Name: ___________________________________

MATH 4315 - Partial differential equations

Final Exam

Due by 5:00pm, December 11

You should complete the exam yourself independently. You are not allowed to ask professors and classmates for help, but you can use any resources you have, including books, articles, and internet.

If there is a formula in the textbook for the solution of an equation, you can just use it and you do not need to derive it yourself.

Total Marks:

Problem 1. (10 points) Solve the first-order equation:

$$y^2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = u, \quad u(x,0) = x.$$

Problem 2. (10 points) Reduce the equation

$$\frac{\partial^2 u}{\partial x^2} - 2\sin x\frac{\partial^2 u}{\partial x \partial y} - \cos^2 x\frac{\partial^2 u}{\partial y^2} - \cos x\frac{\partial u}{\partial y} = 0$$

to canonical form.

Problem 3. (10 points) Consider the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= f(x). \end{aligned}$$

1. Find its solution.
2. Prove that
$$\int_0^\pi |u(x,t)|^2 dx \le e^{-t}\int_0^\pi |f(x)|^2 dx.$$

Problem 4. (10 points) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= f(x). \end{aligned}$$

Problem 5. (10 points) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + e^{-t}, \\ \frac{\partial u}{\partial x}(0,t) &= 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \\ u(x,0) &= 1. \end{aligned}$$

Discuss the limit of the solution as $t \to \infty$.

Problem 6. (10 points) Consider the initial boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \\ u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \end{aligned}$$

1. Find its solution.

2. Prove that the following energy identity

$$\int_0^\pi \left(\left| \frac{\partial u}{\partial t}(x,t) \right|^2 + \left| \frac{\partial u}{\partial x}(x,t) \right|^2 \right) dx = \int_0^\pi \left(\left| \frac{\partial f}{\partial x}(x) \right|^2 + |g(x)|^2 \right) dx, \quad \text{for any } t \geq 0.$$

Problem 7. (10 points) Solve the nonhomogeneous problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) &= 0, \quad u(1,t) = 1, \\ u(x,0) &= 0, \quad \frac{\partial u}{\partial t}(x,0) = x. \end{aligned}$$

Problem 8. (10 points) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\ u(0,y,t) &= u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \\ u(x,y,0) &= 1, \quad \frac{\partial u}{\partial t}(x,y,0) = 0. \end{aligned}$$

Problem 9. (10 points) Solve the Laplace's equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \\ u(0,y,t) &= 2, \quad u(1,y,t) = 1, \quad u(x,0,t) = u(x,1,t) = 0. \end{aligned}$$

Problem 10. (10 points) Solve the Poisson's equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 1, \\ u(0,y,t) &= u(1,y,t) = u(x,0,t) = u(x,1,t) = 0. \end{aligned}$$