Assignment Solutions of Partial Differential Equations

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1 Assignment 1

1.2.3. Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources.

Denote by A the the cross-sectional area.

Physical quantities:

- **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume.
- **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time.
- **Heat sources** $Q(x,t) = 0$.
- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit.
- **Mass density** $\rho(x)$ = mass per unit volume.
- **Fourier's Law**: the heat flux is proportional to the temperature gradient

$$\phi = -K_0 \nabla u. \tag{1}$$

Conservation of heat energy:

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy $= e(x,t)A(x)\Delta x$.
- Heat energy flowing across boundaries per unit time $= \phi(x,t)A(x) - \phi(x+\Delta x,t)A(x+\Delta x)$.

Then

$$\frac{\partial}{\partial t}[e(x,t)A(x)\Delta x] = \phi(x,t)A(x) - \phi(x+\Delta x,t)A(x+\Delta x).$$

Dividing it by Δx and letting Δx go to zero give

$$A(x)\frac{\partial e}{\partial t} = -A(x)\frac{\partial \phi}{\partial x} - \phi(x)\frac{\partial A}{\partial x}. \tag{2}$$

Heat energy per unit mass $= c(x)u(x,t)\rho A\Delta x$. So

$$e(x,t)A(x)\Delta x = c(x)u(x,t)\rho A(x)\Delta x,$$

and then

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows from Fourier's law that

$$c\rho A(x)\frac{\partial u}{\partial t} = A(x)\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + K_0\frac{\partial A}{\partial x}\frac{\partial u}{\partial x}. \tag{3}$$

and then the heat equation

$$A(x)\frac{\partial u}{\partial t} = k\left(A(x)\frac{\partial^2 u}{\partial x^2} + \frac{\partial A}{\partial x}\frac{\partial u}{\partial x}\right), \tag{4}$$

where $k = \frac{K_0}{c\rho}$ is the thermal diffusivity.

1.2.9. Consider a thin one-dimensional rod without source of thermal energy whose lateral surface is not insulated. Let $w(x,t)$ dente the heat energy flowing out of the lateral sides per unit surface area per unit time. Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive the equation for the temperature.

Denote by A the the cross-sectional area, and P the lateral perimeter.

Physical quantities:

- **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume.
- **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time.
- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit.
- **Mass density** $\rho(x)$ = mass per unit volume.
- **Fourier's Law**: the heat flux is proportional to the temperature gradient

$$\phi = -K_0\nabla u. \tag{5}$$

Conservation of heat energy:

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy = $e(x,t)A\Delta x$.
- Heat energy flowing across boundaries per unit time = $\phi(x,t)A - \phi(x+\Delta x,t)A$.
- Heat energy flowing out of the lateral sides per unit time = $w(x,t)P\Delta x = [u(x,t) - \gamma(x,t)]h(x)P\Delta x$, where $h(x)$ is a proportionality.

Then

$$\frac{\partial}{\partial t}[e(x,t)A(x)\Delta x] = \phi(x,t)A(x) - \phi(x+\Delta x,t)A(x+\Delta x) - [u(x,t)-\gamma(x,t)]h(x)P\Delta x.$$

Dividing it by $A\Delta x$ and letting Δx go to zero give

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} - \frac{P}{A}[u(x,t)-\gamma(x,t)]h(x). \tag{6}$$

Heat energy per unit mass $= c(x)u(x,t)\rho A\Delta x$. So

$$e(x,t)A\Delta x = c(x)u(x,t)\rho A\Delta x,$$

and then

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows from Fourier's law that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t)-\gamma(x,t)]h(x). \tag{7}$$

2 Assignment 2

1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:
(a) $Q = 0, \quad u(0) = 0, \quad u(L) = T$.
(f) $Q = K_0 x^2, \quad u(0) = T, \quad u'(L) = 0$.

Solution. (a) Equilibrium satisfies

$$u''(x) = 0,$$

whose general solution is

$$u = c_1 + c_2 x.$$

The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that

$$u = Tx/L.$$

(f) In equilibrium, u satisfies

$$u''(x) = -Q/K_0 = -x^2,$$

whose general solution (by integrating twice) is

$$u = -x^4/12 + c_1 + c_2 x.$$

The boundary condition $u(0) = T$ yields $c_1 = T$, while $u'(L) = 0$ yields $c_2 = L^3/3$. Thus

$$u = -x^4/12 + L^3 x/3 + T.$$

1.4.11. Suppose

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = \beta, \quad \frac{\partial u}{\partial x}(L,t) = 7.$$

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).
(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.

