International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

Boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures

First Published on: 30 April 2007
To cite this Article: Liu, W. and Krstic, M. , 'Boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures', International Journal of Control, 80:6, 1 - 8
To link to this article: DOI: 10.1080/00207170701280895
URL: http://dx.doi.org/10.1080/00207170701280895

International Journal of Control
2007, 1–8, iFirst

Boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures

W. LIU*† and M. KRSTIC‡

†Department of Mathematics, University of Central Arkansas,
201 Donaghey Avenue, Conway, AR 72035, USA
‡Dept of Mechanical and Aerospace Engineering,
University of California, San Diego, La Jolla, CA 92093-0411, USA

(*Received 12 September 2006; in final form 14 February 2007*)

We consider the problem of boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures that are governed by unstable linear reaction-convection-diffusion equations. We extend boundary feedback control laws designed for the one-dimensional reaction-diffusion equation using the backstepping method to this higher-dimensional case. We show that, under certain mathematical conditions on the velocity field, boundary feedback controls similar to the ones for one-dimensional equations also works for the higher dimensional case and exponentially stabilize the homogeneous equilibrium zero at any given decay rate.

1. Introduction

A diffusive fluid mixture consists of diffusive physical quantities and a fluid flow in which the physical quantities are immersed. Typical examples of such physical quantities include fuel in a combustor and chemical pollutants dispersing in the environment. These physical quantities can be described as diffusive scalars. If a scalar like the fuel does not significantly influence the fluid motion, it is called a passive scalar. The scalar usually undergoes three processes: chemical reaction; molecular diffusion (mixing); and convection. These three processes can be mathematically modelled by the reaction-convection-diffusion equation (Bees *et al.* 2001)

$$\begin{aligned} \frac{\partial c}{\partial t} + \nabla\cdot(c\mathbf{v}) &= \kappa\nabla^2 c + ac \quad &&\text{in } \Omega,\\ \kappa\frac{\partial c}{\partial \mathbf{n}} &= 0 &&\text{on } \partial\Omega,\\ c(\mathbf{x}, t_0) &= c^0(\mathbf{x}) &&\text{in } \Omega. \end{aligned}$$

In the above equation, $c = c(\mathbf{x}, t)$ denotes the concentration of the scalar, $c^0(\mathbf{x})$ denotes the initial concentration, $a = a(\mathbf{x}, t)$ is the reaction rate, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_2(\mathbf{x}, t))$ denotes a velocity field, $\kappa > 0$ is the molecular diffusivity of the scalar, Ω is a bounded domain in $\mathbb{R}^3$, $\partial/\partial n$ denotes the normal derivative along the boundary $\partial\Omega$ of Ω, $\nabla = ((\partial/\partial x), (\partial/\partial y), (\partial/\partial x))$ and $\nabla^2 = ((\partial^2/\partial x^2), (\partial^2/\partial y^2), (\partial^2/\partial x^2))$. In this paper, we fix the above no-flux boundary condition for concreteness, but our results here are equally valid for other boundary conditions.

Evidently, if the reaction rate a is large and the diffusivity κ is small, then the homogeneous equilibrium state $c = 0$ is unstable. However, in reality, a certain level of homogeneity of a fluid mixture is often desired. For instance, before fuel is burned in a combustor, it is required to be well mixed so that the combustor has its best efficiency. Hence, it is important to find efficient and practical control strategies to enhance mixing and stabilize the equilibrium $c = 0$.

In fluid mixing, the flow field is a natural mechanism for enhancing mixing (Antonsen *et al.* 1996, Giona *et al.* 2004a,b, Liu and Haller 2004, Haynes and Vanneste 2005, Liu 2005). Therefore, a practical strategy for the enhancement of mixing is to destabilize a flow. Modern approaches of flow destabilization include passive control devices like a backward-facing step (Yang *et al.* 2002), excitation of large-scale coherent structures

*Corresponding author. Email: weijiul@uca.edu

International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170701280895

in the flow through flaps, wall-jets, or other devices (Greenblatt and Wygnanski 2000), active feedback destabilization (Aamo *et al.* 2003) and generation of flow separation (Wang *et al.* 2003).

However, if the reaction rate a is quite large, then this approach of flow destabilization does not work in stabilizing the homogeneous equilibrium state and other control mechanisms for the concentration field have to be introduced. One of them is to control the flux of a physical quantity from the boundary of a domain. For instance, the fuel is constantly injected from the top of a cylinder. The amount of the physical quantity flowing in or out of the domain should be controlled in a way such that the equilibrium 0 is stabilized. This control strategy can be mathematically stated as follows. For simplicity, we assume that $\Omega = [0,1] \times S$ and apply a flux control only on the face $x = 1$, where S is a bounded domain in $\mathbb{R}^2$. Then the boundary control problem is as follows

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{v}) = \kappa \nabla^2 c + ac \quad \text{in } \Omega, \tag{1}$$

$$\kappa \frac{\partial c(1, y, z, t)}{\partial x} = u, \quad (y, z) \in S, \tag{2}$$

$$\kappa \frac{\partial c(0, y, z, t)}{\partial x} = 0, \quad (y, z) \in S, \tag{3}$$

$$\kappa \frac{\partial c}{\partial \mathbf{n}} = 0, \quad 0 < x < 1, \quad (y, z) \in \partial S, \tag{4}$$

$$c(\mathbf{x}, t_0) = c^0(\mathbf{x}) \quad \text{in } \Omega, \tag{5}$$

where u is a control to be designed such that the controlled system is exponentially stable with arbitrarily given decay rate $\gamma > 0$. That is, the concentration c decays to zero exponentially at the rate γ.

