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Strong stabilization of the system of linear elasticity by a Dirichlet boundary feedback

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In this paper, by using the Nagy–Foias–Foguel theory of decomposition of continuous semigroups of contractions, we prove that the system of linear elasticity is strongly stabilizable by a Dirichlet boundary feedback. We also give a concise proof of a theorem of Dafermos about the stability of thermoelasticity.

Keywords: Nagy–Foias–Foguel decomposition; stabilization; linear elasticity.

1. Introduction and main result

Throughout this paper, we denote by Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ of class C^2. We write $H^s(\Omega)$ for the usual Sobolev space for any $s \in \mathbb{R}$. For $s \geqslant 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space and $T > 0$. We denote by $C^k([0, T]; X)$ the space of all k times continuously differentiable functions defined on $[0, T]$ with values in X, and write $C([0, T]; X)$ for $C^0([0, T]; X)$.

We consider the following system of linear and isotropic elasticity:

$$\left.\begin{array}{ll} u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0,\ u'(0) = u^1 & \text{in } \Omega, \end{array}\right\} \tag{1.1}$$

where $u(x, t) = (u_1(x, t), \ldots, u_n(x, t))$, λ, μ are Lamé's constants satisfying

$$\mu > 0, \quad n\lambda + (n+1)\mu > 0. \tag{1.2}$$

By $'$ we denote the derivative with respect to the time variable. We let Δ, ∇, div denote the Laplace, gradient, and divergence operators in the space variables, respectively, and $u(0)$ and $u'(0)$ denote the functions $x \to u(x, 0)$ and $x \to u'(x, 0)$, respectively.

The elastic energy of (1.1) can be defined as

$$E(u,t) = \frac{1}{2}\int_\Omega [|u'(x,t)|^2 + \mu|\nabla u(x,t)|^2 + (\lambda+\mu)|\operatorname{div} u(x,t)|^2]\,\mathrm{d}x. \tag{1.3}$$

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Since

$$E'(u,t) \equiv 0, \tag{1.4}$$

the energy $E(u,t)$ is conservative. Therefore, we need a feedback to stabilize system (1.1). By stabilization we mean that the feedback makes the energy decay to zero as $t \to \infty$. So far, there have been a lot of contributions to the problem (even for the more general case: anisotropic elasticity), notably (Alabau & Komornik, 1999; Lagnese, 1983, 1991; Liu, 1998), to mention a few. However, in all the previous work, Neumann boundary feedbacks were used. As far as we know, Dirichlet boundary feedbacks have not been considered for the system of elasticity in the literature although they were used in the wave equation (see (Lasiecka & Triggiani, 1987)). Therefore, we wish to fill this gap and show that system (1.1) is strongly stabilizable by a Dirichlet boundary feedback.

In order to find a Dirichlet boundary feedback, let us first consider the system of elasticity with non-homogeneous Dirichlet boundary condition

$$\left.\begin{array}{ll} u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ u = \varphi & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0,\ u'(0) = u^1 & \text{in } \Omega, \end{array}\right\} \tag{1.5}$$

and formulate it as an abstract Cauchy problem. In doing so, we define the linear operator A in $(L^2(\Omega))^n$ by

$$Au = -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u, \tag{1.6}$$

with the domain $D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^n$. Obviously, A is a positive self-adjoint operator in $(L^2(\Omega))^n$. We then define the Dirichlet operator D by

$$D\varphi = u, \tag{1.7}$$

where u is the solution of the Lamé system

$$\left.\begin{array}{ll} \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma. \end{array}\right\} \tag{1.8}$$

It is well known from elliptic theory that

$$D : H^s(\Gamma) \to H^{s+1/2}(\Omega) \tag{1.9}$$

is a linear continuous operator for all real s. Let D^* denote the adjoint operator of D. Then

$$D^* : H^{-s}(\Omega) \to H^{-s+1/2}(\Gamma) \tag{1.10}$$

is a linear continuous operator for $0 \leqslant s \leqslant \frac{1}{2}$. Using the operators A and D, (1.5) can be formulated as an abstract equation

$$\left.\begin{array}{ll} u'' = -A(u - D\varphi) & \text{on } (L^2(\Omega))^n, \\ u(0) = u^0,\ u'(0) = u^1. & \end{array}\right\} \tag{1.11}$$