Solution. (a) Integrating the equation, we obtain:

$$\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x\right) dx = \frac{\partial u}{\partial x}\Big|_0^L + \frac{1}{2}L^2 = 7 - \beta + \frac{1}{2}L^2.$$

Integrating in t from 0 to t, we obtain the total thermal energy

$$\int_0^L u(x,t)dx = \int_0^L f(x)dx + \left(7 - \beta + \frac{1}{2}L^2\right) t. \qquad (8)$$

(b) In order for an equilibrium to exist, $\left(7 - \beta + \frac{1}{2}L^2\right) t$ must be 0. So

$$\beta = 7 + \frac{1}{2}L^2.$$

The equilibrium satisfies

$$\phi''(x) + x = 0.$$

Its general solution (after integrating twice) is

$$\phi = -\frac{1}{6}x^3 + c_1 + c_2 x.$$

The boundary condition yields

$$c_2 = 7 + \frac{1}{2}L^2.$$

So

$$\phi = -\frac{1}{6}x^3 + c_1 + \left(7 + \frac{1}{2}L^2\right)x.$$

Since

$$\lim_{t\to\infty} u(x,t) = \phi(x),$$

using (8), we obtain

$$\begin{aligned}
\int_0^L f(x)dx &= \int_0^L u(x,t)dx \\
&= \lim_{t\to\infty}\int_0^L u(x,t)dx \\
&= \int_0^L \phi(x)dx \\
&= \int_0^L \left(-\frac{1}{6}x^3 + c_1 + \left(7 + \frac{1}{2}L^2\right)x\right)dx \\
&= -\frac{1}{24}L^4 + c_1 L + \frac{1}{2}\left(7 + \frac{1}{2}L^2\right)L^2.
\end{aligned}$$

Solving it gives

$$c_1 = \frac{\int_0^L f(x)dx - \frac{7}{2}L^2 - \frac{5}{24}L^4}{L},$$

and then

$$\phi = -\frac{1}{6}x^3 + \left(7 + \frac{1}{2}L^2\right)x + \frac{\int_0^L f(x)dx - \frac{7}{2}L^2 - \frac{5}{24}L^4}{L}.$$

1.5.2. For conduction of thermal energy, the heat flux vector is $\phi = -K_0\nabla u$. If in addition the molecules move at an average velocity $\mathbf{V}$, a process called convection, then $\phi = -K_0\nabla u + c\rho u\mathbf{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

Solution. Physical quantities:

- **Thermal energy density** $e(x,t)$ = the amount of thermal energy per unit volume.
- **Heat flux** $\phi(x,t)$ = the amount of thermal energy flowing across boundaries per unit surface area per unit time.

- **Heat sources** $Q(x,t)$ = heat energy per unit volume generated per unit time.
- **Temperature** $u(x,t)$.
- **Specific heat** c = the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit.
- **Mass density** $\rho(x)$ = mass per unit volume.

Conservation of heat energy:

Rate of change of heat energy in time = Heat energy flowing across boundaries per unit time + Heat energy generated insider per unit time

- heat energy = $\int_R e(x,t)dV$.
- Heat energy flowing across boundaries per unit time = $\oint \phi \cdot \mathbf{n} dS$.
- Heat energy generated insider per unit time = $\int_R Q(x,t)dV = 0$.

Then

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\oint \phi \cdot \mathbf{n} dS.$$

The divergence theorem give

$$\frac{\partial}{\partial t}\int_R e(x,t)dV = -\int_R \nabla \cdot \phi dV.$$

and then

$$\frac{\partial e}{\partial t} = -\nabla \cdot \phi. \tag{9}$$

Heat energy per unit volume = $c(x)u(x,t)\rho$. So

$$e(x,t) = c(x)u(x,t)\rho.$$

It then follows that

$$c\rho\frac{\partial u}{\partial t} = -\nabla \cdot (c\rho u \mathbf{V}) + \nabla \cdot (\mathbf{K_0}\nabla \mathbf{u}). \tag{10}$$

and then the convection-diffusion equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{V}) = k\nabla^2 u, \tag{11}$$

where $k = \frac{K_0}{c\rho}$ is called the thermal diffusivity.

3 Assignment 3

2.3.2. (d) Find the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \frac{d\phi}{dx}(L) = 0. \end{aligned}$$

(i) If $\lambda > 0$, $\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. Thus $\sqrt{\lambda}L = -\frac{\pi}{2} + n\pi$ $(n = 1, 2, \cdots)$. Then the eigenvalues are $\lambda_n = \left(-\frac{\pi}{2} + n\pi\right)^2 / L^2$ and the corresponding eigenfunctions are $\phi_n = \sin\left(\frac{\left(-\frac{\pi}{2}+n\pi\right)x}{L}\right)$ $(n = 1, 2, \cdots)$.

