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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF TIME-DELAYED BURGERS' EQUATION

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Abstract. In this paper, we consider Burgers' equation with a time delay. By using the Liapunov function method, we show that the delayed Burgers' equation is exponentially stable if the delay parameter is sufficiently small. We also give an explicit estimate of the delay parameter in terms of the viscosity and initial conditions, which indicates that the delay parameter tends to zero if the initial states tend to infinity or the viscosity tends to zero. Furthermore, we present numerical simulations for our theoretical results.

1. **Introduction.** In this paper, we are concerned with the problem of asymptotic behavior of solutions of the time-delayed Burgers' equation

$$u_t(x,t) - \epsilon u_{xx}(x,t) + u(x,t-\tau)u_x(x,t) = 0, \qquad 0 < x < 1, \;\; t > 0, \tag{1}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,s) = u_0(x,s), \qquad 0 < x < 1, \; -\tau \le s \le 0, \tag{3}$$

where subscripts denote derivatives, $\epsilon > 0$ denotes the viscosity, $\tau > 0$ is the delay parameter and $u_0(x,s)$ is an initial state in an appropriate function space. Even though Burgers' equation, a one-dimensional version of Navier-Stokes equations, does not model any specific physical flow problem, it would be the first step to understand the turbulence exhibited in a flow.

To explain our motivation of introducing a time delay into Burgers' equation, we consider an ideal one-dimensional flow of fluid with the flow velocity $u = u(x,t)$ (although such a flow does not exist in reality because interesting flows are at least 2D). The rate of change of u "following the fluid", which we denote by $\frac{Du}{Dt}$, is (see [2, page 4])

$$\begin{aligned} \frac{Du}{Dt} &= \frac{d}{dt}u[x(t),t] = \frac{\partial}{\partial t}u(x,t) + \frac{dx(t)}{dt}\frac{\partial}{\partial x}u(x,t) \\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t), \end{aligned} \tag{4}$$

where $x(t)$ is understood to change with time at the local flow velocity $u = \frac{dx}{dt}$, so as to "follow the fluid". However, we might have a delay τ to follow the fluid. In

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this case the rate of change of u "following the fluid" with the delay τ should be

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u[x(t-\tau),t] = \frac{\partial}{\partial t}u(x,t) + \frac{d}{dt}x(t-\tau)\frac{\partial}{\partial x}u(x,t)\\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t-\tau)\frac{\partial}{\partial x}u(x,t).\end{aligned} \tag{5}$$

This clearly shows how we obtain the time-delayed term $u(x,t-\tau)u_x(x,t)$ in Burgers' equation (1).

There has been the existing literature about delayed reaction-diffusion equations, on which our work is based. Indeed, the inequality

$$u_t(x,t) - \Delta u(x,t) \leq u(x,t)(1 - u(x,t-\tau)$$

was investigated by Luckhaus [23], who showed that nonnegative solutions of the Dirichlet problem in a bounded interval remain bounded as time goes to infinity, whereas in a more dimensional domain, in general, this holds only if the delay is not too large. On the other hand, the scalar delay reaction-diffusion equation

$$u_t - \mu\Delta u = f(u(t), u(t-\tau))$$

was studied by Friesecke [14, 15], who showed that in one space dimension, all nonnegative solutions stay bounded as $t \to \infty$ and this ceases to remain true in two or more dimensions: if the dealy is large and the diffusion coefficient small, there exists a large set of trajectories whose total mass tends exponentially to infinity as $t \to \infty$. Moreover, Oliva [25] considered dissipative scalar reaction-diffusion equations that include the ones of the form

$$u_t - \Delta u = f(u(t)),$$

subjected to boundary conditions that include small delays. The author showed that the global unique solutions exist in a convenient fractional power space and, for a sufficiently small delay, all bounded solutions are asymptotic to the set of equilibria as t tends to infinity. The literature reviewed here is only small part of it and lots of others worth mention, for instance, [19, 26, 28]. Even though there have been extensive studies on the delayed reaction-diffusion equations, to our knowledge, it seems that little attention has been paid to a equation that contains a delay term of the form $u(x,t-\tau)u_x(x,t)$.

