JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 211, 258–272 (1997)
ARTICLE NO. AY975459

Exact Internal Controllability for the Semilinear Heat Equation

Weijiu Liu* and Graham H. Williams†

Department of Mathematics, University of Wollongong, Northfields Avenue, Wollongong, NSW 2522, Australia

Submitted by George Leitmann

Received September 16, 1996

Using the Browder–Minty surjective theorem from the theory of monotone operators, we consider the exact internal controllability for the semilinear heat equation. We show that the system is exactly controllable in $L^2(\Omega)$ if the nonlinearities are globally Lipschitz continuous. Furthermore, we prove that the controls depend Lipschitz continuously on the terminal states, and discuss the behaviour of the controls as the nonlinear terms tend to zero in some sense. A variant of the Hilbert Uniqueness Method is presented to cope with the nonlinear nature of the problem.

1. INTRODUCTION

Of recent years, there has been some study on the problem of approximate controllability for the semilinear heat equation. Combining a variational approach and Kakutani's fixed point theorem, Fabre, Puel, and Zuazua [5] studied the approximate controllability for the semilinear heat equation

$$\begin{cases} y' - \Delta y + f(y) = h\chi_\omega & \text{in } Q, \qquad \left(y' = \dfrac{\partial y}{\partial t} \right) \\ y(0) = y^0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma. \end{cases} \tag{1.1}$$

*E-mail address: w.liu@uow.edu.au.
†E-mail address: ghw@uow.edu.au.

0022-247X/97 $25.00

In (1.1), Ω is a bounded domain (nonempty, open, and connected) in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$ (say C^2). $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$ for $T > 0$. $y(0) = y(x,0)$. $h = h(x,t)$ represents the control function, ω is an open and nonempty subset of Ω, and χ_ω is the characteristic function of ω. They proved that this system is approximately controllabel if f is globally Lipschitz.

Naturally, we would ask: Is the semilinear heat equation exactly controllable? That is to say, *for suitable* $T > 0$, *is it possible, for every initial and final data* y^0 *and* z^0 *(given in suitable Hilbert spaces), to find a corresponding control* h *driving the system*

$$\begin{cases} y' - \Delta y + f(t,y) = h\chi_\omega & \text{in } Q, \\ y(0) = y^0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \end{cases} \tag{1.2}$$

to the state z^0 *at time* T, *i.e., such that the solution* $y = y(x,t;h)$ *of* (1.2) *satisfies*

$$y(x,T) = z^0 \qquad \textit{in } \Omega? \tag{1.3}$$

In (1.2), we have allowed f to depend on t as well as y.

It is the purpose of this paper to positively answer this question. To achieve this goal, we will introduce a Monotone Operator Method (abbreviated to MOM). The idea of this method is to first construct a nonlinear, monotone, and continuous operator by coupling a linear heat equation with a semilinear heat equation, and they apply the famous Browder–Minty surjective theorem (see [12, p. 557]) from the theory of monotone operators.

Throughout this paper let Ω be a bounded domain (nonempty, open, and connected) in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$ (say C^2). Let $T > 0$ and set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$.

In the sequel, $H^s(\Omega)$ always denotes the usual Sobolev space and $\|\cdot\|_s$ denotes its norm for any $s \in \mathbb{R}$. Let X be a Banach space. We denote by $C^k([0,T],X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T],X)$ for $C^0([0,T],X)$.

We make the following hypothesis on f:

(H) Assume the function $f(t,y)$ is continuous in t on $[0,T]$ and globally Lipschitz continuous in y on $\mathbb{R}$, that is, there exists a positive constant l such that

$$|f(t,y_1) - f(t,t_2)| \le l|y_1 - y_2|, \qquad \text{for all } y_1, y_2 \in \mathbb{R}. \tag{1.4}$$

The main result of this paper is as follows.

THEOREM 1.1. *Assume* (H) *holds. Then there exists a* $T_0 > 0$ *such that for* $0 < T \leq T_0$ *system* (1.2) *is exactly controllable in* $L^2(\Omega)$ *at time* T, *that is, for any initial state* y^0 *and any terminal state* $z^0 \in L^2(\Omega)$, *there exists an internal control function* $h(x,t) = h(x,t; y^0, z^0) \in L^2(0,T; H^{-1}(\Omega))$ *such that the solution of* (1.2) *with* $\omega = \Omega$ *satisfies* (1.3). *Furthermore, for any fixed* $y^0 \in L^2(\Omega)$, *the control function*

$$h(x,t; y^0, z^0) : L^2(\Omega) \to L^2(0,T; h^{-1}(\Omega))$$

is Lipschitz continuous.

