx =$
   - $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx =$

6. If $f(x) = a_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots + a_n \sin\left(\frac{n\pi x}{L}\right) + \cdots$, then $a_n =$

7. If $f(x) = a_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots + a_n \cos\left(\frac{n\pi x}{L}\right) + \cdots$, then $a_n =$

8. Let $u(x,t) = \phi(x)h(t)$.

- If $u(0,t) = 0$, then $\phi(0) =$
- If $u(L,t) = 0$, then $\phi(L) =$
- If $\frac{\partial u}{\partial x}(0,t) = 0$, then $\frac{\partial \phi}{\partial x}(0) =$
- If $\frac{\partial u}{\partial x}(L,t) = 0$, then $\frac{\partial \phi}{\partial x}(L) =$
- $\frac{\partial u}{\partial t} =$
- $\frac{\partial^2 u}{\partial x^2} =$
- If $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, then the ϕ equation and h equation are
- If $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, then the ϕ equation and h equation are

9. Let $\phi(x,y) = \alpha(x)\beta(y)$.

- If $\phi(0,y) = 0$, then $\alpha(0) =$
- If $\phi(L,y) = 0$, then $\alpha(L) =$
- If $\frac{\partial \phi}{\partial x}(0,y) = 0$, then $\frac{\partial \alpha}{\partial x}(0) =$
- If $\frac{\partial \phi}{\partial x}(L,y) = 0$, then $\frac{\partial \alpha}{\partial x}(L) =$
- If $\phi(x,0) = 0$, then $\beta(0) =$
- If $\phi(x,H) = 0$, then $\beta(H) =$
- If $\frac{\partial \phi}{\partial y}(x,0) = 0$, then $\frac{\partial \beta}{\partial y}(0) =$
- If $\frac{\partial \phi}{\partial y}(x,H) = 0$, then $\frac{\partial \beta}{\partial y}(H) =$
- $\frac{\partial^2 \phi}{\partial x^2} =$
- $\frac{\partial^2 \phi}{\partial y^2} =$
- If $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda\phi$, then the α equation and β equation are

17 Lecture 14 – Nonhomogeneous Problems for the Heat Equation (Section 8.2)

Today:

- Time-independent boundary conditions
- Steady nonhomogeneous equation
- Time-depedendent nonhomogeneous terms

Next:

- Nonhomogeneous Problems for the Wave Equation.

Review:

- Dirichlet boundary value problems:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & (216)\\ u(0,t) &= 0, \quad u(L,t) = 0, & (217)\\ u(x,0) &= f(x). & (218) \end{aligned}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Teaching procedure:

1. Time-independent boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & (219)\\ u(0,t) &= A, \quad u(L,t) = B, & (220)\\ u(x,0) &= f(x). & (221) \end{aligned}$$

2. Steady-state heat equation

$$\begin{aligned} k\frac{d^2 u_e}{dx^2} &= 0, & (222)\\ u_e(0) &= A, \quad u_e(L) = B. & (223) \end{aligned}$$

has the unique solution

$$u_e(x) = A + \frac{B-A}{L}x.$$

3. Conversion of the nonhomogeneous to the homogeneous: Define $v(x,t) = u(x,t) - u_e(x)$. Then v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2}, && (224)\\ v(0,t) &= 0, \quad v(L,t) = 0, && (225)\\ v(x,0) &= f(x) - u_e(x). && (226) \end{aligned}$$

which has a unique solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

where

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_e(x)] \sin\left(\frac{n\pi x}{L}\right) dx.$$

So the solution of the original nonhomogeneous equation is

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

4. Steady nonhomogeneous equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x), && (227)\\ u(0,t) &= A, \quad u(L,t) = B, && (228)\\ u(x,0) &= f(x). && (229) \end{aligned}$$

5. Steady-state heat equation

$$\begin{aligned} k\frac{d^2 u_e}{dx^2} &= -Q(x), && (230)\\ u_e(0) &= A, \quad u_e(L) = B. && (231) \end{aligned}$$

has the unique solution $u_e(x)$.

6. Conversion of the nonhomogeneous to the homogeneous: Define $v(x,t) = u(x,t) - u_e(x)$. Then v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2}, && (232)\\ v(0,t) &= 0, \quad v(L,t) = 0, && (233)\\ v(x,0) &= f(x) - u_e(x). && (234) \end{aligned}$$

which has a unique solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

where

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_e(x)] \sin\left(\frac{n\pi x}{L}\right) dx.$$

So the solution of the original nonhomogeneous equation is

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

7. Example. If $Q(x) = k\sin x$, then $u_e(x) = A + \frac{B-A-\sin L}{L}x + \sin x$.

8. Time-dependent nonhomogeneous problems:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \tag{235}$$

$$u(0,t) = A(t), \quad u(L,t) = B(t), \tag{236}$$

$$u(x,0) = f(x). \tag{237}$$

9. Find a reference temperature distribution $r(x,t)$ such that

$$r(0,t) = A(t), \quad r(L,t) = B(t).$$

For example,

$$r(x,t) = A(t) + \frac{B(t) - A(t)}{L}x.$$

10. Conversion of the nonhomogeneous to the homogeneous: Define $v(x,t) = u(x,t) - r(x,t)$. Then v satisfies

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}, \tag{238}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{239}$$

$$v(x,0) = f(x) - r(x,0). \tag{240}$$

11. Exercises 8.2.1. (c).

18 Lecture 15 – Nonhomogeneous Problems for the Wave Equation and Eigenfunction expansion (Section 8.3)

Today:

- Nonhomogeneous Problems for the Wave Equation
- Eigenfunction expansion

Next:

- Eigenfunction expansion.