If A is extended, with the same symbol, as an operator $(L^2(\Omega))^n \to [D(A)]'$, then (1.11) can be written in the following perturbation form:

$$\left.\begin{array}{ll} u'' = -Au + AD\varphi & \text{on } [D(A)]', \\ u(0) = u^0,\ u'(0) = u^1. & \end{array}\right\} \tag{1.12}$$

In what follows, we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and norm of $(L^2(\Omega))^n$, respectively. Set

$$\mathcal{H} = (L^2(\Omega))^n \times [D(A^{1/2})]'. \tag{1.13}$$

Obviously, we have

$$[D(A^{1/2})]' = (H^{-1}(\Omega))^n, \tag{1.14}$$

and the usual norm $\|u\|_{(H^{-1}(\Omega))^n}$ is equivalent to $\|A^{-1/2}u\|$.

Obviously, in order to stabilize the open loop system (1.12), we have to find a feedback such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,u')\|^2_{\mathcal{H}} \leqslant 0. \tag{1.15}$$

But,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|(u,u')\|^2_{\mathcal{H}} &= 2\mathrm{Re}((u,u')', (u,u'))_{\mathcal{H}} \\ &= 2\mathrm{Re}((u', -Au + AD\varphi), (u,u'))_{\mathcal{H}} \\ &= 2\mathrm{Re}(u', u) - 2\mathrm{Re}(A^{-1/2}Au, A^{-1/2}u') \\ &\quad +2\mathrm{Re}(A^{-1/2}AD\varphi, A^{-1/2}u') \\ &= 2\mathrm{Re}(D\varphi, u'). \end{aligned} \tag{1.16}$$

Therefore, the feedback we are looking for should be taken as

$$\varphi = -D^*u', \tag{1.17}$$

since in this case we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,u')\|^2_{\mathcal{H}} = -2\|D^*u'\|^2_{L^2(\Gamma)} \leqslant 0. \tag{1.18}$$

By using this feedback, the open loop system (1.12) becomes the following closed loop system:

$$\left.\begin{array}{ll} u'' = -Au - ADD^*u' & \text{on } [D(A)]', \\ u(0) = u^0,\ u'(0) = u^1. & \end{array}\right\} \tag{1.19}$$

Setting

$$\Phi = (u, u'), \tag{1.20}$$

$$\mathcal{A}\Phi = (u', -Au - ADD^*u'), \tag{1.21}$$

we can write (1.19) as the following first-order form:

$$\left.\begin{array}{l}\Phi' = \mathcal{A}\Phi,\\ \Phi'(0) = (u^0, u^1)\end{array}\right\} \tag{1.22}$$

on $\mathcal{H}$, where the domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{(u, v) \in \mathcal{H} \ : \ \mathcal{A}(u, v) \in \mathcal{H}\}. \tag{1.23}$$

We define the weak energy $E_w(u, t)$ of (1.19) by

$$E_w(u, t) = \tfrac{1}{2}\|(u(t), u'(t))\|^2_{\mathcal{H}} = \tfrac{1}{2}[\|u(t)\|^2 + \|A^{-1/2}u'(t)\|^2]. \tag{1.24}$$

In comparison with definition (1.3) of energy, it is easy to see that there exists a constant $c > 0$ such that

$$E_w(u, t) \leqslant cE(u, t). \tag{1.25}$$

Thus, we call $E_w(u, t)$ weak energy.

The main result of this paper is as follows.

THEOREM 1.1

(i) (Well-posedness) The operator $\mathcal{A}$ defined by (1.21) generates a strongly continuous semigroup $e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$ and the resolvent operator $R(\omega, \mathcal{A})$ is compact on $\mathcal{H}$ for Re $\omega \geqslant 0$. Therefore, for any $(u^0, u^1) \in \mathcal{H}$, the feedback system (1.19) has a unique solution u with

$$(u, u') \in C([0, \infty); \mathcal{H}). \tag{1.26}$$

Further, the solution orbit

$$\gamma(u^0, u^1) = \bigcup_{t\geqslant 0}(u(t), u'(t)) \tag{1.27}$$

is precompact.

(ii) (Strong stabilization) For any $(u^0, u^1) \in \mathcal{H}$, we have

$$E_w(u, t) \to 0, \qquad \text{as } t \to +\infty \tag{1.28}$$

for the corresponding solution of the feedback system (1.19).