If the reaction rate a is much greater than the diffusivity κ, the change of the concentration inside the domain is quick and this quick change cannot immediately influence the concentration on the face $x = 1$ due to the small diffusivity, which is a measure of diffusion speed. Therefore, to determine the amount of the flux u, one should need the measurement of the concentration c not only on the face $x = 1$ but also inside the domain. Mathematically, this means that $u = u(c(1, y, z, t), c(x, y, z, t))$ is a functional of $c(1, y, z, t)$ and $c(x, y, z, t)$. In fact, we will show that

$$u = -\kappa K(1,1) c(1, y, z, t) - \kappa \int_0^1 \frac{\partial K(1, \xi)}{\partial x} c(\xi, y, z, t)\, \mathrm{d}\xi,$$

where the kernel K needs to be designed.

The problem of boundary feedback control for the reaction-convection-diffusion equation is not new. Important results on feedback stabilization of general parabolic equations include the work of Triggiani (1980), Day (1982), Lasiecka and Triggiani (1983a,b, 1987a,b), Amamm (1989), Burns *et al.* (1996) and Burns and Rubio (1998). These results were obtained mainly by using the abstract semigroup theory or optimal control theory.

Recently, borrowing an idea of backstepping from finite dimensional control systems and an idea of integral transformation from the theory of parabolic partial differential equations (Colton 1977, 1980), a backstepping method was developed to construct explicit feedback control laws for unstable one-dimensional reaction-diffusions (Balogh and Krstic 2001, 2002), but some problems like the well-posedness of a kernel equation were left open. These open problems were solved by Liu (2003). After the resolution of these open problems, the backstepping method has been successfully applied to many other more complicated one-dimensional equations (Aamo *et al.* 2005, Smyshlyaev and Krstic 2004, 2005a), including the linearized Ginzburg-Landau equation.

When applying the backstepping method to higher-dimensional reaction-convection-diffusion equations, great difficulties rise. The first attempt in this aspect was made by Krstic (2005a,b) and Smyshlyaev and Krstic (2005b,c) for constant velocities and reaction rates. Since the constant velocities are not desired in the problems of fluid mixtures, we investigate the varying velocity case in this paper. We will show that, under certain physically reasonable conditions on the velocity field, similar boundary feedback controls designed for the one-dimensional equation also works for the higher dimensional case.

2. Kernel equations

To design a boundary feedback control law, we consider the following integral transformation (Krstic 2005a, Smyshlyaev and Krstic 2005b,c)

$$\theta(x, y, z, t) = c(x, y, z, t) + \int_0^x K(x, \xi, t) c(\xi, y, z, t)\, \mathrm{d}\xi, \tag{6}$$

where the kernel K needs to be determined. We then want to find a kernel K that transforms the controlled problem (1)–(5) into the following exponentially stable equation

$$\frac{\partial \theta}{\partial t} + \nabla \cdot (\theta \mathbf{v}) = \kappa \nabla^2 \theta - \gamma \theta \quad \text{in } \Omega, \tag{7}$$

$$\kappa \frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{8}$$

where γ is any positive constant, standing for the decay rate. The reason why the assumptions on the velocity in the following lemma are needed will be explained in the proof of the lemma.

Lemma 1: *Assume that the reaction rate* $a = a(x, t)$ *and assume that the first component* $v_1 = v_1(x, t)$ *with* $v_1(0, t) = 0$ *of the velocity field does not depend on* y, z, *and the second and third components* $v_i = v_i\ (y, z, t)$ $(i = 2, 3)$ *do not depend on* x. *If the kernel* K *satisfies*

$$\frac{\partial K}{\partial t} + v_1(x,t)\frac{\partial K}{\partial x} + v_1(\xi,t)\frac{\partial K}{\partial \xi} = \kappa\frac{\partial^2 K}{\partial x^2} - \kappa\frac{\partial^2 K}{\partial \xi^2} - \left[\gamma + a(\xi,t) + \frac{\partial v_1(x,t)}{\partial x}\right]K, \quad 0 \le \xi \le x \le 1, \tag{9}$$

$$\frac{\partial K}{\partial \xi}(x,x,t) + \frac{\partial K}{\partial x}(x,x,t) + \frac{\partial}{\partial x}(K(x,x,t)) = \frac{\gamma + a(\xi,t)}{\kappa}, \quad 0 \le x \le 1, \tag{10}$$

$$\frac{\partial K}{\partial \xi}(x,0,t) = 0, \quad 0 \le x \le 1, \tag{11}$$

$$K(0,0,t) = 0. \tag{12}$$

Then the integral transformation (6) *transforms the system* (1)–(5) *with the controller*

$$\frac{\partial c}{\partial x}(1,y,z,t) = -K(1,1,t)c(1,y,z,t) - \int_0^1 \frac{\partial K}{\partial x}(1,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \tag{13}$$

into the system (7)–(8).

Proof: We calculate various derivatives of θ as follows:

$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= \frac{\partial c}{\partial t} + \int_0^x \frac{\partial K}{\partial t}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\quad + \kappa\int_0^x K(x,\xi,t)\left(\frac{\partial^2 c}{\partial \xi^2}(\xi,y,z,t) + \frac{\partial^2 c}{\partial y^2}(\xi,y,z,t) + \frac{\partial^2 c}{\partial z^2}(\xi,y,z,t)\right)\mathrm{d}\xi \\
&\quad + \int_0^x K(x,\xi,t)a(\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\quad - \int_0^x K(x,\xi,t)\left[\frac{\partial}{\partial \xi}[c(\xi,y,z,t)v_1(\xi,t)] + \frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)]\right]\mathrm{d}\xi \\
&\quad - \int_0^x K(x,\xi,t)\frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\,\mathrm{d}\xi \\
&= \frac{\partial c}{\partial t} + \int_0^x \frac{\partial K}{\partial t}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\quad + \kappa K(x,x,t)\frac{\partial c}{\partial x}(x,y,z,t) - \kappa K(x,0,t)\frac{\partial c}{\partial x}(0,y,z,t) \\
&\quad - \kappa\frac{\partial K}{\partial \xi}(x,x,t)c(x,y,z,t) + \kappa\frac{\partial K}{\partial \xi}(x,0,t)c(0,y,z,t) \\
&\quad + \int_0^x \kappa\left(\frac{\partial^2 K}{\partial \xi^2}(x,\xi,t)c(\xi,y,z,t) + K(x,\xi,t) \times\left(\frac{\partial^2 c}{\partial y^2}(\xi,y,z,t) + \frac{\partial^2 c}{\partial z^2}(\xi,y,z,t)\right)\right)\mathrm{d}\xi \\
&\quad + \int_0^x K(x,\xi,t)a(\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\quad - K(x,x,t)v_1(x,t)c(x,y,z,t) \\
&\quad + K(x,0,t)v_1(0,t)c(0,y,z,t) \\
&\quad + \int_0^x c(\xi,y,z,t)v_1(\xi,t)\frac{\partial}{\partial \xi}(K(x,\xi,t))\,\mathrm{d}\xi \\
&\quad - \int_0^x K(x,\xi,t)\left[\frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)] + \frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\right]\mathrm{d}\xi,
\end{aligned} \tag{14}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial c}{\partial x} + K(x,x,t)c(x,y,z,t) + \int_0^x \frac{\partial K}{\partial x}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi, \tag{15}$$

$$\frac{\partial (v_1\theta)}{\partial x} = \frac{\partial (v_1 c)}{\partial x} + \frac{\partial v_1}{\partial x}\int_0^x K(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi + v_1(x,t)\left[K(x,x,t)c(x,y,z,t) + \int_0^x \frac{\partial K}{\partial x}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi\right], \tag{16}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial}{\partial x}(K(x,x,t))c(x,y,z,t) + K(x,x,t)\frac{\partial c(x,y,z,t)}{\partial x} + \frac{\partial K}{\partial x}(x,x,t)c(x,y,z,t) + \int_0^x \frac{\partial^2 K}{\partial x^2}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi, \tag{17}$$

$$\frac{\partial (\theta v_2)}{\partial y} = \frac{\partial (c v_2)}{\partial y} + \int_0^x K(x,\xi,t)\frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)]\,\mathrm{d}\xi, \tag{18}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2 c}{\partial y^2} + \int_0^x K(x,\xi,t)\frac{\partial^2 c}{\partial y^2}(\xi,y,z,t)\,\mathrm{d}\xi, \tag{19}$$

$$\frac{\partial(\theta v_3)}{\partial z} = \frac{\partial(cv_3)}{\partial z} + \int_0^x K(x,\xi,t)\frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\,\mathrm{d}\xi, \tag{20}$$

and

$$\frac{\partial^2\theta}{\partial z^2} = \frac{\partial^2 c}{\partial z^2} + \int_0^x K(x,\xi,t)\frac{\partial^2 c}{\partial z^2}(\xi,y,z,t)\,\mathrm{d}\xi. \tag{21}$$

Note that the independence of v_2, v_3 on x is needed so that the second terms in (18) and (20) can be cancelled by the term

$$-\int_0^x K(x,\xi,t)\left[\frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)] + \frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\right]\mathrm{d}\xi$$

in (14). It then follows from the above equations that

$$\begin{aligned}
&\frac{\partial\theta}{\partial t} + \nabla\cdot(\theta\mathbf{v}) - \kappa\nabla^2\theta + \gamma\theta \\
&\quad= \frac{\partial c}{\partial t} + \int_0^x \frac{\partial K}{\partial t}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad + \kappa K(x,x,t)\frac{\partial c}{\partial x}(x,y,z,t) - \kappa K(x,0,t)\frac{\partial c}{\partial x}(0,y,z,t) \\
&\qquad - \kappa\frac{\partial K}{\partial \xi}(x,x,t)c(x,y,z,t) + \kappa\frac{\partial K}{\partial \xi}(x,0,t)c(0,y,z,t) \\
&\qquad + \int_0^x \kappa\left(\frac{\partial^2 K}{\partial \xi^2}(x,\xi,t)c(\xi,y,z,t) + K(x,\xi,t)\right. \\
&\qquad \left.\times\left(\frac{\partial^2 c}{\partial y^2}(\xi,y,z,t) + \frac{\partial^2 c}{\partial z^2}(\xi,y,z,t)\right)\right)\mathrm{d}\xi \\
&\qquad + \int_0^x K(x,\xi,t)a(\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad - K(x,x,t)v_1(x,t)c(x,y,z,t) + K(x,0,t)v_1(0,t)c(0,y,z,t) \\
&\qquad + \int_0^x c(\xi,y,z,t)v_1(\xi,t)\frac{\partial}{\partial\xi}(K(x,\xi,t))\,\mathrm{d}\xi \\
&\qquad - \int_0^x K(x,\xi,t)\left[\frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)]\right. \\
&\qquad \left. + \frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\right]\mathrm{d}\xi \\
&\qquad + \frac{\partial(v_1 c)}{\partial x} + \frac{\partial v_1}{\partial x}\int_0^x K(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad + v_1(x,t)\left[K(x,x,t)c(x,y,z,t)\right. \\
&\qquad \left. + \int_0^x \frac{\partial K}{\partial x}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi\right] \\
&\qquad + \frac{\partial(cv_2)}{\partial y} + \int_0^x K(x,\xi,t)\frac{\partial}{\partial y}[c(\xi,y,z,t)v_2(y,z,t)]\,\mathrm{d}\xi \\
&\qquad + \frac{\partial(cv_3)}{\partial z} + \int_0^x K(x,\xi,t)\frac{\partial}{\partial z}[c(\xi,y,z,t)v_3(y,z,t)]\,\mathrm{d}\xi \\
&\qquad - \kappa\frac{\partial^2 c}{\partial x^2} - \kappa\frac{\partial}{\partial x}(K(x,x,t))c(x,y,z,t) \\
&\qquad - \kappa K(x,x,t)\frac{\partial c(x,y,z,t)}{\partial x} \\
&\qquad - \kappa\frac{\partial K}{\partial x}(x,x,t)c(x,y,z,t) - \kappa\int_0^x \frac{\partial^2 K}{\partial x^2}(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad - \kappa\frac{\partial^2 c}{\partial y^2} - \kappa\int_0^x K(x,\xi,t)\frac{\partial^2 c}{\partial y^2}(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad - \kappa\frac{\partial^2 c}{\partial z^2} - \kappa\int_0^x K(x,\xi,t)\frac{\partial^2 c}{\partial z^2}(\xi,y,z,t)\,\mathrm{d}\xi \\
&\qquad + \gamma c(x,y,z,t) + \gamma\int_0^x K(x,\xi,t)c(\xi,y,z,t)\,\mathrm{d}\xi \\
&\quad= -\kappa K(x,0,t)\frac{\partial c}{\partial x}(0,y,z,t) + \kappa\frac{\partial K}{\partial\xi}(x,0,t)c(0,y,z,t) \\
&\qquad + K(x,0,t)v_1(0,t)c(0,y,z,t) \\
&\qquad + c\left[a(x,t) + \gamma - \kappa\frac{\partial K}{\partial\xi}(x,x,t) - \kappa\frac{\partial K}{\partial x}(x,x,t) - \kappa\frac{\partial}{\partial x}(K(x,x,t))\right] \\
&\qquad + \int_0^x c(\xi,y,z,t)\left[\frac{\partial K}{\partial t}(x,\xi,t) + \kappa\frac{\partial^2 K}{\partial\xi^2}(x,\xi,t) - \kappa\frac{\partial^2 K}{\partial x^2}(x,\xi,t)\right]\mathrm{d}\xi \\
&\qquad + \int_0^x c(\xi,y,z,t)\left[a(\xi,t) + \gamma + \frac{\partial v_1}{\partial x}\right]K(x,\xi,t)\,\mathrm{d}\xi \\
&\qquad + \int_0^x c(\xi,y,z,t)\left[a(\xi,t) + \gamma + \frac{\partial v_1}{\partial x}\right]K(x,\xi,t)\,\mathrm{d}\xi \\
&\qquad + \int_0^x c(\xi,y,z,t)\left(v_1(\xi,t)\frac{\partial K(x,\xi,t)}{\partial\xi} + v_1(x,t)\frac{\partial K(x,\xi,t)}{\partial x}\right)\mathrm{d}\xi \\
&\quad= 0.
\end{aligned} \tag{22}$$