As remarked in [6], if ω is a proper subset of Ω, the exact internal controllability for the (linear) heat equation is going to be impossible. Thus we cannot expect the exact internal controllability for the semilinear heat equation if ω is a proper subset of Ω.

Compared with existing results, the result obtained here is essentially different from Fabre, Puel, and Zuazua's results [5] since we here consider the exact controllability. In addition, they generalize the relevant theorems of [1, 6, 8, 10] from the linear to the nonlinear case.

The rest of this paper is organised as follows. For completeness, in Section 2 we present some notions and a main theorem about monotone operators. Then we construct a nonlinear operator F in Section 3 and prove the monotonicity of the operator F in Section 4. Theorem 1.1 is proved in Section 5. Finally, we discuss the behaviour of controls as the nonlinear terms tend to zero in some sense in Section 6.

2. PRELIMINARIES

For convenience, we recall some basic notions and a main result about monotone operators. For details, we refer to [4, Chap. 3; 12, Chaps 25–26].

DEFINITION 2.1. (see [12, pp. 472 and 500]). Let X be a real Banach space, X^* its dual space, and $\langle \cdot , \cdot \rangle$ the duality pairing between X and X^*, and let $F : X \to X^*$ be an operator. Then

(i) F is called monotone iff

$$\langle Fu - Fv, u - v \rangle \geq 0, \qquad \text{for all } u, v \in X.$$

(ii) F is called strictly monotone iff

$$\langle Fu - Fv, u - v \rangle > 0, \qquad \text{for all } u, v \in X \text{ with } u \neq v.$$

(iii) F is called strongly monotone iff there is a $\alpha > 0$ such that

$$\langle Fu - Fv, u - v \rangle \geq \alpha \|u - v\|^2, \qquad \textit{for all } u, v \in X.$$

(iv) F is called hemicontinuous iff $\langle F(u + tv), w \rangle$ is continuous in t on $[0, 1]$ for all $u, v, w \in X$.

(v) F is called coercive iff

$$\lim_{\|u\| \to \infty} \frac{\langle Fu, u \rangle}{\|u\|} = +\infty.$$

Obviously, strong monotonicity implies coercivity.

THEOREM 2.2 (Browder-Minty, see [12, p. 557]). *Let X be a real reflexive Banach space, and $F : X \to X^*$ be a monotone, coercive, and hemicontinuous operator. Then F is onto X^*. Furthermore, if F is strongly monotone, then the inverse operator $F^{-1} : X^* \to X$ exists and is Lipschitz continuous.*

3. CONSTRUCTION OF A NONLINEAR OPERATOR

We construct a nonlinear operator F. To do this, we first consider the following problem with a given terminal state $u^T(x)$:

$$\begin{cases} u' + \Delta u = 0 & \text{in } Q, \\ u(x, T) = u^T(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma. \end{cases} \tag{3.1}$$

Concerning problem (3.1), the following results are classical.

LEMMA 3.1 [11, p. 210]. *Δ is the infinitesimal generator of both a C_0 semigroup of contractions and an analytic semigroup of operators on $L^2(\Omega)$.*

LEMMA 3.2 [3, pp. 512–513; p. 210]. (i) *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for all $u^T \in L^2(\Omega)$ there exists a unique solution $u = u(x, t)$ of* (3.1) *with*

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)). \tag{3.2}$$

Moreover,

$$\|u(t)\|_0 \leq \|u^T\|_0, \qquad \forall t \in [0, T]. \tag{3.3}$$

(ii) *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^2. Then for all $u^T \in D(\Delta)$ with*

$$D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \tag{3.4}$$

there exists a unique solution $u = u(x,t)$ *of* (3.1) *with*

$$u \in C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big). \tag{3.5}$$

Moreover, there exists a constant c independent of T such that

$$\|u(t)\|_2 \le c\|u^T\|_2, \qquad \forall t \in [0,T]. \tag{3.6}$$

Using the solution u of (3.1) we then consider the problem with any fixed initial state $y^0(x) \in L^2(\Omega)$:

$$\begin{cases} y' - \Delta y + f(t,y) = u - \Delta u & \text{in } Q, \\ y(x,0) = y^0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma. \end{cases} \tag{3.7}$$

It follows from the assumption (H) and the classical semigroup theory [11, Theorem 1.2, p. 184] that problem (3.7) admits a unique weak solution $y \in C([0,T]; L^2(\Omega))$ since $u - \Delta u \in C([0,T]; L^2(\Omega))$ for $u^T \in D(\Delta)$.