Burgers' equation without delay has been extensively studied (see, e.g., [5, 6, 7, 8, 16, 17, 18, 22, 24]). It has been proved that the equation is globally exponentially stable at least in the norm of H^1. We also recall that uniformly stabilized wave equations and flexible beam equations can be destabilized by a small delay in a feedback control no matter how small the delay is (see, e.g., [4, 9, 10, 11, 12, 13, 20]). So the question is: does a small delay also destabilize Burgers' equation? The answer is No. As in the case of scalar delayed reaction-diffusions, we shall show that the delayed Burgers' equation is still exponentially (but not globally) stable if the delay parameter $\tau = \tau(\epsilon, u_0)$ is sufficiently small. We also give an explicit estimate of τ in terms of ϵ and u_0, which indicates that τ tends to zero if the initial state tends to infinity or $\epsilon \to 0$. Furthermore, we present numerical simulations for our theoretical results.

2. **Exponential Stability.** We now introduce notation used throughout the paper. $H^s(0,1)$ denotes the usual Sobolev space (see [1, 21]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in

$(0,1)$. The norm on $L^2(0,1)$ is denoted by $\|\cdot\|$. Let X be a Banach space and $a<b$. We denote by $C^n([a,b];X)$ the space of n times continuously differentiable functions defined on $[a,b]$ with values in X with the supremum norm and we write $C([a,b];X)$ for $C^0([a,b];X)$.

We first briefly show that problem (1)-(3) is well posed. Define the linear operator A by

$$Aw = \epsilon w_{xx}$$

with the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$. It is well known that A generates an analytic semigroup e^{At} on $L^2(0,1)$. We further define the nonlinear operator $F : C([-\tau,0], H_0^1(0,1)) \to L^2(0,1)$ by

$$F(\varphi) = -\varphi(-\tau)\varphi_x(0) \quad \text{for } \varphi \in C([-\tau,0], H_0^1(0,1)). \tag{6}$$

It is clear that F is locally Lipschitz. Denote

$$u^t(s) = u(t+s), \quad -\tau \le s \le 0.$$

We then transform problem (1)-(3) into the following integral equation

$$u(t) = u_0(t), \quad -\tau \le t \le 0, \tag{7}$$

$$u(t) = e^{At}u_0(0) + \int_0^t e^{A(t-s)}F(u^s)ds, \quad t>0. \tag{8}$$

By Theorem 1 of [25], for every initial value $u_0 = u_0(x,s) \in C([-\tau,0], H_0^1(0,1))$, there exists a $T = T(u_0) > 0$ such that problem (1)-(3) has a unique mild solution u on $[-\tau, T]$ with

$$u \in C([-\tau,T], H_0^1(0,1)).$$

Moreover, if the initial condition is more regular, for instance, Hölder continuous, then u is a classical solution. Furthermore, for any $\tau > 0$, the solution of (1)-(3) does not blow up in finite time. Indeed, integrating by parts, we obtain for $0 \le t \le \tau$

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u_x^2(t)\,dx &= 2\int_0^1 u_x(t)u_{xt}(t)\,dx \\
&= -2\int_0^1 u_{xx}(t)u_t(t)\,dx \\
&= -2\epsilon\int_0^1 u_{xx}^2(t)\,dx + 2\int_0^1 u(t-\tau)u_{xx}(t)u_x(t)\,dx \\
&\le -2\epsilon\int_0^1 u_{xx}^2(t)\,dx + 2\|u_0\|_{C([-\tau,0],H_0^1(0,1))}\int_0^1 |u_{xx}(t)u_x(t)|\,dx \\
&\quad \text{(use Young's inequality))} \\
&\le \epsilon^{-1}\|u_0\|^2_{C([-\tau,0],H_0^1(0,1))}\int_0^1 u_x^2(t)\,dx,
\end{aligned} \tag{9}$$

which implies that

$$\int_0^1 u_x^2(t)\,dx \le M(\|u_0\|_{C([-\tau,0],H_0^1(0,1))}),$$

where $M(\|u_0\|_{C([-\tau,0],H_0^1(0,1))})$ is a positive constant depending on $\|u_0\|_{C([-\tau,0],H_0^1(0,1))}$. Repeating the above procedure, we can prove that for $n\tau \le t \le (n+1)\tau$ $(n =$