From this equation we can see the reason why v_1 is required to be independent of y, z is because we need to assume that the kernel K is independent of y, z. Our other calculations showed that if K depends on all three variables x, y, z, then no result can be obtained. In the above last equation, we have used equations (9)–(12) and the boundary conditions (3) and (4). The homogeneous boundary condition (8) follows from the boundary conditions (3)–(4) and the controller (13). □

If $a = a(x)$ is independent of t, it is clear that the kernel K can be chosen to be independent of t. In this case, the kernel equation is as follows:

$$\begin{aligned}
&\kappa\frac{\partial^2 K}{\partial x^2} - \kappa\frac{\partial^2 K}{\partial\xi^2} - v_1(x)\frac{\partial K}{\partial x} - v_1(\xi)\frac{\partial K}{\partial\xi} \\
&\qquad = [\gamma + a(\xi) + v_1'(x)]K, \quad 0 \le \xi \le x \le 1,
\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial K}{\partial \xi}(x,x)+\frac{\partial K}{\partial x}(x,x)+\frac{\partial}{\partial x}(K(x,x))\\ =\frac{\gamma+a(x)}{\kappa},\quad 0\le x\le 1,\end{aligned}\tag{24}$$

$$\frac{\partial K}{\partial \xi}(x,0)=0,\quad 0\le x\le 1,\tag{25}$$

$$K(0,0)=0.\tag{26}$$

Note that this boundary value problem is the same as the one derived for one-dimensional reaction-diffusion equations (see (Balogh and Krstic (2001, 2002, 2004) and Liu (2003)) if $v_1=0$. □

If $a\in C^1[0,1]$ and $v_1=0$, it was proved by Liu (2003) that the boundary value problem (23)–(26) has a solution which is twice continuously differentiable in $0\le\xi\le x\le 1$. We now show that the same is also true for $v_1\ne 0$.

Lemma 2: *Suppose that* $a=a(x)\in C^1[0,1]$ *and* $v_1=v_1(x)\in C^1[0,1]$. *Then the boundary value problem* (23)–(26) *has an unique solution which is twice continuously differentiable in* $0\le y\le x\le 1$.

Proof: Using the variable changes

$$\eta=x+\xi,\quad \zeta=x-\xi$$

and denoting

$$G(\eta,\zeta)=k(x,\xi)=k\left(\frac{\eta+\zeta}{2},\frac{\eta-\zeta}{2}\right),$$

we transform the problem (23)–(26) into

$$\frac{\partial^2 G}{\partial\eta\partial\zeta}-\varphi_1\frac{\partial G}{\partial\eta}-\varphi_2\frac{\partial G}{\partial\zeta}=\varphi_3 G,\quad 0\le\zeta\le\eta\le 2,\tag{27}$$

$$\frac{\partial G(\eta,\eta)}{\partial\eta}=\frac{\partial G(\eta,\eta)}{\partial\zeta},\quad 0\le\eta\le 2,\tag{28}$$

$$\frac{\partial}{\partial\eta}(G(\eta,0))=\varphi_4(\eta),\quad 0\le\eta\le 2,\tag{29}$$

$$G(0,0)=0.\tag{30}$$

where

$$\begin{aligned}\varphi_1&=\varphi_1(\eta,\zeta)=\frac{1}{4\kappa}\left(v_1\left(\frac{\eta+\zeta}{2}\right)+v_1\left(\frac{\eta-\zeta}{2}\right)\right),\\ \varphi_2&=\varphi_2(\eta,\zeta)=\frac{1}{4\kappa}\left(v_1\left(\frac{\eta+\zeta}{2}\right)+v_1\left(\frac{\eta-\zeta}{2}\right)\right),\\ \varphi_3&=\varphi_3(\eta,\zeta)=\frac{1}{4\kappa}\left(\gamma+a\left(\frac{\eta+\zeta}{2}\right)+v_1'\left(\frac{\eta+\zeta}{2}\right)\right),\\ \varphi_4&=\varphi_4(\eta)=\frac{1}{4\kappa}\left(\gamma+a\left(\frac{\eta}{2}\right)\right).\end{aligned}$$