REMARK 1.1 Due to (1.25), Theorem 1.1 does not imply that the energy $E(u, t)$ defined by (1.3) is strongly stable. Whether $E(u, t)$ is strongly stable by a Dirichlet boundary feedback is an interesing open problem.

The rest of the paper is organized as follows. For completeness, in Section 2, we recall some definitions and theorems from the Nagy–Foias–Foguel theory of decomposition of continuous semigroups of contractions. Then, in Section 3, we give the proof of our main result. Finally, in Section 4, as another interesting application of Nagy–Foias–Foguel theory, we give a concise proof of a theorem of Dafermos: The energy in linear thermoelasticity converges to zero asymptotically if and only if the eigenvalue problem of the Lamé system with divergence free has no nontrivial eigenfunction.

2. Nagy–Foias–Foguel decomposition

For completeness, we recall some definitions and theorems from the Nagy–Foias–Foguel theory of decomposition of continuous semigroups of contractions.

Let H be a Hilbert space. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm of H, respectively.

DEFINITION 2.1 Let H be a Hilbert space. Let S be a linear bounded operator in H and S^* its adjoint. We say that a subspace $V \subset H$ *reduces* S if and only if

$$SV \subset V \text{ and } S^*V \subset V. \tag{2.1}$$

DEFINITION 2.2 A linear bounded operator S in H is

(i) *Unitary* if

$$S^*S = SS^* = I, \tag{2.2}$$

(ii) *Completely non-unitary* (cnu) if there exists no subspace other than $\{0\}$ reducing S to a unitary operator.

The following is the Nagy–Foias decomposition theorem.

THEOREM 2.3 (Benchimol, 1978) Let H be a Hilbert space and $S(t)$ a continuous semigroup of contractions on H. Let A be the infinitesimal generator of $S(t)$ with domain $D(A)$. Then H can be decomposed into an orthogonal sum

$$H = H_{\mathrm{u}} \oplus H_{\mathrm{cnu}}, \tag{2.3}$$

where H_{u} and H_{cnu} are reducing subspaces for $S(t)$, such that

(i) the restriction $S_{\mathrm{u}}(t) = S(t)|_{H_{\mathrm{u}}}$ of $S(t)$ to H_{u} is a unitary semigroup;
(ii) the restriction $S_{\mathrm{cnu}}(t) = S(t)|_{H_{\mathrm{cnu}}}$ of $S(t)$ to H_{cnu} is a cnu semigroup;
(iii) this decomposition (where, of course, H_{u} or H_{cnu} can be trivial) is unique and H_{u} can be characterized by

$$H_{\mathrm{u}} = \{x \in H : \|S(t)x\| = \|S^*(t)x\| = \|x\| \text{ for all } t \geqslant 0\}. \tag{2.4}$$

Moreover

$$H_{\mathrm{u}} = \overline{D(A) \cap H_{\mathrm{u}}}. \tag{2.5}$$

DEFINITION 2.4 A continuous semigroup $S(t)$ on a Hilbert space H is said to be

(i) *weakly stable* if $(S(t)x, y) \to 0$ as $t \to +\infty$ for any $x, y \in H$;
(ii) *strongly stable* if $\|S(t)x\| \to 0$ as $t \to +\infty$ for any $x \in H$.

Set

$$W = \{x \in H : S(t)x \rightharpoonup 0 \text{ (weakly) as } t \to +\infty\}, \tag{2.6}$$

where $S(t)x \rightharpoonup 0$ (weakly) as $t \to +\infty$ means that $(S(t)x, y) \to 0$ as $t \to +\infty$ for any $y \in H$.

The following is the Nagy–Foias–Foguel decomposition theorem.

THEOREM 2.5 (Benchimol, 1978) Let H be a Hilbert space and $S(t)$ a continuous semigroup of contractions on H. Then H can be decomposed into an orthogonal sum

$$H = W_{\mathrm{u}} \oplus W^{\perp} \oplus H_{\mathrm{cnu}}, \tag{2.7}$$

where W_{u}, $W^{\perp}$ and H_{cnu} are reducing subspaces for $S(t)$, such that

$$H_{\mathrm{u}} = W_{\mathrm{u}} \oplus W^{\perp}, \tag{2.8}$$

$$W = W_{\mathrm{u}} \oplus H_{\mathrm{cnu}}. \tag{2.9}$$

Further, we have

(i) $S(t)$ is completely nonunitary and weakly stable on H_{cnu};
(ii) $S(t)$ is unitary and weakly stable on W_{u};
(iii) $S(t)$ is unitary on $W^{\perp}$, and $\forall x \in W^{\perp}$, $S(t)x \not\rightharpoonup 0$ and $S^*(t)x \not\rightharpoonup 0$ as $t \to +\infty$.