Review:

- Dirichlet boundary value problems:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, &(241)\\ u(0,t) &= 0, \quad u(L,t) = 0, &(242)\\ u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). &(243) \end{aligned}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

where

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

Teaching procedure:

1. Time-independent nonhomogeneous problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} + Q(x), &(244)\\ u(0,t) &= A, \quad u(L,t) = B, &(245)\\ u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). &(246) \end{aligned}$$

2. Steady-state wave equation

$$c^2 \frac{d^2 u_e}{dx^2} = -Q(x), \tag{247}$$

$$u_e(0) = A, \quad u_e(L) = B. \tag{248}$$

has the unique solution $u_e(x)$.

3. Conversion of the nonhomogeneous to the homogeneous: Define $v(x,t) = u(x,t) - u_e(x)$. Then v satisfies

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \tag{249}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{250}$$

$$v(x,0) = f(x) - u_e(x), \quad \frac{\partial v}{\partial t}(x,0) = g(x). \tag{251}$$

which has a unique solution

$$v(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

where

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_e(x)] \sin\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

So the solution of the original nonhomogeneous equation is

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

4. Time-dependent nonhomogeneous problems:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t), \tag{252}$$

$$u(0,t) = A(t), \quad u(L,t) = B(t), \tag{253}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{254}$$

5. Find a reference vibration $r(x,t)$ such that

$$r(0,t) = A(t), \quad r(L,t) = B(t).$$

For example,

$$r(x,t) = A(t) + \frac{B(t) - A(t)}{L} x.$$

6. Conversion of the nonhomogeneous to the homogeneous: Define $v(x,t) = u(x,t) - r(x,t)$. Then v satisfies

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= c^2\frac{\partial^2 v}{\partial x^2} + Q(x,t) - \frac{\partial^2 r}{\partial t^2} + c^2\frac{\partial^2 r}{\partial x^2}, && (255)\\
v(0,t) &= 0, \quad v(L,t) = 0, && (256)\\
v(x,0) &= f(x) - r(x,0), \quad \frac{\partial v}{\partial t}(x,0) = g(x) - \frac{\partial r}{\partial t}(x,0). && (257)
\end{aligned}$$

7. Eigenfunction expansion

$$\begin{aligned}
\frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} + Q(x,t), && (258)\\
v(0,t) &= 0, \quad v(L,t) = 0, && (259)\\
v(x,0) &= f(x). && (260)
\end{aligned}$$

The corresponding eigenvalue problem is

$$\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi, && (261)\\
\phi(0) &= 0, \quad \phi(L) = 0. && (262)
\end{aligned}$$

Assume that $v(x,t)$ can be expanded as the series of the eigenfunctions

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Plugging it into the heat equation gives

$$\sum_{n=1}^{\infty} a_n'(t)\phi_n(t) = k\sum_{n=1}^{\infty} a_n(t)\phi_n''(x) + Q(x,t) = -k\sum_{n=1}^{\infty}\lambda_n a_n(t)\phi_n(x) + Q(x,t).$$

Multiplying the equation by $\phi_m(x)$ and integrating from 0 to L gives

$$a_m'(t) = -k\lambda_m a_m(t) + \frac{\int_0^L Q(x,t)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

To get the initial condition for a_m, we use the initial condition $v(x,0) = f(x)$ to get

$$f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

So

$$a_m(0) = \frac{\int_0^L f(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}, \quad m = 1, 2, \cdots,$$

Once solving the initial value problem for a_m, we obtain the solution $v(x,t)$.

8. Example. Solve

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + t, \tag{263}$$

$$v(0,t) = 0, \quad v(\pi,t) = 0, \tag{264}$$

$$v(x,0) = \sin x. \tag{265}$$

19 Review of Fourier Series and 1D Wave Equation

19.1 Main Topics

1. The basic techniques in the hand-out 16 that must be known.

2. Orthogonality of the set of eigenfunctions

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cdots, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right), \cdots\right\}.$$

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \cdots, \cos\left(\frac{n\pi x}{L}\right), \cdots\right\}.$$

$$\left\{\sin\left(\frac{\pi x}{L}\right), \cdots, \sin\left(\frac{n\pi x}{L}\right), \cdots\right\}.$$

3. **Fourier series** of $f(x)$ on $[-L, L]$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{266}$$

Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \tag{267}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, \tag{268}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. \tag{269}$$

4. **Fourier sine series**: If f is odd, that is, $f(-x) = -f(x)$, then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = 0, \tag{270}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0, \quad n \geq 1, \tag{271}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1. \tag{272}$$

So

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right). \tag{273}$$

5. **Fourier cosine series**: If f is even, that is, $f(-x) = f(x)$, then

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx, && (274)\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1, && (275)\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0, \quad n \geq 1. && (276)
\end{aligned}
$$

So

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right). \tag{277}$$

6. **Fourier Convergence Theorem**.

7. Calculation of Fourier series.

8. Sketch of Fourier series.

9. How to solve Dirichlet initial boundary value problem:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, && (278)\\
u(0,t) &= 0, \quad u(L,t) = 0, && (279)\\
u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). && (280)
\end{aligned}
$$

10. How to solve Neumann initial boundary value problem:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, && (281)\\
\frac{\partial u}{\partial x}(0,t) &= 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, && (282)\\
u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). && (283)
\end{aligned}
$$

19.2 Review Exercises

Exercises 3.2: 3.2.2. (b), (d); Exercises 3.3: 3.3.1. (b), 3.3.2 (c); 3.3.5 (a).

Solve Dirichlet initial boundary value problem:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, && (284)\\
u(0,t) &= 0, \quad u(1,t) = 0, && (285)\\
u(x,0) &= x(1-x), \quad \frac{\partial u}{\partial t}(x,0) = 0. && (286)
\end{aligned}
$$

Solve Neumann initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{287}$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \tag{288}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 1. \tag{289}$$

Evaluate

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

20 Lecture 16 – Eigenfunction expansion for the wave equation

Today:

- Eigenfunction expansion for the wave equation

Next:

- Characteristics for first-order wave equations (Section 12.2).