To transform the differential equation (27) into an integral equation, we first find $G(\eta,\eta)$. The equation (27) can be written as

$$\begin{aligned}\frac{\partial}{\partial\zeta}\left[\frac{\partial G}{\partial\eta}e^{-\int_0^\zeta\varphi_1(\eta,\tau)\,\mathrm{d}\tau}\right]-e^{-\int_0^\zeta\varphi_1(\eta,\tau)\,\mathrm{d}\tau}\varphi_2\frac{\partial G}{\partial\zeta}\\ =e^{-\int_0^\zeta\varphi_1(\eta,\tau)\,\mathrm{d}\tau}\varphi_3 G.\end{aligned}\tag{31}$$

Integrating this equation with respect to ζ from 0 to η gives

$$\begin{aligned}\frac{\partial G(\eta,\eta)}{\partial\eta}&=\frac{\partial G(\eta,0)}{\partial\eta}e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}+\varphi_2(\eta,\eta)G(\eta,\eta)\\ &\quad+e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\varphi_3(\eta,\zeta)e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}G(\eta,\zeta)\,\mathrm{d}\zeta\\ &\quad-e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\frac{\partial}{\partial\zeta}\left(e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}\varphi_2(\eta,\zeta)\right)G(\eta,\zeta)\,\mathrm{d}\zeta\\ &=\varphi_4(\eta)e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}+\varphi_2(\eta,\eta)G(\eta,\eta)\\ &\quad+e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\varphi_3(\eta,\zeta)e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}G(\eta,\zeta)\,\mathrm{d}\zeta\\ &\quad-e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\frac{\partial}{\partial\zeta}\left(e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}\varphi_2(\eta,\zeta)\right)G(\eta,\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

It then follows from (28) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}[G(\eta,\eta)]&=\frac{\partial}{\partial\eta}[G(\eta,\eta)]+\frac{\partial}{\partial\zeta}[G(\eta,\eta)]\\ &=2\frac{\partial}{\partial\eta}[G(\eta,\eta)]\\ &=2\varphi_4(\eta)e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}+2\varphi_2(\eta,\eta)G(\eta,\eta)\\ &\quad+2e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\varphi_3(\eta,\zeta)e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}G(\eta,\zeta)\,\mathrm{d}\zeta\\ &\quad-2e^{\int_0^\eta\varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\eta\frac{\partial}{\partial\zeta}\left(e^{-\int_0^\zeta\varphi_1(\eta,\tau)\mathrm{d}\tau}\varphi_2(\eta,\zeta)\right)\\ &\quad\times G(\eta,\zeta)\,\mathrm{d}\zeta.\end{aligned}\tag{33}$$

Solving this equation for $G(\eta,\eta)$ gives

$$\begin{aligned}G(\eta,\eta)&=2e^{2\int_0^\eta\varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\eta\varphi_4(s)e^{\int_0^s\varphi_1(s,\tau)-2\varphi_2(\tau,\tau)\mathrm{d}\tau}\mathrm{d}s\\ &\quad+2e^{2\int_0^\eta\varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\eta e^{\int_0^s(\varphi_1(s,\tau)-2\varphi_2(\tau,\tau)\mathrm{d}\tau}\\ &\quad\times\int_0^s\varphi_3(s,\zeta)e^{-\int_0^\zeta\varphi_1(s,\tau)d\tau}G(s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s\\ &\quad-2e^{2\int_0^\eta\varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\eta e^{\int_0^s(\varphi_1(s,\tau)-2\varphi_2(\tau,\tau)\mathrm{d}\tau}\\ &\quad\times\int_0^s\frac{\partial}{\partial\zeta}\left(e^{-\int_0^\zeta\varphi_1(s,\tau)d\tau}\varphi_2(s,\zeta)\right)G(s,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s.\end{aligned}\tag{34}$$

We are now ready to transform the differential equation (27) into an integral equation. Integrating the equation (31) with respect to ζ from 0 to ζ gives

$$\begin{aligned}\frac{\partial G(\eta,\zeta)}{\partial \eta} &= \varphi_4(\eta)e^{\int_0^\zeta \varphi_1(\eta,\tau)\mathrm{d}\tau} + \varphi_2(\eta,\zeta)G(\eta,\zeta)\\ &+ e^{\int_0^\zeta \varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\zeta \varphi_3(\eta,s)e^{-\int_0^s \varphi_1(\eta,\tau)\mathrm{d}\tau}G(\eta,s)\,\mathrm{d}s\\ &- e^{\int_0^\zeta \varphi_1(\eta,\tau)\mathrm{d}\tau}\int_0^\zeta \frac{\partial}{\partial s}\Big(e^{-\int_0^s \varphi_1(\eta,\tau)\mathrm{d}\tau}\varphi_2(\eta,s)\Big)G(\eta,s)\,\mathrm{d}s.\end{aligned}$$