Let $S(t)$ be a continuous semigroup on a Banach space X. We define the set $\gamma(x) = \bigcup_{t\geqslant 0} S(t)x$ to be the positive orbit corresponding to the initial state $x \in X$. Set

$$X_b = \{x \in X : \gamma(x) \text{ is bounded}\}, \tag{2.10}$$

$$X_c = \{x \in X : \gamma(x) \text{ is precompact}\}. \tag{2.11}$$

THEOREM 2.6 (Walker, 1980, p. 179, Theorem 5.2) Let a linear operator $A : D(A) \to X$ be the infinitesimal generator of a continuous semigroup $S(t)$ on a Banach space X.

(i) If there exists a compact linear operator $P : X \to X$ such that

$$Px \in D(A) \text{ and } PAx = APx \quad \text{for all } x \in D(A), \tag{2.12}$$

then $x \in X_b$ implies $Px \in X_c$; further, if $X_b = X$, then $\overline{R(P)} \subset X_c$, where $R(P)$ denotes the range of P.
(ii) If $J_\omega = (I - \omega A)^{-1}$ is compact for some $\omega > 0$, then $X_c = X_b$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first give some basic properties about the domain $D(\mathcal{A})$ and the Dirichlet operator D. By (1.9) and (1.10), we deduce that

$$DD^* : (L^2(\Omega))^n \to (H^1(\Omega))^n. \tag{3.1}$$

If $(u, v) \in D(\mathcal{A})$, then we have $v \in (L^2(\Omega))^n$ and $A(u + DD^*v) \in (D(A^{1/2}))'$, and then $u + DD^*v \in D(A^{1/2}) = (H_0^1(\Omega))^n$. It therefore follows from (3.1) that $u \in (H^1(\Omega))^n$. In conclusion, we have

$$D(\mathcal{A}) \subset (H^1(\Omega))^n \times (L^2(\Omega))^n. \tag{3.2}$$

On the other hand, we have for $u \in D(A)$

$$D^*Au = -\mu\frac{\partial u}{\partial \nu} - (\lambda + \mu)\nu \operatorname{div} u \quad \text{on } \Gamma. \tag{3.3}$$

In fact, we have

$$
\begin{aligned}
(v, D^*Au)_{L^2(\Gamma)} &= (Dv, Au) \\
&= -\left(v, \mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu \operatorname{div} u\right)_{L^2(\Gamma)} - (ADv, u) \\
&= -\left(v, \mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\nu \operatorname{div} u\right)_{L^2(\Gamma)}, \qquad (3.4)
\end{aligned}
$$

which implies (3.3).

LEMMA 3.1

(i) The operator $\mathcal{A}$ defined by (1.21) is dissipative on $\mathcal{H}$.
(ii) The resolvent operator $R(\omega, \mathcal{A})$ of $\mathcal{A}$ is compact as an operator on $\mathcal{H}$ for any ω with $\operatorname{Re}\omega \geqslant 0$.
(iii) The operator $(I + \omega^2 A^{-1} + \omega DD^*)$ is boundedly invertible on $(L^2(\Omega))^n$ for any ω with $\operatorname{Re}\omega \geqslant 0$.

Proof. (i) For any $(u, v) \in D(\mathcal{A})$, we have

$$
\begin{aligned}
\operatorname{Re}(\mathcal{A}(u, v), (u, v)) &= \operatorname{Re}((v, -Au - ADD^*v), (u, v))_{\mathcal{H}} \\
&= \operatorname{Re}(v, u) - \operatorname{Re}(A^{-1/2}Au, A^{-1/2}v) \\
&\quad -\operatorname{Re}(A^{-1/2}ADD^*v, A^{-1/2}v) \\
&= -\|D^*v\|^2_{L^2(\Gamma)}. \qquad (3.5)
\end{aligned}
$$

Thus, $\mathcal{A}$ is dissipative.