Review:

- Eigenfunction expansion of the heat equation

$$\begin{aligned}
\frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} + Q(x,t), && (290)\\
v(0,t) &= 0, \quad v(L,t) = 0, && (291)\\
v(x,0) &= f(x). && (292)
\end{aligned}$$

The solution $v(x,t)$ can be expanded as the series of the eigenfunctions

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $\lambda_n = \frac{n^2\pi^2}{L^2}$ and

$$a_n'(t) = -k\lambda_n a_n(t) + \frac{\int_0^L Q(x,t)\phi_n(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

$$a_n(0) = \frac{\int_0^L f(x)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}, \quad n = 1, 2, \cdots,$$

Teaching procedure:

1. Solve

$$a_n''(t) = -c^2 n^2 a_n(t) + bt.$$

$$a_n(0) = b_1, \quad a_n'(0) = b_2.$$

2. Eigenfunction expansion for

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + Q(x,t), \tag{293}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{294}$$

$$v(x,0) = f(x), \quad \frac{\partial v}{\partial t}(x,0) = g(x). \tag{295}$$

The corresponding eigenvalue problem is

$$-\frac{d^2\phi}{dx^2} = \lambda\phi, \tag{296}$$

$$\phi(0) = 0, \quad \phi(L) = 0, \tag{297}$$

which has the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Assume that $v(x,t)$ can be expanded as the series of the eigenfunctions

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

Plugging it into the wave equation gives

$$\sum_{n=1}^{\infty} a_n''(t)\phi_n(t) = c^2 \sum_{n=1}^{\infty} a_n(t)\phi_n''(x) + Q(x,t) = -c^2 \sum_{n=1}^{\infty} \lambda_n a_n(t)\phi_n(x) + Q(x,t).$$

Multiplying the equation by $\phi_m(x)$ and integrating from 0 to L gives

$$a_m''(t) = -c^2\lambda_m a_m(t) + \frac{\int_0^L Q(x,t)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

To get the initial condition for a_m, we use the initial condition $v(x,0) = f(x)$ and $\frac{\partial v}{\partial t}(x,0) = g(x)$ to get

$$f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x),$$

$$g(x) = \sum_{n=1}^{\infty} a_n'(0)\phi_n(x).$$

So

$$a_m(0) = \frac{\int_0^L f(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}, \quad m = 1, 2, \cdots,$$

$$a_m'(0) = \frac{\int_0^L g(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}, \quad m = 1, 2, \cdots,$$

Once solving the initial value problem for a_m, we obtain the solution $v(x,t)$.

3. Example. Solve

$$
\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= 4\frac{\partial^2 v}{\partial x^2} + t, && (298)\\
v(0,t) &= 0, \quad v(\pi,t) = 0, && (299)\\
v(x,0) &= \sin x, \quad \frac{\partial v}{\partial t}(x,0) = \sin(x). && (300)
\end{aligned}
$$

21 Lecture 17 – Method of Characteristics for the first order wave equation (Section 12.2 and Chapter 2 of Dr. Arrigo's book)

Today:

- Method of Characteristics for the first order wave equation

Next:

- Characteristics for first-order quasi-linear equations (Section 12.2).

Review:

- Normal vector of a surface.
- parametric equations.
- Chain rule for partial derivatives

Teaching procedure:

1. **Solution of The first-order wave equation**

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, && (301) \\ u(x,0) &= f(x). && (302) \end{aligned}$$

Please try the method of separation of variable to see why the method does not work for this so simple equation.

The idea of solving this equation is to reduce the PDE to an ODE. For this we construct a curve in the $x-t$ plane

$$x = x(s), \quad t = t(s).$$

Then consider the PDE along this curve by setting

$$z(s) = u(x(s), t(s)).$$

To construct such a curve, we compute

$$\frac{dz}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds}.$$

If we take

$$\frac{dx}{ds} = c, \quad \frac{dt}{ds} = 1,$$

we obtain

$$\frac{dz}{ds} = 0,$$

a simple ODE we can solve easily! Before we solve it, let us add initial conditions to these equations. From the initial condition $u(x, 0) = f(x)$, we can see $x(0)$ can be at any point on the x-axis and $t(0)$ is always equal to 0. So we have $x(0) = r$ (any number) and $t(0) = 0$. Then $z(0) = u(x(0), t(0)) = u(r, 0) = f(r)$. We then have the following initial value problem

$$\frac{dx}{ds} = c, \quad \frac{dt}{ds} = 1, \quad \frac{dz}{ds} = 0,$$

$$x(0) = r, \quad t(0) = 0, \quad z(0) = f(r).$$

Solving it gives

$$x = cs + r, \quad t = s, \quad z(s) = f(r),$$

and then

$$z(s) = f(x - cs) = f(x - ct).$$

So we obtain the solution

$$u(x, t) = z = f(x - ct).$$

2. **Important notions**.

- **Characteristic curve** (line): for any fixed r, the parametric equation

  $$x = cs + r, \quad t = s, \quad z(s) = f(r),$$

  gives a curve (straight line in this case) in space, called **Characteristic curve**.
- **Characteristic equations**:

  $$\frac{dx}{ds} = c, \quad \frac{dt}{ds} = 1, \quad \frac{dz}{ds} = 0.$$

- **Characteristic method**: The method of using **Characteristic equations** to solve first PDEs is called **Characteristic method**.
- The **Domain of dependence** of $u(x, t)$ on the initial values is the single point $r = x - ct$.

- The **Range of Influence** of the initial values at a particular point r is the characteristic line $x = ct + r$.