(ii) If $\lambda = 0$, $\phi = c_1 + c_2 x$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2 = 0$. Thus $\lambda = 0$ is not an eigenvalue.

(ii) If $\lambda < 0$, $\phi = c_1 \exp(\sqrt{-\lambda}x) + c_2 \exp(-\sqrt{-\lambda}x)$. $\phi(0) = 0$ implies $c_1 + c_2 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_1\sqrt{-\lambda}\exp(\sqrt{-\lambda}L) - c_2\sqrt{-\lambda}\exp(-\sqrt{-\lambda}L) = 0$. Solving this system for c_1, c_2 gives $c_1 = c_2 = 0$. Thus $\lambda < 0$ is not an eigenvalue.

2.3.3. (c) Solve the initial boundary value problems:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, && (12) \\ u(0,t) &= 0, \quad u(L,t) = 0, && (13) \\ u(x,0) &= 2\cos\left(\frac{3\pi x}{L}\right). && (14) \end{aligned}$$

Solution. The problem has infinite series solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

The initial condition yields

$$2\cos\left(\frac{3\pi x}{L}\right) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

So

$$c_n = \frac{2}{L}\int_0^L 2\cos\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx.$$

2.3.8. **Solution**. (a) The equation for the equilibrium is

$$\begin{aligned} k\frac{d^2u}{dx^2} - \alpha u &= 0, && (15) \\ u(0) &= 0, \quad u(L) = 0. && (16) \end{aligned}$$

The general solution is $u = c_1 \exp(\sqrt{\frac{\alpha}{k}}x) + c_2 \exp(-\sqrt{\frac{\alpha}{k}}x)$. $u(0) = 0$ implies $c_1 + c_2 = 0$, while $u(L) = 0$ implies $c_1 \exp(\sqrt{\frac{\alpha}{k}}L) + c_2 \exp(-\sqrt{\frac{\alpha}{k}}L) = 0$. Solving this system for c_1, c_2 gives $c_1 = c_2 = 0$. Thus $u = 0$.

(b) Separation of variable, $u = \phi(x)G(t)$, yields two ODEs:

$$\frac{dG}{dt} = -(\lambda k + \alpha)G$$

and

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \phi(0) &= 0, \quad \phi(L) = 0. \end{aligned}$$

The G-equation has solution

$$G(t) = Ce^{-\alpha t}e^{-\lambda k t}.$$

The eigenvalue problem has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots \tag{17}$$

and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots \tag{18}$$

Thus by superposition,

$$u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

The initial condition gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

which gives

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Since

$$\lim_{t\to\infty} e^{-\frac{kn^2\pi^2 t}{L^2}} = 0, \quad \lim_{t\to\infty} e^{-\alpha t} = 0,$$

we have

$$\lim_{t\to\infty} u(x,t) = 0.$$

The $u(x,t)$ converges to the only equilibrium 0.

4 Assignment 4

2.4.1. (a). The solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{L/2}^{L} dx = \frac{1}{2}, \\ a_n &= \frac{2}{L}\int_{L/2}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

2.4.1. (b). The solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{kn^2\pi^2 t}{L^2}}.$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L \left(6 + 4\cos\left(\frac{3\pi x}{L}\right)\right) dx = 6, \\ a_3 &= \frac{2}{L}\int_0^L \left(6 + 4\cos\left(\frac{3\pi x}{L}\right)\right)\cos\left(\frac{n\pi x}{L}\right) dx = 4, \end{aligned}$$

and others are 0.

2.4.2. **Solution**.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{19}$$

$$\frac{\partial u}{dx}(0,t) = 0, \quad u(L,t) = 0, \tag{20}$$

$$u(x,0) = f(x). \tag{21}$$

Look for a solution of the form of separation of variables:

$$u(x,t) = \phi(x)G(t), \tag{22}$$

Substitute the above expression into the equation (19), we obtain

$$\phi(x)G'(t) = k\phi''(x)G(t),$$

and then

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \tag{23}$$

where λ is constant to be determined. The boundary condition (20) yields that

$$\frac{\phi u}{dx}(0) = \phi(L) = 0.$$

We then have an eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dx^2} &= \lambda\phi, \\ \frac{\phi}{dx}(0) &= 0, \quad \phi(L) = 0. \end{aligned}$$

Auxiliary equations:

$$m^2 = -\lambda.$$

- Case 1: $\lambda < 0$. Distinct real roots $m_1 = \sqrt{-\lambda}$ and $m_2 = -\sqrt{-\lambda}$:

  $$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

  The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.