(ii) We first show that the resolvent $R(\omega, \mathcal{A})$ of $\mathcal{A}$ exists as an operator on $\mathcal{H}$ for any $\omega \geqslant 0$.

If $\omega = 0$, then the resolvent of

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -ADD^* \end{pmatrix} \qquad (3.6)$$

is given by

$$\mathcal{A}^{-1} = \begin{pmatrix} -DD^* & -A^{-1} \\ I & 0 \end{pmatrix}. \qquad (3.7)$$

We now consider the case where $\omega > 0$. Since

$$
\begin{aligned}
\begin{pmatrix} I & 0 \\ -(1/\omega)A & I \end{pmatrix}(\omega I - \mathcal{A}) &= \begin{pmatrix} I & 0 \\ -(1/\omega)A & I \end{pmatrix}\begin{pmatrix} \omega I & -I \\ A & \omega I + ADD^* \end{pmatrix} \\
&= \begin{pmatrix} \omega I & -I \\ 0 & (1/\omega)A + \omega I + ADD^* \end{pmatrix}, \qquad (3.8)
\end{aligned}
$$

we deduce that $\omega I - \mathcal{A}$ is boundedly invertible on $\mathcal{H}$ if and only if $(1/\omega)A + \omega I + ADD^*$ is boundedly invertible on $(L^2(\Omega))^n$, and then if and only if $I + \omega^2 A^{-1} + \omega DD^*$ is boundedly invertible on $(L^2(\Omega))^n$ because A is boundedly invertible on $(L^2(\Omega))^n$. The latter is true since $I + \omega^2 A^{-1} + \omega DD^*$ is strictly positive bounded and $\omega^2 A^{-1} + \omega DD^*$ is compact on $(L^2(\Omega))^n$.

We next show that the resolvent operator $R(\omega, \mathcal{A})$ of $\mathcal{A}$ exists as an operator on $\mathcal{H}$ for any ω with Re $\omega > 0$. By the dissipativeness of $\mathcal{A}$, we deduce that

$$\|(\omega I - \mathcal{A})(u, v)\|_{\mathcal{H}} \geqslant \omega \|(u, v)\|_{\mathcal{H}}, \quad \forall\, \omega > 0, \tag{3.9}$$

which implies that

$$\|R(\omega, \mathcal{A})\| \leqslant \frac{1}{\omega} \quad \forall\, \omega > 0. \tag{3.10}$$

It therefore follows from (Pazy, 1983, Remark 5.4, p. 20) that

$$\|R(\omega, \mathcal{A})\| \leqslant \frac{1}{\operatorname{Re} \omega} \quad \forall \operatorname{Re} \omega > 0. \tag{3.11}$$

It remains to show that $R(ir, \mathcal{A})$ exists for any real number r. By (3.8), it suffices to show that $I + (ir)^2 A^{-1} + irDD^*$ is boundedly invertible on $(L^2(\Omega))^n$ for any real number r. For this, we first show that the equation

$$(I + (ir)^2 A^{-1} + irDD^*)u = 0 \tag{3.12}$$

has only solution 0. In fact, taking inner product with u, we obtain

$$\|u\|^2 - r^2(A^{-1}u, u) + ir\|D^*u\|^2_{L^2(\Gamma)} = 0. \tag{3.13}$$

Since $\|u\|^2 - r^2(A^{-1}u, u)$ is real, we have

$$D^*u = 0. \tag{3.14}$$

It therefore follows from (3.12) that

$$Au = r^2 u, \quad u \in D(A). \tag{3.15}$$

By (3.3), we deduce that

$$\mu \frac{\partial u}{\partial \nu} + (\lambda + \mu)\nu \operatorname{div} u = -D^*Au = r^2 D^*u = 0 \quad \text{on } \Gamma. \tag{3.16}$$

Consequently, it follows from the unique continuation property of the Lamé system (see Ang *et al.*, 1998, Corollary, p. 373) that $u = 0$. Thus, $I + (ir)^2 A^{-1} + irDD^*$ is injective. Further, $(ir)^2 A^{-1} + irDD^*$ is compact. Therefore, $I + (ir)^2 A^{-1} + irDD^*$ is boundedly invertible on $(L^2(\Omega))^n$ for any real number r.