Integrating this equation with respect to η from ζ to η and using (34), we obtain the following integral equation

$$\begin{aligned}G(\eta,\zeta) &= 2e^{2\int_0^\zeta \varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\zeta \varphi_4(s)e^{\int_0^s(\varphi_1(s,\tau)-2\varphi_2(\tau,\tau))\mathrm{d}\tau}\mathrm{d}s\\ &+ 2e^{2\int_0^\zeta \varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\zeta e^{\int_0^s(\varphi_1(s,\tau)-2\varphi_2(\tau,\tau))\mathrm{d}\tau}\\ &\times \int_0^s \varphi_3(s,r)e^{-\int_0^r \varphi_1(s,\tau)\mathrm{d}\tau}G(s,r)\,\mathrm{d}r\,\mathrm{d}s\\ &- 2e^{2\int_0^\zeta \varphi_2(\tau,\tau)\mathrm{d}\tau}\int_0^\zeta e^{\int_0^s(\varphi_1(s,\tau)-2\varphi_2(\tau,\tau))d\tau}\\ &\times \int_0^s \frac{\partial}{\partial r}\Big(e^{-\int_0^r \varphi_1(s,\tau)\mathrm{d}\tau}\varphi_2(s,r)\Big)G(s,r)\,\mathrm{d}r\,\mathrm{d}s\\ &+ \int_\zeta^\eta \varphi_4(r)e^{\int_0^\zeta \varphi_1(r,\tau)\mathrm{d}\tau}\mathrm{d}r + \int_\zeta^\eta \varphi_2(r,\zeta)G(r,\zeta)\,\mathrm{d}r\\ &+ \int_\zeta^\eta e^{\int_0^\zeta \varphi_1(r,\tau)\mathrm{d}\tau}\int_0^\zeta \varphi_3(r,s)e^{-\int_0^s \varphi_1(r,\tau)\mathrm{d}\tau}G(r,s)\,\mathrm{d}s\,\mathrm{d}r\\ &- \int_\zeta^\eta e^{\int_0^\zeta \varphi_1(r,\tau)\mathrm{d}\tau}\int_0^\zeta \frac{\partial}{\partial s}\\ &\times\Big(e^{-\int_0^s \varphi_1(r,\tau)d\tau}\varphi_2(r,s)\Big)G(r,s)\,\mathrm{d}s\,\mathrm{d}r. \end{aligned} \tag{35}$$

As in the proof of Lemma 2.2 of Liu (2003), by the method of successive approximations we can show that this equation has a unique continuous solution. Moreover, it follows from (35) that G is twice continuously differentiable because a and $v_1 \in C^1[0,1]$. □

For the time-dependent kernel problem (9)–(12), it was shown (Colton 1977, 1980, Smyshlyaev and Krstic 2004) that if a is analytic in t and $v_1 = 0$, then the boundary value problem has an unique solution for small time. Whether or not the time-dependent problem has a solution for all time $t > 0$ is open even in the case of $v_1 = 0$. We do not want to go further to study the existence of a solution of (9)–(12) for small time because this smallness on time does not meet the requirement for stabilization problem, in which the main concern is about the large time behavior of the concentration c.

3. Exponential stabilization

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (Evans 1998) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. The L^2 norm of a function $f(\mathbf{x}) \in L^2(\Omega)$ is denoted by

$$\|f\| = \left(\int_\Omega |f(\mathbf{x})|^2 \mathrm{d}V\right)^{1/2}.$$

Let K be the solution of the boundary value problem (23)–(26). Then the following feedback control law

$$\begin{aligned}\frac{\partial c}{\partial x}(1,y,z,t) &= -K(1,1)c(1,y,z,t)\\ &\quad - \int_0^1 \frac{\partial K(1,\xi)}{\partial x}c(\xi,y,z,t)\,\mathrm{d}\xi\end{aligned} \tag{36}$$

exponentially stabilizes the equilibrium 0 of the problem (1)–(5).

Theorem 1: *Assume that $\gamma > 0$ is any positive constant and $a = a(x) \in C^1[0,1]$ is any function. Assume that the velocity $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the following conditions:*

(i) *the velocity satisfies the no-penetration condition on the boundary of Ω: $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$;*
(ii) *$v_1 = v_1(x)$ is independent of y, z, t;*
(iii) *$v_2 = v_2(y,z,t)$ and $v_3 = v_3(y,z,t)$ are independent of x;*
(iv) *$v_0 = \max_{t\geq t_0, \mathbf{x}\in\Omega}\{|\mathrm{div}(\mathbf{v}(\mathbf{x},t)|, |\nabla\mathbf{v}(\mathrm{x},t)|, |\nabla\mathrm{div}(\mathbf{v}(\mathbf{x},t))|\} < \infty.$*

Then, for arbitrary initial data $c^0(x) \in H^1(0,1)$, the problem (1)–(5) with the controller (36) has a unique solution that satisfies

$$\|c(t)\|_{H^1} \leq M\|c^0\|_{H^1}e^{-(\gamma-2v_0)(t-t_0)}, \tag{37}$$

where M is a positive constant independent of c^0.

Proof: Since the proof of this theorem is similar to the one of Theorem 3.1 of Liu (2003), we give here just a sketch. By Lemma 3.3 of Liu (2003), if K is the solution of the problem (23)–(26), then the linear bounded operator K: $H^i(\Omega) \to H^i(\Omega)$ $(i = 0,1,2)$ defined by

$$\begin{aligned}w(x,y,z) &= (Ku)(x,y,z) = u(x,y,z)\\ &\quad + \int_0^x K(x,\xi)u(\xi,y,z)\,\mathrm{d}\xi \quad \text{for} \quad u \in H^i(\Omega)\end{aligned} \tag{38}$$

has a linear bounded inverse K^{-1}: $H^i(\Omega) \to H^i(\Omega)$ $(i = 0,1,2)$. Therefore, it suffices to show that the solution of the problem (7)–(8) satisfies

$$\|\theta(t)\|_{H^1} \leq \|\theta(t_0)\|_{H^1}e^{-(\gamma-2v_0)(t-t_0)}. \tag{39}$$