$1, 2, \dots)$

$$\int_0^1 u_x^2(t)\,dx \le M(n,\ \|u_0\|_{C([-\tau,0],H_0^1(0,1))}).$$

In summary, we have proved

THEOREM 2.1. *For any initial condition $u_0 = u_0(x,s) \in C([-\tau,0],H_0^1(0,1))$. problem* (1)-(3) *has a unique global mild solution u on $[-\tau,\infty)$ with*

$$u \in C([-\tau,\infty), H_0^1(0,1)).$$

To state our main result about the exponential stability, we introduce the following notations. For a given initial condition $u_0 = u_0(x,s) \in C([-\tau,0],H_0^1(0,1))$, denote

$$\begin{aligned} K &= K(u_0) \\ &= \sup_{-\tau\le s\le 0} \|u_{0x}(s)\| \\ &\quad + \sqrt{8[\|u_0(0)\|^2 + \|u_{0x}(0)\|^2] \exp\left[\epsilon^{-1}\Big(\|u_{0x}\|^2_{L^2_\tau} + \|u_0(0)\|^2\Big)\right]}, \quad (10) \\ \sigma &= \sigma(\epsilon, u_0) \\ &= \sup\Big\{\delta > 0\ :\ [\|u_0(0)\|^2 + \|u_{0x}(0)\|^2]\exp\left[\epsilon^{-1}e^{\omega\tau}\Big(\|u_{0x}\|^2_{L^2_\tau} + \omega^{-1}\|u_0(0)\|^2\Big)\right] \\ &\qquad \le K^2/4 \ \text{ for } 0 \le \tau \le \delta\Big\}, \quad (11) \\ \tau_0 &= \tau_0(\epsilon, u_0) = \min\Big\{\sigma, \frac{(\sqrt{5}-1)\epsilon}{2K^2}\Big\}, \quad (12) \\ \omega &= \omega(\epsilon,\tau,K) = \epsilon - \sqrt{\tau(\epsilon K^2 + \tau K^4)} > 0, \quad \text{for } 0 \le \tau < \tau_0, \quad (13) \end{aligned}$$

where $\|\cdot\|$ denotes the L^2 norm and

$$\|u_{0x}\|^2_{L^2_\tau} = \int_{-\tau}^0\int_0^1 u_{0x}^2(x,s)dxds.$$

In (13), we have $\omega > 0$ for $0 \le \tau < \tau_0$ because

$$\epsilon - \sqrt{\tau(\epsilon K^2 + \tau K^4)} > 0$$

is equivalent to

$$K^4\tau^2 + \epsilon K^2\tau - \epsilon^2 < 0,$$

which in turn is equivalent to

$$-\frac{(\sqrt{5}+1)\epsilon}{2K^2} < \tau < \frac{(\sqrt{5}-1)\epsilon}{2K^2}.$$

THEOREM 2.2. *For any initial condition $u_0 = u_0(x,s) \in C([-\tau,0],H_0^1(0,1))$, let $\tau_0 = \tau_0(\epsilon,u_0)$ be given by* (12). *Then, for $\tau < \tau_0$, the solution of* (1)-(3) *satisfies*

$$\|u_x(t)\| \le \frac{K}{2}e^{-\omega t/2}, \quad \forall\, t \ge 0. \tag{14}$$

Proof. Let

$$T_0 = \sup\{\delta : \|u_x(t)\| \le K \ \text{ on } 0 \le t \le \delta\}. \tag{15}$$