Now we define for any fixed initial state $y^0(x) \in L^2(\Omega)$ the nonlinear operator $F(y^0,\cdot): D(\Delta) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$F(y^0, u^T) = y(x,T). \tag{3.8}$$

4. MONOTONICITY OF THE NONLINEAR OPERATOR F

We will prove that the operator F defined by (3.8) is Lipschitz continuous and strongly monotone.

LEMMA 4.1. *Assume* (H) *holds. Then the operator F defined by* (3.8) *is Lipschitz continuous, that is, there exists a positive constant* $c = c(T)$ *such that*

$$\|F(y_2^0, u_2^T) - F(y_1^0, u_1^T)\|_0 \le c\big(\|y_2^0 - y_1^0\|_0 + \|u_2^T - u_1^T\|_0\big),$$

for any u_2^T, $u_1^T \in D(\Delta)$ *and any* y_2^0, $y_1^0 \in L^2(\Omega)$.

Proof. Let u_1 and u_2 be solutions of (3.1) with terminal states u_1^T and $u_2^T \in D(\Delta)$, and y_1 and y_2 be the solutions of (3.7) corresponding to u_1, y_1^0 and u_2, y_2^0. Then we have

$$\begin{cases} (y_2 - y_1)' - \Delta(y_2 - y_1) + f(t,y_2) - f(t,y_1) & \\ \qquad = u_2 - u_1 - \Delta(u_2 - u_1) & \text{in } Q, \\ y_2(x,0) - y_1(x,0) = y_2^0 - y_1^0 & \text{in } \Omega, \\ y_2 - y_1 = 0 & \text{on } \Sigma. \end{cases} \tag{4.2}$$

Multiplying (4.2) by $y_2 - y_1$ and integrating over $Q_t = \Omega \times (0, t)$, we obtain

$$\frac{1}{2}\int_\Omega |y_2(t) - y_1(t)|^2 dx - \frac{1}{2}\int_\Omega |y_2^0 - y_1^0|^2 dx + \int_{Q_t} |\nabla(y_2 - y_1)|^2 dxdt$$
$$= \int_{Q_t} (y_2 - y_1)(u_2 - u_1)\,dxdt + \int_{Q_t} \nabla(u_2 - u_1)\cdot\nabla(y_2 - y_1)\,dxdt$$
$$- \int_{Q_t} (f(t, y_2) - f(t, y_1))(y_2 - y_1)\,dxdt$$
$$\leq \frac{1}{2}\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt$$
$$+ \frac{1}{2}\int_{Q_t} \left(|y_2 - y_1|^2 + |\nabla(y_2 - y_1)|^2\right) dxdt + l\int_{Q_t} |y_2 - y_1|^2 dxdt. \tag{4.3}$$

Thus,

$$\frac{1}{2}\int_\Omega |y_2(t) - y_1(t)|^2 dx$$
$$\leq \frac{1}{2}\int_\Omega |y_2(t) - y_1(t)|^2 dx + \frac{1}{2}\int_{Q_t} |\nabla(y_2 - y_1)|^2 dxdt$$
$$\leq \frac{1}{2}\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt + \frac{1}{2}\int_\Omega |y_2^0 - y_1^0|^2 dx$$
$$+ (2l + 1)\frac{1}{2}\int_0^t \int_\Omega |y_2(t) - y_1(t)|^2 dxdt. \tag{4.4}$$

Gronwall's inequality (see [7, p. 36]) gives

$$\int_\Omega |y_2(t) - y_1(t)|^2 dx$$
$$\leq e^{(2l+1)t}\left(\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt + \int_\Omega |y_2^0 - y_1^0|^2 dx\right). \tag{4.5}$$

But there exists a positive constant c such that

$$\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt \le c\|u_2^T - u_1^T\|_0^2. \tag{4.6}$$