Multiplying (7) by θ, integrating over Ω by parts, and using the boundary conditions on θ and the no-penetration boundary condition on $\mathbf{v}$, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\theta(t)|^2 \mathrm{d}V &= -\kappa\int_\Omega |\nabla\theta(t)|^2 dV - \gamma\int_\Omega |\theta(t)|^2\,\mathrm{d}V\\ &\quad -\frac{1}{2}\int_\Omega |\theta(t)|^2 \mathrm{div}(\mathbf{v})\,\mathrm{d}V\\ &\le -(\gamma - v_0)\int_\Omega |\theta(t)|^2\,\mathrm{d}V.\end{aligned} \tag{40}$$

Multiplying (7) by $\nabla^2\theta$, integrating over Ω, and using the boundary conditions on θ and $\mathbf{v}$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\theta(t)|^2 \mathrm{d}V\\ &\quad = -\kappa\int_\Omega |\nabla^2\theta(t)|^2 \mathrm{d}V - \gamma\int_\Omega |\nabla\theta(t)|^2 \mathrm{d}V\\ &\qquad + \int_\Omega \left(\nabla\theta\,\nabla\mathbf{v}\nabla\theta + \theta\,\nabla\theta\cdot\nabla\mathrm{div}(\mathbf{v}) + \frac{1}{2}|\nabla\theta(t)|^2 \mathrm{div}(v)\right)\mathrm{d}V\\ &\le v_0\int_\Omega |\theta(t)|^2 \mathrm{d}V - (\gamma - 2v_0)\int_\Omega |\nabla\theta(t)|^2\,\mathrm{d}V.\end{aligned}$$

Therefore (39) follows from (40) and (41). □

The flux controller (36) can be replaced by the following concentration controller

$$c(1, y, z, t) = -\int_0^1 K(1, \xi)c(\xi, y, z, t)\,\mathrm{d}\xi. \tag{42}$$

Theorem 2: *Assume that $\gamma > 0$ is any positive constant and $a = a(x) \in C^1[0, 1]$ is any function. Assume that the velocity $\mathbf{v} = (v_1, v_2, v_3)$ satisfies the following conditions*:

(i) *the velocity satisfies the no-penetration condition on the boundary of Ω: $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$;*
(ii) *$v_1 = v_1(x)$ is independent of y, z, t;*
(iii) *$v_2 = v_2(y, z, t)$ and $v_3 = v_3(y, z, t)$ are independent of x;*
(iv) $v_0 = \max_{t\ge t_0, \mathbf{x}\in\Omega}\{|\mathrm{div}(\mathbf{v}(\mathbf{x},t)|, |\nabla\mathbf{v}(\mathbf{x},t)|, |\nabla\mathrm{div}(\mathbf{v}(\mathbf{x},t))|\} < \infty$.

Then, for arbitrary initial data $c^0(x) \in H^1(0, 1)$, the problem (1)–(5) with the controller (42) has a unique solution that satisfies

$$\|c(t)\|_{H^1} \le M\|c^0\|_{H^1} e^{-(\gamma - 2v_0)(t-t_0)}, \tag{43}$$

where M is a positive constant independent of c^0.

It is clear that if $a = a(y)$, $v_2 = v_2(y)$, $v_1 = v_1(x, z, t)$, and $v_3 = v_3(x, z, t)$ (or $a = a(z)$, $v_3 = v_3(z)$, $v_1 = v_1(x, y, t)$, and $v_2 = v_2(x, y, t)$), then similar controllers on the face $y = 1$ (or $z = 1$) can be designed.

The assumptions (ii) and (iii) in Theorems 1 and 2 are restrictive. However, we can relax them somewhat. We can allow the general dependence $v_1(x, y, z, t)$, $v_2(x, y, z, t)$, $v_3(x, y, z, t)$ on all four arguments, as long as $(\partial v_1/\partial y)$, $(\partial v_1/\partial z)$, $(\partial v_2/\partial y)$, $(\partial v_3/\partial y)$ are allowed to be small, uniformly in all four arguments. Then, we would perform the control design based on $v_1(x, 0, 0, t)$, $v_2(0, y, z, t)$, $v_3(0, y, z, t)$ and exploit the robustness of our design to a small parametric perturbation in the plant coefficients, which holds due to the fact that our nominal system is exponentially stable. The target system would have to be derived from scratch and is much more complicated than (7), (8). It would include the contribution of the small perturbations arising from the dependence of v_1 on y and z, and v_2 and v_3 on x. This derivation would involve both the direct backstepping transformation and its inverse, as well as estimates of bounds on both of the transformations' kernels. Finally, a Lyapunov analysis of the perturbed system would demonstrate robustness to sufficiently small perturbations that arise in the target system. The whole analysis would take at least 5–8 pages, therefore it is not included here but only outlined.

Acknowledgements

This work was supported by the start-up fund and the University Research Council Fund of the University of Central Arkansas.

References

O.M. Aamo, M. Krstic and T.R. Bewley, "Control of mixing by boundary feedback in 2D channel flow", *Automatica*, 39, pp. 1597–1606, 2003.

O.M. Aamo, A. Smyshlyaev and M. Krstic, "Boundary control of the linearized Ginzburg-Landau model of vortex shedding", *SIAM J. Control. Optim.*, 43, pp. 1953–1971, 2005.

H. Amamm, "Feedback stabilization of linear and semilinear parabolic systems", in *Semigroup Theory and Applications*, P. Clement, S. Invernizzi, E. Mitidieri, I.I. Vrabie, Eds, *Lecture Notes in Pure and Appl. Math. 116*, Marcel Dekker, New York, pp. 21–57, 1989.

T.M. Antonsen, Z. Fan, E. Ott and E. Garcia-Lopez, "The role of passive scalars in the determination of power spectra of passive scalars", *Phys. Fluids*, 8, pp. 3094–3104, 1996.