3. **Geometrical interpretation** of the first-order wave equation. The surface $z = u(x,t)$ in xtz-space corresponding to solutions of the PDE is called **integral surface**. The equation implies that the vector $\mathbf{v} = (c, 1, 0)$, called **characteristic vector**, is perpendicular to the normal vector of the surface $\mathbf{n} = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, -1)$.

4. Example. Solve

$$\begin{aligned} \frac{\partial u}{\partial t} + 4\frac{\partial u}{\partial x} &= 0, \\ u(x,0) &= \cos x. \end{aligned}$$

22 Lecture 18 – Method of Characteristics for the first-order quasi-linear equations (Section 12.6.1, handouts, and Chapter 2 of Dr. Arrigo's book)

Today:

- Method of Characteristics for first-order quasi-linear equations

Next:

- Characteristics for the second-order wave equations (Section 12.3).

Review:

- Method of Characteristics for the first-order wave equation

Teaching procedure:

1. **First-order quasi-linear equations**

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u), \tag{303}$$

$$u(x,0) = f(x). \tag{304}$$

Consider the curve $x = x(s), y = y(s)$ defined by the equations

$$\frac{dx}{ds} = a(x,y,u), \quad \frac{dy}{ds} = b(x,y,u).$$

Set $z(s) = u(x(s), y(s))$. Then along this curve, we have

$$\frac{dz}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial x} = c(x,y,u) = c(x,y,z).$$

Thus we obtain **Characteristic equations**:

$$\frac{dx}{ds} = a(x,y,z), \quad \frac{dy}{ds} = b(x,y,z), \quad \frac{dz}{ds} = c(x,y,z)$$

with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad z(0) = f(r).$$

Solving it gives

$$x = x(s,r), \quad y = y(s,r), \quad z = z(s,r).$$

The curve of this parametric equation for any fixed r is called **Characteristic curve**. Solving the first two equation

$$x = x(s,r), \quad y = y(s,r)$$

for s, r gives

$$s = s(x,y), \quad r = r(x,y).$$

We then obtain the solution

$$u = z = z(s(x,y), r(x,y)).$$

2. **Geometrical interpretation**. The surface $z = u(x, y)$ in xyz-space corresponding to solutions of the PDE is called **integral surface**. The equation implies that the vector $\mathbf{v} = (a(x, y, u), b(x, y, u), c(x, y, u))$, called **characteristic vector**, is perpendicular to the normal vector of the surface $\mathbf{n} = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1)$.

3. Example 1. Solve

$$\begin{aligned} y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 1, \\ u(x, 0) &= \cos x. \end{aligned}$$

The Characteristic equations are

$$\frac{dx}{ds} = y, \quad \frac{dy}{ds} = 1, \quad \frac{dz}{ds} = 1$$

with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad z(0) = \cos(r).$$

The solution is

$$x = \frac{1}{2}s^2 + r, \quad y = s, \quad z = s + \cos(r).$$

So the solution of the PDE is

$$u = z = y + \cos\left(x - \frac{1}{2}y^2\right).$$

4. Example 2. Solve

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= u^2, \\ u(x, 0) &= e^x. \end{aligned}$$

The Characteristic equations are

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 1, \quad \frac{dz}{ds} = z^2$$

with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad z(0) = e^r.$$

The solution is

$$x = s + r, \quad y = s, \quad z = \frac{1}{e^{-r} - s}.$$

So the solution of the PDE is

$$u = z = \frac{1}{e^{y-x} - y}.$$

23 Exercises of Characteristics

Use the method of characteristics to solve

(1)

$$\begin{aligned} y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} &= xu, \\ u(x,0) &= e^{-x^2}. \end{aligned}$$

(2)

$$\begin{aligned} x\frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial y} &= u, \\ u(x,0) &= x^2. \end{aligned}$$

(1) **Solution**. The Characteristic equations are

$$\frac{dx}{ds} = y, \quad \frac{dy}{ds} = x, \quad \frac{dz}{ds} = xz$$

with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad z(0) = e^{-r^2}.$$

We then have

$$\frac{dy}{dx} = \frac{x}{y}.$$

$$ydy = xdx.$$

$$y^2 = x^2 + c.$$

The initial condition gives $c = -r^2$. So

$$y^2 = x^2 - r^2.$$

Also

$$\frac{dz}{dy} = \frac{xz}{x} = z.$$

So $z = e^{y-r^2}$. We then have the solution

$$u = e^{y+y^2-x^2}.$$

(2) **Solution**. The Characteristic equations are

$$\frac{dx}{ds} = x, \quad \frac{dy}{ds} = 2z, \quad \frac{dz}{ds} = z$$

with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad z(0) = r^2.$$

Solving it gives

$$x = re^s, \quad z = r^2e^s, \quad y = 2r^2(e^s - 1).$$

We then have

$$\frac{z}{x} = \frac{r^2e^s}{re^s} = r.$$

$$e^s = \frac{x}{r} = \frac{x}{\frac{z}{x}} = \frac{x^2}{z}.$$

Thus the solution is

$$y = 2\frac{z^2}{x^2}\left(\frac{x^2}{z} - 1\right).$$

24 Lecture 19 – Reduction of Second Order Equations to Canonical Form (Chapter 3 of Dr. Arrigo's book)

Today:

- Type of second order equations
- Canonical form
- Reduction of hyperbolic equations

Next:

- Reduction of parabolic equations
- Reduction of elliptic equations

Review:

- chain rule of partial derivative
- Classification of conic section of the form:

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C are constant. It is
a **hyperbola** if $B^2 - AC > 0$,
a **parabola** if $B^2 - AC = 0$,
an **ellipse** if $B^2 - AC < 0$.

Teaching procedure:

1. Classification of second order equations:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y), \tag{305}$$

where A, B, C are given functions of x, y. It is said to be
hyperbolic if $B^2 - AC > 0$,
parabolic if $B^2 - AC = 0$,
elliptic if $B^2 - AC < 0$.