Therefore, (4.1) follows from (4.5) and (4.6). ▮

Remark 4.2. By Lemma 4.1, for any fixed $y^0 \in L^2(\Omega)$, the operator $F(y^0, \cdot)$ can be extended to $L^2(\Omega)$. To do so, let $u^T \in L^2(\Omega)$ and $\{u_n^T\}$ be a sequence in $D(\Delta)$ such that $u_n^T \to u^T$ in $L^2(\Omega)$. Then it follows from Lemma 4.1 that $\{F(y^0, u_n^T)\}$ is a Cauchy sequence in $L^2(\Omega)$. We set

$$F(y^0, u^T) = \lim_{n\to\infty} F(y^0, u_n^T).$$

It is clear that $F(y^0, u^T)$ is independent of the choice of the sequence $\{u_n^T\}$. Furthermore, it is easy to show that the extension of $F(y^0, \cdot)$ is still Lipschitz. From now on, F is thought of as an operator defined on $L^2(\Omega) \times L^2(\Omega)$.

To prove the strong monotonicity of $F(y^0, \cdot)$ for any fixed $y^0 \in L^2(\Omega)$, we need the exponential decay rate for solutions of the heat equation.

LEMMA 4.3. *Let u be the solution of* (3.1). *Then there is a constant $\delta > 0$ such that*

$$\int_\Omega |u(t)|^2 dx \le e^{-\delta(T-t)} \int_\Omega |u(T)|^2 dx, \qquad \textit{for } t \in [0, T]. \tag{4.7}$$

Proof. Since

$$\begin{aligned}
\frac{d}{dt}\left[e^{\delta(T-t)} \int_\Omega |u(t)|^2 dx\right] &= 2e^{\delta(T-t)} \int_\Omega u(t)u'(t)\,dx - \delta e^{\delta(T-t)} \int_\Omega |u(t)|^2 dx \\
&= 2e^{\delta(T-t)} \int_\Omega |\nabla u(t)|^2 dx - \delta e^{\delta(T-t)} \int_\Omega |u(t)|^2 dx \\
&\ge 2e^{\delta(T-t)} \int_\Omega |\nabla u(t)|^2 dx - \delta\beta e^{\delta(T-t)} \int_\Omega |\nabla u(t)|^2 dx \\
&= (2 - \delta\beta) e^{\delta(T-t)} \int_\Omega |\nabla u(t)|^2 dx \\
&> 0,
\end{aligned}$$

if $\delta < 2/\beta$, we obtain

$$\int_\Omega |u(t)|^2 dx \le e^{-\delta(T-t)} \int_\Omega |u(T)|^2 dx, \qquad \text{for } t \in [0, T].$$

Here we have used Poincaré's inequality (see [2, p. 127]) and β is Poincaré's constant. This is valid for the solution u of (3.1) by the definition of $D(\Delta)$. ∎

LEMMA 4.4. *Assume* (H) *holds. Suppose l or T is so small that*

$$M(l,T) = 1 + \frac{l^2}{2l+1}\left(1 - e^{(2l+1)T}\right) > 0. \tag{4.8}$$

Then for any fixed $y^0 \in L^2(\Omega)$ the operator $F(y^0, \cdot): L^2(\Omega) \to L^2(\Omega)$ is strongly monotone.

Proof. We first assume u_1^T, $u_2^T \in D(\Delta)$. Multiplying (4.2) by $u_2 - u_1$, integrating over Q, and noting $y_2^0 = y_1^0 = y^0$, we obtain

$$\begin{aligned}
&\int_\Omega \left(F(y^0, u_2^T) - F(y^0, u_1^T)\right)\left(u_2^T - u_1^T\right) dx \\
&\quad = \int_\Omega (y_2(T) - y_1(T))\left(u_2^T - u_1^T\right) dx \\
&\quad = \int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt \\
&\qquad - \int_Q (f(t, y_2) - f(t, y_1))(u_2 - u_1)\, dxdt \\
&\quad \geq \int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt - \frac{1}{2}\int_Q |u_2 - u_1|^2 dxdt \\
&\qquad - \frac{l^2}{2}\int_Q |y_2 - y_1|^2 dxdt \\
&\quad \geq \frac{1}{2}\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt \qquad \text{(use (4.5))} \\
&\qquad - \frac{l^2}{2}\int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt \int_0^T e^{(2l+1)t} dt \\
&\quad \geq \frac{1}{2} M(l,T) \int_Q \left(|u_2 - u_1|^2 + |\nabla(u_2 - u_1)|^2\right) dxdt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_2(T) - u_1(T)|^2 dx - \frac{1}{2}\int_\Omega |u_2(0) - u_1(0)|^2 dx \\
&\qquad - \int_Q |\nabla(u_2 - u_1)|^2 dxdt = 0.
\end{aligned}$$