A. Balogh and M. Krstic, "Infinite-step backstepping for a heat equation-like PDE with arbitrarily many unstable eigenvalues", in *Proceedings of the 2001 American Control Conference*, Vol. 3, IEEE Press, Piscataway, NJ, pp. 2480–2485, 2001.

A. Balogh and M. Krstic, "Infinite dimensional backstepping—style feedback transformations for a heat equation with an arbitrary level of instability", *Euro. J. Cont.*, 8, pp. 165–176, 2002.

A. Balogh and M. Krstic, "Stability of partial difference equations governing control gains in infinite-dimensional backstepping", *Syst. Cont. Lett.*, 51, pp. 151–164, 2004.

M.A. Bees, A.J. Pons, P.G. Sorensen and F. Sagues, "Chemoconvection: a chemically driven hydrodynamics instability", *J. Chem. Phys.*, 114, pp. 1932–1943, 2001.

J.A. Burns and D. Rubio, "A distributed parameter control approach to sensor location for optimal feedback control of thermal processes", in *Proceedings of the 36th IEEE Conference on*

Decision and Control, Vol. 3, IEEE Press, Piscataway, NJ, pp. 2243–2247, 1998.

J.A. Burns, D. Rubio and B.B. King, "Regularity of feedback operators for boundary control of thermal processes", in *Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace*, Daytona Beach, FL, pp. 67–73, 1996.

D. Colton, "The solution of initial-boundary value problems for parabolic equations by the method of integral operators", *J. Differential Equations*, 26, pp. 181–190, 1977.

D. Colton, *Analytic Theory of Partial Differential Equations*, London: Pitman Press, 1980.

W.A. Day, "A decreasing property of solutions of parabolic equations with applications to thermoelasticity", *Quart. Appl. Math.*, 40, pp. 468–475, 1982/83.

L.C. Evans, *Partial Differential Equations*, American Mathematical Society: Providence, Rhode Island, 1998.

M. Giona, S. Cerbelli and V.Vitacolonna, "Universality and imaginary potentials in advection diffusion equations in closed flows", *J. Fluid Mech.*, 513, pp. 221–237, 2004a.

M. Giona, V. Vitacolonna, S. Cerbelli and A. Adrover, "Advection diffusion in nonchaotic closed flows: non-Hermitian operators, universality, and localization", *Phys. Rev. E*, 70, pp. 0462241–046224-12, 2004b.

D. Greenblatt and I.J. Wygnanski, "The control of flow separation by periodic excitation", *Progress in Aerospace Sci.*, 36, pp. 487–545, 2000.

P.H. Haynes and J. Vanneste, "What controls the decay of passive scalars in smooth flows?", *Phys. Fluids*, 17, pp. 0971031–097103-16, 2005.

M. Krstic, "Lyapunov adaptive stabilization of parabolic PDEs: Part IA benchmark for boundary control", *IEEE Conference on Decision and Control*, Seville, Spain, pp. 3164–3170, 2005a.

M. Krstic, "Lyapunov adaptive stabilization of parabolic PDEs: Part II Output feedback and other benchmark problems", *IEEE Conference on Decision and Control*, Seville, Spain, pp. 3170–3176, 2005b.

I. Lasiecka and R. Triggiani, "Stabilization of Neumann boundary feedback of parabolic equations: the case of trace in the feedback loop", *Appl. Math. Optim.*, 10, pp. 307–350, 1983a.

I. Lasiecka and R. Triggiani, "Stabilization and structural assignment of Dirichlet boundary feedback parabolic equations", *SIAM J. Control Optim.*, 21, pp. 766–803, 1983b.

I. Lasiecka and R. Triggiani, "The regulator problem for parabolic equations with Dirichlet boundary control. I. Riccati's feedback synthesis and regularity of optimal solution", *Appl. Math. Optim.*, 16, pp. 147–168, 1987a.

I. Lasiecka and R. Triggiani, "The regulator problem for parabolic equations with Dirichlet boundary control. II. Galerkin approximation", *Appl. Math. Optim.*, 16, pp. 187–216, 1987b.

W. Liu, "Boundary feedback stabilization of an unstable heat equation", *SIAM J. Cont. Optim.*, 42, pp. 1033–1043, 2003.

W. Liu, "Does a fast mixer really exist?", *Physical Review E*, 72, pp. 0163121–016312-10, 2005.

W. Liu and G. Haller, "Strange eigenmodes and decay of variance in the mixing of diffusive tracers", *Physica D*, 188, pp. 1–39, 2004.

A. Smyshlyaev and M. Krstic, "Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations", *IEEE Transactions on Automatic Cont.*, 49, pp. 2185–2202, 2004.

A. Smyshlyaev and M. Krstic, "Backstepping observers for a class of parabolic PDEs", *Syst. and Cont. Lett.*, 54, pp. 613–625, 2005a.

A. Smyshlyaev and M. Krstic, "Passive identifiers for boundary adaptive control of 3D reaction-advection-diffusion PDEs", *IEEE Conference on Decision and Control*, Seville, Spain, pp. 1515–1521, 2005b.

A. Smyshlyaev and M. Krstic, "Output feedback adaptive controllers with swapping identifiers for two unstable PDEs with infinite relative degree", *IEEE Conference on Decision and Control*, Seville, Spain, pp. 1509–1515, 2005c.

R. Triggiani, "Boundary feedback stabilization of parabolic equations", *Appl. Math. Optim.*, 6, pp. 201–220, 1980.

Y. Wang, G. Haller, A. Banaszuk and G. Tadmor, "Closed-loop Lagrangian separation control in a bluff body shear flow model", *Phys. Fluids*, 15, pp. 2251–2266, 2003.

W.M. Yang, S.K. Chou, C. Shu, Z.W. Li and H. Xue, "Combustion in micro-cylindrical combustors with and without a backward facing step", *Applied Thermal Eng.*, 22, pp. 1777–1787, 2002.