2. **Canonical form** of second order equations:

   - Hyperbolic Canonical form

$$\frac{\partial^2 u}{\partial x \partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y),$$

or

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y),$$

- Parabolic Canonical form

$$\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y),$$

- Elliptic Canonical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y),$$

3. **Reduction of hyperbolic equations** ($B^2 - AC > 0$). We now want to find a change of variables

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

to reduce the hyperbolic equation to its canonical form. We assume that the inverse transform of the above

$$x = x(\xi,\eta), \quad y = y(\xi,\eta)$$

exits. Then

$$u(x,y) = u(x(\xi,\eta), y(\xi,\eta)) = u(\xi,\eta).$$

To find the above desired change of variables, we now compute derivatives of u and then substitute them into the equation (305).

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}, \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}, \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}.
\end{aligned}$$

Substituting them into the equation (305) gives

$$A_1\frac{\partial^2 u}{\partial \xi^2} + 2B_1\frac{\partial^2 u}{\partial \xi \partial \eta} + C_1\frac{\partial^2 u}{\partial \eta^2} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial}\eta\right) = 0, \tag{306}$$

where

$$\begin{aligned}
A_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y}, \\
B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}, \\
C_1 &= A\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial y}.
\end{aligned}$$

To reduce the equation (306) to the canonical form, we need to choose ξ and η such that $A_1 = C_1 = 0$. This motivate us to consider the equation

$$A\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x} + 2B\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + C\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y} = 0. \tag{307}$$

Dividing the equation by $\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y}$ gives

$$A\left(\frac{\partial \phi}{\partial x}/\frac{\partial \phi}{\partial y}\right)^2 + 2B\frac{\partial \phi}{\partial x}/\frac{\partial \phi}{\partial y} + C = 0.$$

Solving the equation gives

$$\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

We then obtain two equations

$$A\frac{\partial \phi}{\partial x} + (B + \sqrt{B^2 - AC})\frac{\partial \phi}{\partial y} = 0, \tag{308}$$

and

$$A\frac{\partial \phi}{\partial x} + (B - \sqrt{B^2 - AC})\frac{\partial \phi}{\partial y} = 0. \tag{309}$$

Let $\phi_1(x, y)$ and $\phi_2(x, y)$ are two solutions of these two equation, respectively. If we choose

$$\xi = \phi_1(x, y), \quad \eta = \phi_2(x, y),$$

then we have $A_1 = C_1 = 0$ because $\phi_1(x, y)$ and $\phi_2(x, y)$ satisfy the equation (307).

The equation (307) is called the **characteristic equation** of (305) and the curves on the xy plane defined by $\phi_1(x, y) =$ constant and $\phi_2(x, y) =$ constant is called the **characteristic curve** of (305).

4. **Example**. Determine the type of each of the following equation, reduce it to canonical form, and then find its general solution.

$$3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0.$$

   (a) $A = C = 3$ and $B = 5$. So $B^2 - AC = 25 - 9 = 16 > 0$. Thus the equation is hyperbolic.

   (b) Solving

$$3\frac{\partial \phi}{\partial x} + 9\frac{\partial \phi}{\partial y} = 0,$$

   and

$$3\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

we obtain

$$\xi = \phi_1 = y - 3x, \quad \eta = \phi_2 = y - \frac{1}{3}x.$$

Then we have

$$\begin{aligned}
A_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y} \\
&= 3 \cdot (-3)^2 + 10 \cdot (-3) + 3 \\
&= 0, \\
B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 3 \cdot (-3)/(-3) + 5(-3 - 1/3) + 3 \\
&= -32/5, \\
C_1 &= A\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 3(-1/3)^2 - 10/3 + 3 \\
&= 0.
\end{aligned}$$

So the canonical form is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

(c) Integrating the above canonical equation with respect to η, we obtain

$$\frac{\partial u}{\partial \xi} = f(\xi).$$

Integrating it with respect to ξ gives the general solution

$$u = F(\xi) + G(\eta) = F(y - 3x) + G(y - x/3).$$

5. **Reduction of parabolic equations** ($B^2 = AC$). In this case, equations (308) and (309) become one equation
$$A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y} = 0.$$
Let $\phi(x, y)$ be the solution of the equation and $\eta(x, y)$ be any other twice differentiable function. If we choose
$$\xi = \phi(x, y), \quad \eta = \eta(x, y),$$
then we have $A_1 = 0$ because $\phi(x, y)$ satisfies the equation (307). Also
$$\begin{aligned}
B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
&= \left(A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y}\right)\frac{\partial \eta}{\partial x} + \left(B\frac{\partial \phi}{\partial x} + C\frac{\partial \phi}{\partial y}\right)\frac{\partial \eta}{\partial y} \\
&= \left(B\frac{\partial \phi}{\partial x} + \frac{B^2}{A}\frac{\partial \phi}{\partial y}\right)\frac{\partial \eta}{\partial y} \\
&= \frac{B}{A}\left(A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y}\right)\frac{\partial \eta}{\partial y} \\
&= 0.
\end{aligned}$$

6. **Example**. Determine the type of each of the following equation, reduce it to canonical form, and then find its general solution.
$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$
   (a) $A = C = 1$ and $B = 1$. So $B^2 - AC = 1 - 1 = 0$. Thus the equation is parabolic.
   (b) Solving
   $$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0,$$
   we obtain
   $$\xi = \phi = x - y$$
   Let
   $$\eta = x.$$

Then we have

$$\begin{aligned}
A_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y} \\
&= 1^2 + 2\cdot(-1) + (-1)^2 \\
&= 0, \\
B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 1 + (-1) + 0 \\
&= 0, \\
C_1 &= A\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 1.
\end{aligned}$$