Since $\|u_x(0)\| < K$ and $\|u_x(t)\|$ is continuous, we have $T_0 > 0$. We shall prove that $T_0 = +\infty$. For this, we argue by contradiction. If $T_0 < +\infty$, then we have

$$\|u_x(t)\| \le K, \quad \forall\ -\tau \le t < T_0 \tag{16}$$

and

$$\|u_x(T_0)\| = K. \tag{17}$$

Using equations (1)-(3), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^2(t)\,dx &= 2\int_0^1 u(t)u_t(t)\,dx \\
&= 2\epsilon\int_0^1 u(t)u_{xx}(t)\,dx - 2\int_0^1 u(t)u(t-\tau)u_x(t)\,dx \\
&\text{(note that } \textstyle\int_0^1 u^2(t)u_x(t)\,dx = 0) \\
&= -2\epsilon\int_0^1 u_x^2(t)\,dx - 2\int_0^1 [u(t-\tau)-u(t)]u(t)u_x(t)\,dx \\
&\text{(note that } |u(x,t)| \le \|u_x(t)\| \text{ for } 0 \le x \le 1) \\
&\le -2\epsilon\int_0^1 u_x^2(t)\,dx + 2\int_0^1 u_x^2(t)\,dx\Big(\int_0^1 |u(t-\tau)-u(t)|^2\,dx\Big)^{1/2} \\
&= -2\epsilon\int_0^1 u_x^2(t)\,dx + 2\int_0^1 u_x^2\,dx\Big(\int_0^1 \Big|\int_{t-\tau}^t u_s(s)ds\Big|^2\,dx\Big)^{1/2} \\
&\le -2\epsilon\int_0^1 u_x^2(t)\,dx + 2\sqrt{\tau}\int_0^1 u_x^2(t)\,dx\Big(\int_0^1\int_{t-\tau}^t u_s^2(s)ds\,dx\Big)^{1/2}.
\end{aligned} \tag{18}$$

We now want to estimate $\int_0^1\int_{t-\tau}^t u_s^2(s)ds\,dx$. Since

$$\begin{aligned}
\epsilon\frac{d}{dt}\int_0^1 u_x^2(t)\,dx &= 2\epsilon\int_0^1 u_x(t)u_{xt}(t)\,dx \\
&= -2\epsilon\int_0^1 u_{xx}(t)u_t(t)\,dx \\
&= -2\int_0^1 u_t^2(t)\,dx - 2\int_0^1 u(t-\tau)u_x(t)u_t(t)\,dx,
\end{aligned} \tag{19}$$

we have for $0 \le t \le T_0$ that

$$\begin{aligned}
&\epsilon\int_0^1 u_x^2(t)\,dx + 2\int_{t-\tau}^t\int_0^1 u_s^2(s)\,dxds \\
&\quad= \epsilon\int_0^1 u_x^2(t-\tau)\,dx - 2\int_{t-\tau}^t\int_0^1 u(s-\tau)u_x(s)u_s(s)\,dxds \\
&\quad\le \epsilon K^2 + 2K\Big(\int_{t-\tau}^t\int_0^1 u_x^2(s)\,dxds\Big)^{1/2}\Big(\int_{t-\tau}^t\int_0^1 u_s^2(s)\,dxds\Big)^{1/2} \\
&\quad\le \epsilon K^2 + \tau K^4 + \int_{t-\tau}^t\int_0^1 u_s^2(s)\,dxds,
\end{aligned} \tag{20}$$

which implies that

$$\int_{t-\tau}^t\int_0^1 u_s^2(s)\,dxds \le \epsilon K^2 + \tau K^4, \quad \forall\, 0 \le t \le T_0. \tag{21}$$

It therefore follows from (18) that for $0 \leq t \leq T_0$

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 u^2(t)\,dx &\leq -2\epsilon\int_0^1 u_x^2(t)\,dx + 2\sqrt{\tau}\sqrt{\epsilon K^2+\tau K^4}\int_0^1 u_x^2(t)\,dx \\
&\leq -2\Big(\epsilon - \sqrt{\tau(\epsilon K^2+\tau K^4)}\Big)\int_0^1 u_x^2(t)\,dx \\
&= -2\omega\int_0^1 u_x^2(t)\,dx \\
&\text{(since } \textstyle\int_0^1 u^2(t)\,dx \leq \int_0^1 u_x^2(t)\,dx) \\
&\leq -2\omega\int_0^1 u^2(t)\,dx,
\end{aligned}
\tag{22}
$$

where ω is defined by (13). Solving the above inequality gives

$$
\int_0^1 u^2(t)\,dx \leq e^{-2\omega t}\int_0^1 u_0(x,0)^2\,dx, \quad \forall\, 0 \leq t \leq T_0. \tag{23}
$$