Thus, it follows from Lemma 4.3 that

$$\begin{aligned}
&\int_\Omega \left(F\left(y^0, u_2^T\right) - F\left(y^0, u_1^T\right)\right)\left(u_2^T - u_1^T\right) dx \\
&\quad \geq \frac{1}{2} M(l,T) \frac{1}{2} \left(\int_\Omega |u_2(T) - u_1(T)|^2 dx - \int_\Omega |u_2(0) - u_1(0)|^2 dx \right) \\
&\quad \geq \frac{1}{4} M(l,T) \left(\int_\Omega |u_2(T) - u_1(T)|^2 dx - e^{-\delta T} \int_\Omega |u_2(T) - u_1(T)|^2 dx \right) \\
&\quad = \frac{1}{4} M(l,T)(1 - e^{-\delta T}) \int_\Omega |u_2(T) - u_1(T)|^2 dx. \qquad (4.9)
\end{aligned}$$

By taking a limit, we can show (4.9) holds for any $u_1^T, u_2^T \in L^2(\Omega)$. Thus we have proved $A(y^0, \cdot)$ is strongly monotone. ∎

Remark 4.5. There are many function $f(t, y)$ that satisfy the conditions of Lemmas 4.1 and 4.4. For example,

$$f(t, y) = \lambda t \sin y,$$

provided λ is so small that $1 + (\lambda^2 T^2/(2\lambda T + 1))(1 - e^{(2\lambda T + 1)T}) > 0$.

5. PROOF OF THEOREM 1.1

Lemma 4.4 shows that the constant l is required to be small enough so that F is strongly monotone. To overcome this drawback, we introduce a *Domain Expansion Method* to prove Theorem 1.1 This method is general and can be applied to other situations.

Proof of Theorem 1.1. For $\tau > 0$, we introduce a function f_τ by

$$f_\tau(t, u) = \frac{1}{\tau^2} f\left(\frac{t}{\tau^2}, u\right), \qquad (5.1)$$

and a domain

$$\Omega(\tau) = \{\tau x : x \in \Omega\}.$$

Set

$$Q_1(\tau) = \Omega(\tau) \times (0,1),$$

$$\Sigma_1(\tau) = \partial\Omega(\tau) \times (0,1) = \{(\tau x, t) : (x, t) \in \Gamma \times (0,1)\}.$$

Instead of (3.1) and (3.7), we consider

$$\begin{cases} u' + \Delta u = 0 & \text{in } Q_1(\tau), \\ u(x,1) = u^1(x) & \text{in } \Omega(\tau), \\ u = 0 & \text{on } \Sigma_1(\tau), \end{cases} \tag{5.2}$$

and

$$\begin{cases} w' - \Delta w + f_\tau(t,w) = u - \Delta u & \text{in } Q_1(\tau), \\ w(x,0) = w^0(x) & \text{in } \Omega(\tau), \\ w = 0 & \text{on } \Sigma_1(\tau), \end{cases} \tag{5.3}$$

where

$$w^0(x) = y^0\left(\frac{x}{\tau}\right), \qquad x \in \Omega(\tau). \tag{5.4}$$

Then the operator F defined by (3.8) becomes

$$F(w^0, u^1) = w(x,1). \tag{5.5}$$

The constant l in (H) for f_τ now is l/τ^2, and the constant M defined by (4.8) now is

$$M\left(\frac{l}{\tau^2}, 1\right) = 1 + \frac{l^2}{\tau^2(2l + \tau^2)}\left(1 - \exp\left(\frac{2l + \tau^2}{\tau^2}\right)\right).$$

Let τ_0 be such that $M(l/\tau_0^2, 1) > 0$. Then we have $M(l/\tau^2, 1) > 0$ for $\tau \geq \tau_0$. It therefore follows from Lemmas 4.1 and 4.4 that F is continuous and strongly monotone on $L^2(\Omega(\tau))$. It then follows from Theorem 2.2 that for any $z^0 \in L^2(\Omega)$ there exists $u^1 \in L^2(\Omega(\tau))$ such that

$$F(w^0, u^1) = v^0,$$

where

$$v^0(x) = z^0\left(\frac{x}{\tau}\right), \qquad x \in \Omega(\tau). \tag{5.6}$$