So the canonical form is

$$\frac{\partial^2 u}{\partial \eta^2} = 0.$$

(c) Integrating the above canonical equation with respect to η, we obtain

$$\frac{\partial u}{\partial \eta} = f(\xi).$$

Integrating it with respect to η gives the general solution

$$u = \eta f(\xi) + g(\xi) = x f(x-y) + g(x-y).$$

7. **Reduction of elliptic equations** ($B^2 - AC < 0$). In this case, equations (308) and (309) become complex equation

$$A\frac{\partial \phi}{\partial x} + (B + i\sqrt{AC - B^2})\frac{\partial \phi}{\partial y} = 0.$$

Let $\phi(x,y) = \phi_1(x,y) + i\phi_2(x,y)$ be the complex solution of the equation and substitute it into the equation (307). We then obtain

$$A\frac{\partial(\phi_1 + i\phi_2)}{\partial x}\frac{\partial(\phi_1 + i\phi_2)}{\partial x} + 2B\frac{\partial(\phi_1 + i\phi_2)}{\partial x}\frac{\partial(\phi_1 + i\phi_2)}{\partial y} + C\frac{\partial(\phi_1 + i\phi_2)}{\partial y}\frac{\partial(\phi_1 + i\phi_2)}{\partial y} = 0.$$

Separating the real and imaginary parts gives

$$A\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_1}{\partial x} + 2B\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_1}{\partial y} + C\frac{\partial \phi_1}{\partial y}\frac{\partial \phi_1}{\partial y} = A\frac{\partial \phi_2}{\partial x}\frac{\partial \phi_2}{\partial x} + 2B\frac{\partial \phi_2}{\partial x}\frac{\partial \phi_2}{\partial y} + C\frac{\partial \phi_2}{\partial y}\frac{\partial \phi_2}{\partial y}$$

and

$$A\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} + B\left(\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_1}{\partial y}\frac{\partial \phi_2}{\partial x}\right) + C\frac{\partial \phi_1}{\partial y}\frac{\partial \phi_2}{\partial y} = 0$$

So we can choose

$$\xi = \phi_1(x,y), \quad \eta = \phi_2(x,y),$$

then we have $A_1 = C_1$ and $B_1 = 0$.

8. **Example**. Determine the type of each of the following equation, reduce it to canonical form, and then find its general solution if possible.

$$5\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(a) $A = 5, C = 1$ and $B = 2$. So $B^2 - AC = 4 - 5 = -1$. Thus the equation is elliptic.

(b) Solving

$$5\frac{\partial \phi}{\partial x} + (2+i)\frac{\partial \phi}{\partial y} = 0,$$

we obtain

$$\phi = x - 2y + iy$$

Let

$$\xi = x - 2y, \quad \eta = y.$$

Then we have

$$\begin{aligned}
A_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y} \\
&= 5 + 4 \cdot (-2) + (-2)^2 \\
&= 1, \\
B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 5 \cdot 0 + 2 - 2 \\
&= 0, \\
C_1 &= A\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x} + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\frac{\partial \eta}{\partial y}\frac{\partial \eta}{\partial y} \\
&= 1.
\end{aligned}$$

So the canonical form is

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0.$$

(c) Can we find its general solution?

25 Lecture 20 – Method of Characteristics for the one-dimensional wave equation)

Today:

- General solutions
- initial value problems
- Initial boundary value problems

Next:

- Review

Review:

- Characteristic equation

$$A\frac{\partial \phi}{\partial x} + (B + \sqrt{B^2 - AC})\frac{\partial \phi}{\partial y} = 0.$$

$$A\frac{\partial \phi}{\partial x} + (B - \sqrt{B^2 - AC})\frac{\partial \phi}{\partial y} = 0.$$

Teaching procedure:

1. General solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Solving

$$c\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial t} = 0$$

$$c\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

gives

$$\xi = x - ct, \quad \eta = x + ct$$

. Then the wave equation reduces to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

Integrating twice gives the general solution

$$u = F(x - ct) + G(x + ct).$$

2. Initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Initial condition gives

$$\begin{aligned} f(x) &= F(x) + G(x), \\ g(x) &= -cF'(x) + cG'(x). \end{aligned}$$

Integrating the second equation gives

$$\begin{aligned} F(x) + G(x) &= f(x), \\ -F(x) + G(x) &= \frac{1}{c}\int_0^x g(z)dz + k. \end{aligned}$$

Solving it gives

$$\begin{aligned} F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)dz - \frac{k}{2}, \\ G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)dz + \frac{k}{2}. \end{aligned}$$

and then **d'Alembert's solution**

$$\begin{aligned} u(x,t) &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\left[\int_0^{x+ct} g(z)dz - \int_0^{x-ct} g(z)dz\right] \\ &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz, \end{aligned}$$

3. Example. Solve the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = x.$$

Solution.

$$\begin{aligned} u(x,t) &= \frac{x+t+x-t}{2} + \frac{1}{2}\int_{x-t}^{x+t} zdz \\ &= x + \frac{1}{4}\left[(x+t)^2 - (x+t)^2\right], \end{aligned}$$

4. **An important function equation**: If

$$u(x,t) = F(x-ct) + G(x+ct).$$

then

$$u(x_0,t_0) - u(x_0+c\xi, t_0+\xi) - u(x_0-c\eta, t_0+\eta) + u(x_0+c\xi-c\eta, t_0+\xi+\eta) = 0$$

for any x_0 and t_0. Geometrically, for any parallelogram $A(x_0,t_0), B(x_0+c\xi, t_0+\xi), C(x_0+c\xi-c\eta, t_0+\xi+\eta), D(x_0-c\eta, t_0+\eta)$ we have

$$u(A) + u(C) = u(B) + u(D). \tag{310}$$

5. Example. Solve the initial boundary value problem using (310)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = 0, \quad u(\pi,t) = 1,$$
$$u(x,0) = 1, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