By the first part of (22), we have

$$
\frac{d}{dt}\int_0^1 u^2(t)\,dx + 2\omega\int_0^1 u_x^2(t)\,dx \leq 0. \tag{24}
$$

Multiplying (24) by $e^{\omega t}$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\Big(e^{\omega t}\int_0^1 u^2(t)\,dx\Big) + 2\omega e^{\omega t}\int_0^1 u_x^2(t)\,dx &\leq \omega e^{\omega t}\int_0^1 u^2(t)\,dx \\
&\leq \omega e^{-\omega t}\int_0^1 u_0(x,0)^2\,dx.
\end{aligned}
\tag{25}
$$

Integrating the above inequality from 0 to T_0 gives

$$
e^{\omega T_0}\int_0^1 u^2(T_0)\,dx + 2\omega\int_0^{T_0} e^{\omega t}\int_0^1 u_x^2(t)\,dxdt \leq (2-e^{-\omega T_0})\int_0^1 u_0(x,0)^2\,dx, \tag{26}
$$

which implies that

$$
\omega\int_0^{T_0} e^{\omega t}\int_0^1 u_x^2(t)\,dxdt \leq \int_0^1 u_0(x,0)^2\,dx. \tag{27}
$$

Consequently, we have

$$
\begin{aligned}
\int_0^{T_0} e^{\omega t}\int_0^1 u_x^2(t-\tau)\,dxdt &= \int_{-\tau}^{T_0-\tau} e^{\omega(s+\tau)}\int_0^1 u_x^2(s)\,dxds \\
&\leq \int_{-\tau}^{0} e^{\omega(s+\tau)}\int_0^1 u_{0x}^2(s)\,dxds \\
&\quad + \int_0^{T_0} e^{\omega(s+\tau)}\int_0^1 u_x^2(s)\,dxds \\
&\leq e^{\omega\tau}\int_{-\tau}^{0}\int_0^1 u_{0x}^2(s)\,dxds + \omega^{-1}e^{\omega\tau}\int_0^1 u_0(x,0)^2\,dx.
\end{aligned}
\tag{28}
$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 u_x^2(t)\,dx &= 2\int_0^1 u_x(t)u_{xt}(t)\,dx \\ &= -2\epsilon\int_0^1 u_{xx}^2(t)\,dx + 2\int_0^1 u_{xx}(t)u(t-\tau)u_x(t)\,dx \\ &\le \frac{1}{\epsilon}\int_0^1 u_x^2(t-\tau)\,dx\int_0^1 u_x^2(t)\,dx.\end{aligned} \tag{29}$$

Using Lemma 2.1 below with

$$\begin{aligned} y &= \int_0^1 u_x^2(t)\,dx, \\ g &= \frac{1}{\epsilon}\int_0^1 u_x^2(t-\tau)\,dx, \\ h &= 0, \\ \delta &= \omega, \\ C_1 &= \epsilon^{-1}\Big(e^{\omega\tau}\int_{-\tau}^0\int_0^1 u_{0x}^2(s)\,dxds + \omega^{-1}e^{\omega\tau}\int_0^1 u_0(x,0)^2\,dx\Big) \quad \text{(by (28))}, \\ C_2 &= 0, \\ C_3 &= \omega^{-1}\int_0^1 u_0(x,0)^2\,dx \quad \text{(by (27))}, \end{aligned}$$

it follows that for $0 \le t \le T_0$

$$\begin{aligned}\int_0^1 u_x^2(t)\,dx \le \int_0^1 &[u_0(x,0)^2 + u_{0x}(x,0)^2]\,dx \\ &\times \exp\Big[\epsilon^{-1}\Big(e^{\omega\tau}\int_{-\tau}^0\int_0^1 u_{0x}^2(s)\,dxds + \omega^{-1}e^{\omega\tau}\int_0^1 u_0(x,0)^2\,dx\Big)\Big]e^{-\omega t} \\ &\le \frac{K^2}{4}e^{-\omega t}.\end{aligned} \tag{30}$$