- Case 2: $\lambda = 0$. Repeated real roots $m_1 = m_2 = 0$:

  $$\phi = c_1 + c_2 x.$$

  The boundary conditions imply that $c_1 = c_2 = 0$. So no non-zero solutions exist and then $\lambda < 0$ is not an eigenvalue.

- Case 3: $\lambda > 0$. Conjugate complex roots $m_1 = i\sqrt{\lambda}$ and $m_2 = -i\sqrt{\lambda}$:

  $$\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

  $\phi'(0) = 0$ implies that $c_2 = 0$. $\phi(L) = 0$ gives

  $$\cos(\sqrt{\lambda}L) = 0.$$

  So $\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$ $(n = 0, 1, 2, \cdots)$ and then we obtain the eigenvalues

  $$\lambda_n = \frac{\left(\frac{\pi}{2} + n\pi\right)^2}{L^2}, \quad n = 0, 1, 2, \cdots \tag{24}$$

  and the corresponding eigenfunctions

  $$\phi_n = \cos\left(\frac{\left(\frac{\pi}{2} + n\pi\right)x}{L}\right), \quad n = 0, 1, 2, \cdots \tag{25}$$

On the other hand, it follows from (23) that

$$\frac{dG}{dt} = -\lambda k G, \tag{26}$$

which has solutions

$$G(t) = ce^{-\lambda k t} = ce^{-\frac{(\frac{\pi}{2}+n\pi)^2 kt}{L^2}}.$$

We then derive the infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\left(\frac{\pi}{2}+n\pi\right)x}{L}\right) e^{-\frac{(\frac{\pi}{2}+n\pi)^2 kt}{L^2}}.$$

The initial condition gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\left(\frac{\pi}{2}+n\pi\right)x}{L}\right). \tag{27}$$

To determine a_n, we multiply (27) by $\cos\left(\frac{(\frac{\pi}{2}+n\pi)x}{L}\right)$ and integrate from 0 to L. We then find

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\left(\frac{\pi}{2}+n\pi\right)x}{L}\right)dx, \quad n \geq 1. \tag{28}$$

2.5.1. (c) Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{29}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad u(L,y) = g(Y), \quad u(x,0) = 0, \quad u(x,H) = 0, \tag{30}$$

Look for a solution of the form of separation of variables:

$$u(x,y) = h(x)\phi(y), \tag{31}$$

Substitute the above expression into the equation (29), we obtain

$$\phi(y)h''(x) + \phi''(y)h(x) = 0,$$

and then

$$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \lambda, \tag{32}$$

where λ is constant to be determined. The boundary condition yields that

$$\phi(0) = \phi(H) = 0, \quad h'(0) = 0.$$

We then have an eigenvalue problem

$$
\begin{aligned}
-\frac{d^2\phi}{dx^2} &= \lambda\phi, \\
\phi(0) &= 0, \quad \phi(H) = 0,
\end{aligned}
$$

which has the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{H^2}, \quad n = 1, 2, \cdots \tag{33}$$

and the corresponding eigenfunctions

$$\phi_n = \sin\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, \cdots \tag{34}$$

On the other hand, it follows from (32) that

$$h''(x) = \frac{n^2\pi^2}{H^2} h(x), \quad h'(0) = 0. \tag{35}$$

The general solutions are

$$h(x) = c_1 e^{\frac{n\pi x}{H}} + c_2 e^{-\frac{n\pi x}{H}}.$$

The boundary condition $h'(0) = 0$ gives

$$c_1 \frac{n\pi}{H} - c_2 \frac{n\pi}{H} = 0.$$

So

$$h(y) = c_1 \left(e^{\frac{n\pi x}{H}} + e^{-\frac{n\pi x}{H}}\right).$$

We then derive the infinite series solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{H}\right)\left(e^{\frac{n\pi x}{H}} + e^{-\frac{n\pi x}{H}}\right).$$

The boundary condition $u(L, y) = g(y)$ gives

$$g(y) = u(L, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{H}\right)\left(e^{\frac{n\pi L}{H}} + e^{-\frac{n\pi L}{H}}\right). \tag{36}$$

To determine a_n, we multiply the above by $\sin\left(\frac{n\pi y}{H}\right)$ and integrate from 0 to H. We then find

$$a_n = \frac{2\left(e^{\frac{n\pi L}{H}} + e^{-\frac{n\pi L}{H}}\right)}{H} \int_0^H g(y) \sin\left(\frac{n\pi y}{H}\right) dy, \quad n \geq 1. \tag{37}$$

5 Assignment 5

See the solutions in the textbook.