26 Review of Nonhomogeneous Problems

1. Procedure for solving Steady nonhomogeneous equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x), \tag{311}$$
$$u(0,t) = A, \quad u(L,t) = B, \tag{312}$$
$$u(x,0) = f(x). \tag{313}$$

   (a) Find Steady-state solution

$$k\frac{d^2 u_e}{dx^2} = -Q(x), \tag{314}$$
$$u_e(0) = A, \quad u_e(L) = B. \tag{315}$$

   (b) Convert the nonhomogeneous to the homogeneous by defining $v(x,t) = u(x,t) - u_e(x)$

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}, \tag{316}$$
$$v(0,t) = 0, \quad v(L,t) = 0, \tag{317}$$
$$v(x,0) = f(x) - u_e(x). \tag{318}$$

   (c) Solve the homogeneous equation to obtain

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

   where

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_e(x)] \sin\left(\frac{n\pi x}{L}\right) dx.$$

   (d) Write down the solution of the original nonhomogeneous equation

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

2. Procedure for solving time-dependent nonhomogeneous problems:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \tag{319}$$
$$u(0,t) = A(t), \quad u(L,t) = B(t), \tag{320}$$
$$u(x,0) = f(x). \tag{321}$$

(a) Find a reference temperature distribution $r(x,t)$

$$r(x,t) = A(t) + \frac{B(t) - A(t)}{L} x.$$

(b) Convert the nonhomogeneous to the homogeneous by defining $v(x,t) = u(x,t) - r(x,t)$

$$\begin{aligned} \frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2} = k\frac{\partial^2 v}{\partial x^2} + Q_1(x,t), && (322) \\ v(0,t) &= 0, \quad v(L,t) = 0, && (323) \\ v(x,0) &= f(x) - r(x,0) = f_1(x). && (324) \end{aligned}$$

(c) Use eigenfunction expansion

$$v(x,t) = \sum_{m=1}^{\infty} a_m(t)\phi_m(x),$$

where

$$\phi_m(x) = \sin\left(\frac{m\pi x}{L}\right)$$

$$a_m'(t) = -k\lambda_m a_m(t) + \frac{\int_0^L Q_1(x,t)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

$$a_m(0) = \frac{\int_0^L f_1(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}, \quad m = 1, 2, \cdots,$$

3. All these procedures can be applied to the wave equation.

4. Exercises.

(a) Solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sin x, && (325) \\ u(0,t) &= 1, \quad u(\pi,t) = 2, && (326) \\ u(x,0) &= 0. && (327) \end{aligned}$$

(b) Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + 1, && (328) \\ u(0,t) &= 1, \quad u(\pi,t) = 2, && (329) \\ u(x,0) &= 0, \quad \frac{\partial u}{\partial t}(x,0) = 0. && (330) \end{aligned}$$

(c) Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin x e^{-2t}, \tag{331}$$

$$u(0,t) = 1, \quad u(\pi,t) = 0, \tag{332}$$

$$u(x,0) = 0. \tag{333}$$

(d) Solve

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin x e^{-2t}, \tag{334}$$

$$u(0,t) = 1, \quad u(\pi,t) = 0, \tag{335}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0. \tag{336}$$

27 Review of First-order quasi-linear equations

1. Procedure for solving First-order quasi-linear equations

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u), \tag{337}$$

$$u(x,0) = f(x). \tag{338}$$

   (a) Solve the characteristic equations:

$$\frac{dx}{ds} = a(x,y,u), \quad \frac{dy}{ds} = b(x,y,u), \quad \frac{du}{ds} = c(x,y,u)$$

   with the initial conditions

$$x(0) = r, \quad y(0) = 0, \quad u(0) = f(r)$$

   to obtain

$$x = x(s,r), \quad y = y(s,r), \quad u = u(s,r).$$

   (b) Solve the first two equations

$$x = x(s,r), \quad y = y(s,r)$$

   for s, r to obtain

$$s = s(x,y), \quad r = r(x,y),$$

   and then the solution

$$u = u(s(x,y), r(x,y)).$$

2. Working problems. Solve the following equations (from Dr. Arrigo's tests or sample tests):

   (a) $xu_x - yu_y = 2u$.

   (b) $yu_x - u_y = 1$.

   (c) $tu_t - u_x = 2u$.

   (d) $yu_x + xu_y = xu, \quad u(x,0) = e^{-x^2}$.

   (e) $xu_x + 2uu_y = u, \quad u(x,0) = x^2$.

   (f) $xu_x + (x+2y)u_y = 2u, \quad u(x,0) = \frac{x^2}{x+1}$.

28 Final Review

28.1 Eigenvalue Problems

1. The eigenvalue problem with Dirichlet boundary conditions

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0 \end{aligned}$$

   has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{339}$$

   and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{340}$$

2. The eigenvalue problem with Neumann boundary conditions

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{d\phi}{dx}(0) &= 0, \quad \frac{d\phi}{dx}(L) = 0 \end{aligned}$$

   has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{341}$$

   and the corresponding eigenfunctions

$$\phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{342}$$

3. Further Study Problems:

   (a) Prove Poincaré's inequality: For any

$$u(x) = \sum_{i=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

   we have

$$\int_0^L |u'(x)|^2 dx \geq \frac{\pi^2}{L^2}\int_0^L |u(x)|^2 dx.$$

   (b) (Ref. page 64 of the textbook) Show that the eigenvalue problem with periodic boundary conditions

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(-L) &= \phi(L), \quad \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L) \end{aligned}$$

has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{343}$$

and the corresponding eigenfunctions

$$\phi_0 = 1, \quad \phi_{2n-1} = \cos\left(\frac{n\pi x}{L}\right), \quad \phi_{2n} = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{344}$$

(c) (Ref. page 199 of the textbook) Discuss the eigenvalue problem with boundary conditions of the third kind

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \frac{d\phi}{dx}(L) = -h\phi(L) \quad (h > 0). \end{aligned}$$

(d) Find the eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{aligned} -k\frac{d^2\phi}{dx^2} + v_0\frac{d\phi}{dx} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0. \end{aligned}$$

(e) Solve the eigenvalue problem 5.3.9. (c) on page 169 of the textbook.