Hence

$$\|u_x(T_0)\| \le \frac{K}{2}e^{-\omega T_0/2},$$

which is in contradiction with (17). Therefore, we have proved that $T_0 = +\infty$ and then (14) follows from (30). □

LEMMA 2.1. *Let g, h and y be three positive and integrable functions on (t_0, T) such that y' is integrable on (t_0, T). Assume that*

$$\frac{dy}{dt} \le gy + h \quad \text{for } t_0 \le t \le T, \tag{31}$$

$$\int_{t_0}^T g(s)ds \le C_1, \tag{32}$$

$$\int_{t_0}^T e^{\delta s}h(s)ds \le C_2, \tag{33}$$

$$\int_{t_0}^T e^{\delta s}y(s)ds \le C_3, \tag{34}$$

where δ, C_1, C_2 *and* C_3 *are positive constants. Then*

$$y(t) \leq [C_2 + \delta C_3 + y(t_0)]e^{C_1}e^{-\delta(t-t_0)} \quad \text{for } t_0 \leq t \leq T. \tag{35}$$

Proof. Multiplying (31) by $e^{\delta t}$, we obtain

$$\frac{d}{dt}(e^{\delta t}y) \leq e^{\delta t}gy + e^{\delta t}h + \delta e^{\delta t}y \quad \text{for } t \geq t_0. \tag{36}$$

By Gronwall's inequality (see, e.g., [27, p.90]), we deduce

$$\begin{aligned} e^{\delta t}y(t) &\leq e^{\delta t_0}y(t_0)\exp\Big(\int_{t_0}^{t} g(s)ds\Big) \\ &\quad + \int_{t_0}^{t}\Big(e^{\delta s}h(s) + \delta e^{\delta s}y(s)\Big)\exp\Big(-\int_{t}^{s} g(\tau)d\tau\Big)ds \\ &\leq (C_2 + \delta C_3)e^{C_1} + e^{\delta t_0 + C_1}y(t_0), \end{aligned} \tag{37}$$

which implies (35). □

REMARK 1. Note that there is no boundedness assumption on the solution u in the above theorem because of the nature of Burgers' equation. Usually, we need a boundedness assumption such as

$$\sup_{t\geq 0}\sup_{0\leq x\leq 1}\|u(x,t)\| < \infty$$

to obtain further stability results in the study of long-time behavior of delayed systems due to the complex nature of other equations(see, e.g., [14, Theorem 1], [25, Theorem 5]).

REMARK 2. It can been seen from (12) that τ tends to zero if the initial state tends to infinity or $\epsilon \to 0$.

3. **Numerical Simulations.** In this section we give numerical simulations for the theoretical results of the last section. The approximation scheme we used here for problem (1)-(3) is the central difference approximation (see, e.g., [3, Chap.2]):

$$\begin{aligned} \frac{1}{\delta}(u_{i,j+1} - u_{i,j}) &= \frac{\epsilon}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \\ &\quad - u_0(i/20, j\delta - \tau)(u_{i+1,j} - u_{i,j})/h, \quad \text{if } j\delta - \tau \leq 0, \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{1}{\delta}(u_{i,j+1} - u_{i,j}) &= \frac{\epsilon}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \\ &\quad - u_{i,j-K}(u_{i+1,j} - u_{i,j})/h, \quad \text{if } j\delta - \tau > 0, \end{aligned} \tag{39}$$

where

$$\delta = \frac{3}{15000}, \tag{40}$$

$$h = \frac{1}{20}, \tag{41}$$

$$u_{i,j} = u(ih, j\delta), \qquad i = 0, 1, \ldots, 20, \quad j = 0, 1, \ldots, 15000, \tag{42}$$

and K denotes the largest integer less than τ/δ. The initial condition we take here is

$$u_0(x,s) = 20(1-s)\sin(5\pi x).$$

For simplicity, we take $\epsilon = 1$. Since $r = \frac{\delta\epsilon}{h^2} = \frac{4}{50} < \frac{1}{2}$, our difference scheme is convergent (see, e.g., [3, p.45]).