Finally, compactness of $R(\omega, \mathcal{A})$ follows from the compactness of the embedding of $D(\mathcal{A})$ into $\mathcal{H}$.

(iii) This is the direct consequence of (ii) and (3.8). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Part (i) of Theorem 1.1 follows immediately from the Lumer–Phillips theorem (Pazy, 1983, Theorem 4.3, p. 14), Theorem 2.6 and Lemma 3.1.

To prove part (ii), we apply Nagy–Foias–Foguel's theory. By Theorem 2.5, $\mathcal{H}$ can be decomposed into an orthogonal sum

$$\mathcal{H} = W_{\mathrm{u}} \oplus W^{\perp} \oplus \mathcal{H}_{\mathrm{cnu}} \tag{3.17}$$

such that $S(t) = \mathrm{e}^{\mathcal{A}t}$ is

(i) completely non-unitary and weakly stable on $\mathcal{H}_{\mathrm{cnu}}$;
(ii) unitary and weakly stable on W_{u};
(iii) unitary on $W^{\perp}$, and for all $x \in W^{\perp}$, $S(t)x \not\to 0$ and $S^*(t)x \not\to 0$ as $t \to +\infty$. Since the positive orbit $\gamma(u^0, u^1)$ is precompact for every $(u^0, u^1) \in \mathcal{H}$, it follows that $\mathrm{e}^{\mathcal{A}t}$ is actually strongly stable on $\mathcal{H}_{\mathrm{cnu}}$. To complete the proof, it suffices to prove that $\mathcal{H}_{\mathrm{u}} = W_{\mathrm{u}} \oplus W^{\perp} = \{0\}$. If $\mathcal{H}_{\mathrm{u}} \neq \{0\}$, then, by Stone's theorem (Pazy, 1983, Theorem 10.8, p. 41), $i\mathcal{A}$ is self-adjoint on $\mathcal{H}_{\mathrm{u}}$ since $\mathrm{e}^{\mathcal{A}t}$ is unitary on $\mathcal{H}_{\mathrm{u}}$. In addition, $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, $\mathcal{A}$ must have eigenvalues and the eigenvalues must be on the imaginary axis. This contradicts Lemma 3.1 (ii). □

REMARK 3.1 It was shown in (Lasiecka & Triggiani, 1987) that the wave equation is uniformly stabilizable by a Dirichlet boundary feedback. When we tried to apply the method of (Lasiecka & Triggiani, 1987) to the system of elasticity, we encountered some difficulties that cannot be handled. Therefore, whether the system of elasticity is uniformly stabilizable by a Dirichlet boundary feedback or not is an open problem.

4. Proof of Dafermos's theorem

Consider the system of equations of thermoelasticity

$$\left.\begin{array}{ll} u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega \times (0,\infty), \\ \theta' - \Delta\theta + \beta \operatorname{div} u' = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0, \quad \theta = 0 & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0, \ u'(0) = u^1, \ \theta(0) = \theta^0 & \text{in } \Omega, \end{array}\right\} \tag{4.1}$$

in the absence of external forces and heat sources, where α, $\beta > 0$ are the coupling parameters.

The thermoelastic energy of (4.1) can be defined as

$$E(u,\theta,t) = \frac{1}{2}\int_{\Omega}\Big[|u'(x,t)|^2 + \mu|\nabla u(x,t)|^2 \\ +(\lambda+\mu)|\operatorname{div} u(x,t)|^2 + \frac{\alpha}{\beta}|\theta(x,t)|^2\Big]\,\mathrm{d}x. \tag{4.2}$$

Here we have used the notation

$$|\nabla u(x,t)|^2 = \sum_{i,j=1}^{n}\left|\frac{\partial u_i}{\partial x_j}\right|^2. \tag{4.3}$$

It is easy to verify that

$$E'(u,\theta,t) = -\frac{\alpha}{\beta}\int_{\Omega} |\nabla\theta(x,t)|^2 \, dx. \tag{4.4}$$

Therefore, the energy $E(u,\theta,t)$ decreases on $(0,\infty)$, but, in general, does not tend to zero as $t \to \infty$. In fact, Dafermos in his pioneering work Dafermos (1968) obtained the following famous theorem.