Moreover, $F^{-1}(w^0, \cdot\,]$ is Lipschitz continuous. Then we solve problem (5.2) with the terminal state u^1. Thus we have found an internal control function

$$u - \Delta u \in L^2\big(0,1; H^{-1}(\Omega(\tau))\big)$$

such that the solution of (5.3) satisfies

$$w(x,1) = v^0, \qquad x \in \Omega(\tau).$$

Setting

$$y(x,t) = w(\tau x, \tau^2 t), \qquad x \in \Omega,\ t \geq 0,$$
$$h(x,t;y^0,z^0) = \tau^2\big[u(\tau x, \tau^2 t) - \Delta u(\tau x, \tau^2 t)\big], \tag{5.7}$$

then y satisfies (1.2) and

$$y\left(x, \frac{1}{\tau^2}\right) = z^0, \qquad x \in \Omega.$$

Set

$$T_0 = \frac{1}{\tau_0^2}.$$

Then we have proved that for $0 < T = 1/\tau^2 \leq T_0$ system (1.2) is exactly controllable in $L^2(\Omega)$ at time T.

Furthermore, it follows from (5.7) and the Lipschitz continuity of F^{-1} that (the following c's denoting various constants)

$$\begin{aligned}
&\|h(x,t;y^0,z_1^0) - h(x,t;y^0,z_2^0)\|^2_{L^2(0,T;H^{-1}(\Omega))} \\
&\quad \leq c\|u_1 - u_2 + \Delta(u_2 - u_1)\|^2_{L^2(0,1;H^{-1}(\Omega(\tau)))} \\
&\quad \leq c\|u_1 - u_2\|^2_{L^2(0,1;H_0^1(\Omega(\tau)))} \\
&\quad \leq c\|u_1^1 - u_2^1\|^2_{0,\Omega(\tau)} \\
&\quad = c\|F^{-1}(w^0, v_1^0) - F^{-1}(w^0, v_2^0)\|^2_{0,\Omega(\tau)} \\
&\quad \leq c\|v_1^0 - v_2^0\|u^2_{0,\Omega(\tau)} \\
&\quad \leq c\|z_1^0 - z_1^0\|^2_{0,\Omega}. \qquad \blacksquare
\end{aligned}$$

We call the method of the above proof the *Domain Expansion Method*.

Remark 5.1. It should be understood that the solution of (5.3) for $v^0 = F(w^0, u^1) \notin \{F(w^0, u^1) : u^1 \in D(\Delta)\}$ is the limit of $\{w_n\}$ in $C([0,T]; L^2(\Omega(\tau)))$, where the w_n are the weak solutions of (5.3) with u replaced by the solutions u_n of (5.2) with terminal states $u_n^1 \in D(\Delta)$ such that $u_n^T \to u^T$ in $L^2(\Omega(\tau))$.

6. BEHAVIOUR OF CONTROLS AS $f \to 0$

Let c_1, c_2 with $c_1 < c_2$ be two fixed constants. It is known from Theorem 1.1 that if $\varepsilon \in [c_1, c_2]$, then we can find $T > 0$ independent of ε such that for any terminal state $z^0 \in L^2(\Omega)$ there exists an internal control

$h_\varepsilon(x,t)$ such that the solution y_ε, which depends on ε, of

$$\begin{cases} y'_\varepsilon - \Delta y_\varepsilon + \varepsilon f(t, y_\varepsilon) = h_\varepsilon & \text{in } Q, \\ y_\varepsilon(x,0) = y^0 & \text{in } \Omega, \\ y_\varepsilon = 0 & \text{on } \Sigma, \end{cases} \tag{6.1}$$

satisfies

$$y_\varepsilon(x,T) = z^0 \qquad \text{iin } \Omega. \tag{6.2}$$

We now study the behaviour of $h_\varepsilon(x,t)$ as $\varepsilon \to 0$ if $0 \in [c_1, c_2]$. This is a kind of nonlinear perturbation, which is motivated by Lions' work [9], where the problems of linear perturbation have been studied in detail.