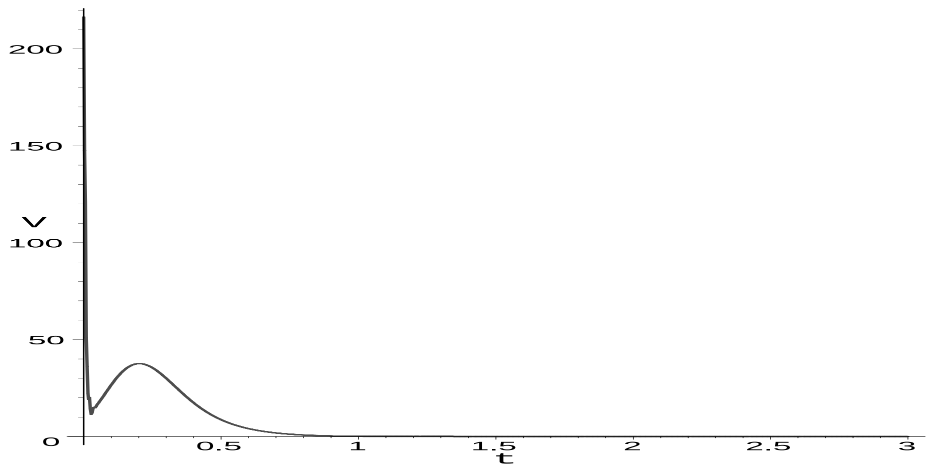

FIGURE 1. H^1 norm of an approximate solution with $\epsilon = 1,\ \tau = 0.9$ and $u_0(x,s) = 20(1-s)\sin(5\pi x)$.

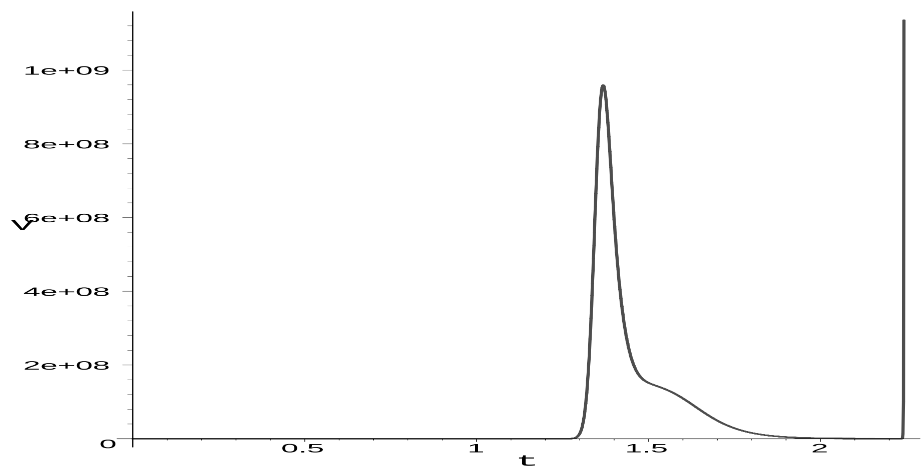

FIGURE 2. H^1 norm of an approximate solution with $\epsilon = 1,\ \tau = 1$ and $u_0(x,s) = 20(1-s)\sin(5\pi x)$.

In Figures 1 and 2, V denotes the H^1 norm given by $V(t) = \left(\int_0^1 u^2(x,t)dx\right)^{1/2}$. It can be seen from Figures 1 that the H^1 norm of approximate solution of (1)-(3) with $\tau = 0.9$ decays to zero quickly. When τ increases to 1, Figures 2 shows that the H^1 norm of approximate solution oscillatorily grows up. This accords with our theoretical results.

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