THEOREM 4.1 (Dafermos, 1968) The energy $E(u,\theta,t)$ of every solution of (4.1) converges to zero as $t \to \infty$ if and only if the following eigenvalue problem of the Lamé system has no non-trivial eigenfunction:

$$\left.\begin{array}{ll} -\mu\Delta\phi - (\lambda+\mu)\nabla \operatorname{div}\phi = \omega^2\phi & \text{in } \Omega, \\ \operatorname{div}\phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{array}\right\} \tag{4.5}$$

Theorem 4.1 was originally proved by using LaSalle's invariance principle. We give here a new concise proof via Nagy–Foias–Foguel theory.

We introduce a function space as follows:

$$\mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega). \tag{4.6}$$

In the sequel, we use the following energy norm on $\mathcal{H}$:

$$\|(u,v,\theta)\|_{\mathcal{H}} = \left(\int_{\Omega}\left[\mu|\nabla u|^2 + (\lambda+\mu)|\operatorname{div}(u)|^2 + |v|^2 + \frac{\alpha}{\beta}|\theta|^2\right] dx\right)^{1/2} \tag{4.7}$$

for $(u,v,\theta) \in \mathcal{H}$, which is equivalent to the usual one induced by $(H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$ when (1.2) is satisfied.

Proof of Theorem 4.1 We define the linear operator A on $\mathcal{H}$ by

$$A(u,v,\theta) = (v,\ \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u - \alpha\nabla\theta,\ \Delta\theta - \beta \operatorname{div} u') \tag{4.8}$$

with domain

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \times (H^2(\Omega) \cap H_0^1(\Omega)). \tag{4.9}$$

It is well known that A generates a continuous semigroup $S(t)$ of contractions on $\mathcal{H}$. Thus, by Theorem 2.5, $\mathcal{H}$ can be decomposed into an orthogonal sum

$$\mathcal{H} = W_{\mathrm{u}} \oplus W^{\perp} \oplus \mathcal{H}_{\mathrm{cnu}}, \tag{4.10}$$

such that $S(t)$ is

(i) completely nonunitary and weakly stable on $\mathcal{H}_{\mathrm{cnu}}$;
(ii) unitary and weakly stable on W_{u};

(iii) unitary on $W^{\perp}$, and for all $x \in W^{\perp}$, $S(t)x \not\to 0$ and $S^*(t)x \not\to 0$ as $t \to +\infty$. Since $J_\omega = (I - \omega A)^{-1}$ is compact on $\mathcal{H}$ for all $\omega > 0$, by Theorem 2.6, the positive orbit $\gamma(u^0, u^1, \theta^0)$ is precompact for every $(u^0, u^1, \theta^0) \in \mathcal{H}$. Therefore, $S(t)$ is actually strongly stable on $\mathcal{H}_{\mathrm{cnu}}$ and W_{u}. But $S(t)$ is also unitary on W_{u}. This implies that $\|S(t)(u^0, u^1, \theta^0)\|_{\mathcal{H}} \equiv \|(u^0, u^1, \theta^0)\|_{\mathcal{H}}$ for all $t \geqslant 0$ and $(u^0, u^1, \theta^0) \in W_{\mathrm{u}}$. Thus, we have $W_{\mathrm{u}} = 0$. Consequently, $\mathcal{H}_{\mathrm{u}} = W^{\perp}$. Hence, the energy $E(u, \theta, t)$ of every solution of (1.1) converges to zero as $t \to \infty$ if and only if $\mathcal{H}_{\mathrm{u}} = \{0\}$.

We now want to identify $\mathcal{H}_{\mathrm{u}}$ and $\mathcal{H}_{\mathrm{cnu}}$. Let the energy $E(t) = E(u, \theta, t)$ of system (4.1) be defined by (4.2). By a straightforward calculation, we obtain

$$E'(t) = -\frac{\alpha}{\beta} \int_{\Omega} |\nabla\theta(x, t)|^2 \, \mathrm{d}x. \tag{4.11}$$

It follows therefore from Theorem 2.3 that

$$\begin{aligned} \mathcal{H}_{\mathrm{u}} &= \{(u^0, u^1, \theta^0) \in \mathcal{H} : E(t) \equiv E(0)\} \\ &= \{(u^0, u^1, \theta^0) \in \mathcal{H} : E'(t) \equiv 0\} \\ &= \{(u^0, u^1, \theta^0) \in \mathcal{H} : \nabla\theta \equiv 0 \text{ on } \Omega \times (0, \infty)\} \\ &= \{(u^0, u^1, \theta^0) \in \mathcal{H} : \theta \equiv 0 \text{ on } \Omega \times (0, \infty)\}. \end{aligned} \tag{4.12}$$