Let f be replaced by εf in the proof of Theorem 1.1. Obviously, the operator F defined by (5.5) now depends on ε. So we write F_ε for F. For a fixed $z^0 \in L^2(\Omega)$, let u_ε^1 be the solution of the operator equation

$$F_\varepsilon(w^0, u^1) = v^0, \tag{6.3}$$

and $u_\varepsilon(x,t)$ be the solution of (5.2) with $u^1 = u_\varepsilon^1$, where w^0, v^0 are given by (5.4) and (5.6), respectively. Taking $u_2^T = u_\varepsilon^1$ and $u_1^T = 0$ and $\Omega = \Omega(\tau)$ in (4.9), we obtain

$$\begin{aligned} \int_{\Omega(\tau)} \big(v^0 - F_\varepsilon(w^0, 0)\big) u_\varepsilon^1 \, dx &= \int_{\Omega(\tau)} \big(F_\varepsilon(w^0, u_\varepsilon^1) - F_\varepsilon(w^0, 0)\big) u_\varepsilon^1 \, dx \\ &\geq \frac{1}{4} M(l\varepsilon/\tau^2, 1)(1 - e^{-\delta}) \int_{\Omega(\tau)} |u_\varepsilon^1|^2 dx \\ &\geq \frac{1}{8}(1 - e^{-\delta}) \int_{\Omega(\tau)} |u_\varepsilon^1|^2 dx, \end{aligned} \tag{6.4}$$

for $\varepsilon \in [c_1, c_2]$ if τ is sufficiently large so that $M(l\varepsilon/\tau^2, 1) > 1/2$ for $\varepsilon \in [c_1, c_2]$.

On the other hand, as in the proof of Lemma 4.1, we can prove for the solution $w_\varepsilon(x,t) = F_\varepsilon(w^0, 0)$ of (5.3) with $u - \Delta u = 0$

$$\begin{aligned} &\int_{\Omega(\tau)} |w_\varepsilon(x,t)|^2 dx \\ &\quad \leq \exp\left[\frac{\varepsilon(2l + \tau^2)t}{\tau^2}\right] \\ &\qquad \times \left[\int_{\Omega(\tau)} |w^0(x)|^2 dx + \frac{\varepsilon m(\Omega(\tau))}{\tau^4} \int_0^1 |f(t/\tau^2, 0)|^2 dt\right], \\ &\qquad\qquad \text{for } t \in [0,1]. \end{aligned} \tag{6.5}$$

In fact, multiplying (5.3) by w_ε, integrating over $Q_t(\tau) = \Omega(\tau) \times (0, t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega(\tau)} |w_\varepsilon(x,t)|^2 dx - \frac{1}{2}\int_{\Omega(\tau)} |w^0(x)|^2 dx + \int_{Q_t(\tau)} |\nabla w_\varepsilon|^2 dxdt \\
&\quad = -\varepsilon \int_{Q_t(\tau)} f_\tau(t, w_\varepsilon) w_\varepsilon \, dxdt \\
&\quad = -\varepsilon \int_{Q_t(\tau)} [f_\tau(t, w_\varepsilon) - f_\tau(t, 0)] w_\varepsilon \, dxdt - \varepsilon \int_{Q_t(\tau)} f_\tau(t, 0) w_\varepsilon \, dxdt \\
&\quad \le \frac{\varepsilon m(\Omega(\tau))}{2\tau^4} \int_0^1 |f(t/\tau^2, 0)|^2 dt + \frac{\varepsilon(2l + \tau^2)}{2\tau^2} \int_{Q_t(\tau)} |w_\varepsilon|^2 dxdt,
\end{aligned}$$

where $m(\Omega(\tau))$ denotes the Lebesgue measure of $\Omega(\tau)$. Thus,

$$\begin{aligned}
&\int_{\Omega(\tau)} |w_\varepsilon(x,t)|^2 dx \\
&\qquad \le \int_{\Omega(\tau)} |w^0(x)|^2 dx + \frac{\varepsilon m(\Omega(\tau))}{\tau^4} \int_0^1 |f(t/\tau^2, 0)|^2 dt \\
&\qquad\quad + \frac{\varepsilon(2l + \tau^2)}{\tau^2} \int_0^t \int_{\Omega(\tau)} |w_\varepsilon(x,t)|^2 dxdt,
\end{aligned}$$

which, by Gronwall's inequality, gives (6.5).