(f) Study the Sturm-Liouville eigenvalue problems, Section 5.3 (optional)

(g) Study the Bessel's differential equation, section 7.7.4 (optional)

28.2 The Heat Equation

1. Dirichlet boundary value problems:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{345}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{346}$$

$$u(x,0) = f(x). \tag{347}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

2. Neumann boundary value problem:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, &(348)\\ \frac{\partial u}{dx}(0,t) &= 0, \quad \frac{\partial u}{dx}(L,t)=0, &(349)\\ u(x,0) &= f(x). &(350)\end{aligned}$$

has the infinite series solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}},$$

where

$$\begin{aligned}a_0 &= \frac{1}{L}\int_0^L f(x)dx, &(351)\\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1. &(352)\end{aligned}$$

3. Method of eigenfunction expansion for the nonhomogeneous problem

$$\begin{aligned}\frac{\partial v}{\partial t} &= k\frac{\partial^2 v}{\partial x^2} + Q(x,t), &(353)\\ v(0,t) &= 0, \quad v(L,t)=0, &(354)\\ v(x,0) &= f(x). &(355)\end{aligned}$$

$$v(x,t) = \sum_{m=1}^{\infty} a_m(t)\phi_m(x),$$

where

$$\phi_m(x) = \sin\left(\frac{m\pi x}{L}\right)$$

$$a'_m(t) = -k\lambda_m a_m(t) + \frac{\int_0^L Q(x,t)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

$$a_m(0) = \frac{\int_0^L f(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}, \quad m = 1,2,\cdots,$$

4. Further study problems:

   (a) Problem 5.3.4. (b) on page 168 of the textbook.

(b) Solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + au, && (356) \\ u(0,t) &= 0, \quad u(L,t) = 0, && (357) \\ u(x,0) &= f(x), && (358) \end{aligned}$$

where a is a constant. Determine the range of a for which the solution may grow exponentially and the range of a for which the solution decay to zero exponentially.

(c) Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && (359) \\ u(0,t) &= 0, \quad \frac{\partial u}{\partial x}(1,t) = -\frac{1}{2}u(1,t), && (360) \\ u(x,0) &= f(x), && (361) \end{aligned}$$

Show that

$$\|u(t)\| \le \|f\|e^{-\frac{\pi^2 t}{8}},$$

where $\|\cdot\|$ denotes the L^2 norm defined by

$$\|u(t)\| = \sqrt{\int_0^1 |u(x,t)|^2 dx}.$$

(d) Problem 1.5.3. (b) and (e) on page 29 of the textbook.

(e) Problem 1.5.4. (b) and (e) on page 30 of the textbook.

(f) Problem 5.4.3. on page 173 of the textbook (optional).

(g) Problem 8.3.3. on page 359 of the textbook (optional).

28.3 The Wave Equation

1. The Dirichlet initial boundary value problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, && (362) \\ u(0,t) &= 0, \quad u(L,t) = 0, && (363) \\ u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) && (364) \end{aligned}$$

has infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right),$$

where

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\\
b_n\frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

2. The Neumann initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{365}$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \tag{366}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{367}$$

has infinite series solution:

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)\cos\left(\frac{n\pi x}{L}\right),$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx,\\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n\ge 1\\
b_0 &= \frac{1}{L}\int_0^L g(x)dx.\\
b_n\frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n\ge 1.
\end{aligned}$$

3. The Method of Eigenfunction expansion for

$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2} + Q(x,t), \tag{368}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{369}$$

$$v(x,0) = f(x), \quad \frac{\partial v}{\partial t}(x,0) = g(x). \tag{370}$$

The corresponding eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Solution:

$$v(x,t)=\sum_{n=1}^{\infty}a_n(t)\phi_n(x).$$

where

$$a_m''(t)=-c^2\lambda_m a_m(t)+\frac{\int_0^L Q(x,t)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx}.$$

$$a_m(0)=\frac{\int_0^L f(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx},\quad m=1,2,\cdots,$$

$$a_m'(0)=\frac{\int_0^L g(x)\phi_m(x)dx}{\int_0^L \phi_m^2(x)dx},\quad m=1,2,\cdots.$$

4. Further study problems:

   (a) Problem 5.8.6. (c) on page 210 of the textbook.

   (b) Solve

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \tag{371}$$

$$v(0,t) = 0,\quad v(1,t)=\sin t, \tag{372}$$

$$v(x,0) = 0,\quad \frac{\partial v}{\partial t}(x,0)=0. \tag{373}$$

   (c) Problem 8.5.2. on page 371 of the textbook.

   (d) Problem 7.7.1. on page 315 of the textbook (optional).

28.4 Laplace's equation and Poisson's equation

Further study problems:

1. Problem 2.5.1. (e) on page 85 of the textbook.
2. Problem 7.3.7. (c) on page 288 of the textbook.
3. Problem 8.6.6. on page 379 of the textbook (use eigenfunction expansion).
4. Problem 2.5.8. (a) on page 87 of the textbook (optional).
5. Problem 8.6.3. (a) on page 378 of the textbook (optional).

28.5 Method of Characteristics

Further study problems:

1. Problem 12.6.8. (a) and (e) on page 582 of the textbook.
2. Problem 12.6.9. (a) and (c) on page 582 of the textbook.
3. Problem 12.4.2. on page 556 of the textbook (optional).