Thus, by (4.1), we deduce that $(u^0, u^1, \theta^0) \in D(A) \cap \mathcal{H}_{\mathrm{u}}$ if and only if

$$\left.\begin{array}{ll} u'' - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u = 0 & \text{in } \Omega \times (0, \infty), \\ \theta = \operatorname{div} u' = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \Gamma \times (0, \infty), \\ u(0) = u^0, \ u'(0) = u^1, \ \theta(0) = \theta^0 & \text{in } \Omega, \end{array}\right\} \tag{4.13}$$

and then

$$-(\lambda + 2\mu)\Delta \operatorname{div} u = 0 \quad \text{in } \Omega \times (0, \infty)\cdot \tag{4.14}$$

From (4.13) and (4.14), we conclude that

$$D(A) \cap \mathcal{H}_{\mathrm{u}} = \{(u^0, u^1, 0) \in D(A) \cap \mathcal{H} : \Delta \operatorname{div} u^0 = 0, \ \operatorname{div} u^1 = 0\}. \tag{4.15}$$

We denote by ν the unit normal on Γ directed towards the exterior of Ω. It therefore follows from (Temam, 1977, Theorem 1.4, p. 15) that

$$\begin{aligned} \mathcal{H}_{\mathrm{u}} &= \overline{D(A) \cap \mathcal{H}_{\mathrm{u}}} \\ &= \{(u^0, u^1, 0) \in \mathcal{H} : \Delta \operatorname{div} u^0 = 0, \ \operatorname{div} u^1 = 0, \ u^1 \cdot \nu|_{\Gamma} = 0\}, \end{aligned} \tag{4.16}$$

and

$$\mathcal{H}_{\mathrm{cnu}} = \{(u^0, u^1, \theta^0) \in \mathcal{H} : \operatorname{curl} u^0 = 0, \ u^1 = \nabla p, \ p \in H^1(\Omega)\}. \tag{4.17}$$

It is clear that $\mathcal{H}_{\mathrm{u}} = \{0\}$ if and only if $D(A) \cap \mathcal{H}_{\mathrm{u}} = \{0\}$, and if and only if (4.13) has no non-trivial solution. Further, it is easy to see that (4.13) has no non-trivial solution if and only if the eigenvalue problem (4.5) has no non-trivial eigenfunction. This completes the proof of Theorem 4.1. □

As a by-product of the above proof, we obtain the following.

THEOREM 4.2 Let $S(t)$ be the continuous semigroup of contractions generated by system (4.1) on $\mathcal{H}$. Let $\mathcal{H}_u$ and $\mathcal{H}_{\text{cnu}}$ be given by (4.16) and (4.17), respectively. Then $\mathcal{H}$ can be decomposed into an orthogonal sum

$$\mathcal{H} = \mathcal{H}_{\text{u}} \oplus \mathcal{H}_{\text{cnu}}$$

such that $\mathcal{H}_{\text{u}}$ and $\mathcal{H}_{\text{cnu}}$ are reducing subspaces for $S(t)$ and

(i) the restriction $S_{\text{u}}(t) = S(t)|_{\mathcal{H}_{\text{u}}}$ of $S(t)$ to $\mathcal{H}_{\text{u}}$ is a unitary semigroup and conservative;
(ii) the restriction $S_{\text{cnu}}(t) = S(t)|_{\mathcal{H}_{\text{cnu}}}$ of $S(t)$ to $\mathcal{H}_{\text{cnu}}$ is a cnu semigroup and strongly stable.

Theorem 4.2 shows that if the initial data $(u^0, u^1, \theta^0) \in \mathcal{H}_{\text{u}}$, then $\theta \equiv 0$. Hence no thermal damping exists in the elastic body, and consequently the energy $E(t)$ is conservative.

In Dafermos (1968), it was pointed out that (4.5) has no non-trivial eigenfunction 'generically' for smooth domains. However, when Ω is a ball, (4.5) does have non-trivial eigenfunctions (see (Lions & Zuazua, 1996, p.228)).

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