Relations (6.4) and (6.5) show that $\{u_\varepsilon^1\}$ is bounded in $L^2(\Omega(\tau))$ for $c_1 \le \varepsilon \le c_2$. Furthermore, since $\|u_\varepsilon\|_{-1} \le \|u_\varepsilon\|_0$ and Δ is an isomorphism from $H_0^1(\Omega(\tau))$ onto $H^{-1}(\Omega(\tau))$, it follows from (5.7) that there exists a constant $c > 0$ such that

$$\begin{aligned}
\|h_\varepsilon\|^2_{L^2(0,T;H^{-1}(\Omega))} &\le c\|u_\varepsilon - \Delta u_\varepsilon\|^2_{L^2(0,1;H^{-1}(\Omega(\tau)))} \\
&\le c\int_0^1 \|u_\varepsilon\|^2_{1,\Omega(\tau)} dt. \qquad (6.6)
\end{aligned}$$

Moreover, by Theorem 3 of [3, p. 520], there is a constant $c > 0$ such that

$$\int_{Q_1(\tau)} \left(|u_\varepsilon|^2 + |\nabla u_\varepsilon|^2\right) dxdt \le c\|u_\varepsilon^1\|^2_{0,\Omega(\tau)}. \qquad (6.7)$$

It therefore follows from (6.6), (6.7), and the boundedness of $\{u_\varepsilon^1 : c_1 \le \varepsilon \le c_2\}$ that the set $\{h_\varepsilon : c_1 \le \varepsilon \le c_2\}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$.

Let $0 \in [c_1, c_2]$. Let the subsequence $\{h_{\varepsilon_i}\}$ of $\{h_\varepsilon : c_1 \le \varepsilon \le c_2\}$ be such that $h_{\varepsilon_i} \to h$ weakly in $L^2(0, T; H^{-1}(\Omega))$ as $\varepsilon_i \to 0$.

Let y be the solution of

$$\begin{cases} y' - \Delta y = h & \text{in } Q, \\ y(0) = y^0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \end{cases} \tag{6.8}$$

then

$$y(T) = z^0 \qquad \text{in } \Omega. \tag{6.9}$$

Indeed, subtracting (6.8) from (6.1), we obtain

$$\begin{cases} (y_\varepsilon - y)' - \Delta(y_\varepsilon - y) + \varepsilon f(t, y_\varepsilon) = h_\varepsilon - h & \text{in } Q, \\ y_\varepsilon(0) - y(0) = 0 & \text{in } \Omega, \\ y_\varepsilon - y = 0 & \text{on } \Sigma. \end{cases} \tag{6.10}$$

For any $\theta^T \in L^2(\Omega)$, let θ be the solution of

$$\begin{cases} \theta' + \Delta\theta = 0 & \text{in } Q, \\ \theta(T) = \theta^T & \text{in } \Omega, \\ \theta = 0 & \text{on } \Sigma. \end{cases} \tag{6.11}$$

Multiplying (6.10) by θ and integrating by parts, we obtain

$$\int_\Omega (z^0 - y(T))\theta^T dx + \int_Q \varepsilon f(t, y_\varepsilon)\theta dx dt = \int_Q (h_\varepsilon - h)\theta dx. \tag{6.12}$$

In addition, it follows from the assumption (H) that

$$\|\varepsilon f(t, y_\varepsilon)\|_0 \le \varepsilon l \|y_\varepsilon\|_0 + \varepsilon |f(t,0)| [m(\Omega)]^{1/2}. \tag{6.13}$$

Therefore, letting $\varepsilon_i \to 0$ in (6.12), we obtain

$$\int_\Omega (z^0 - y(T))\theta^T dx = 0. \tag{6.14}$$

This implies (6.9) because θ^T is arbitrary in $L^2(\Omega)$.

In summary, we have proved

THEOREM 6.1. *Let h_ε be internal control functions obtained as in Theorem 1.1, driving system (6.1) from an initinal state $y^0 \in L^2(\Omega)$ to a terminal state $z^0 \in L^2(\Omega)$. Then for any fixed constants c_1, c_2 with $c_1 < c_2$, the set $\{h_\varepsilon : c_1 \le \varepsilon \le c_2\}$ is relatively weakly compact in $L^2(0, T; H^{-1}(\Omega))$. Furthermore, if $0 \in [c_1, c_2]$, then any weak limit h of a subsequence $[h_{\varepsilon_i}\}$ of $\{h_\varepsilon : c_1 \le \varepsilon \le c_2\}$ in $L^2([0, T]; H^{-1}(\Omega))$ as $\varepsilon_i \to 0$ is an internal control function driving system (6.8) from the initinal state y^0 to the terminal state z^0.*

ACKNOWLEDGMENTS

The first author is grateful for the Overseas Postgraduate Research Scholarship from Australian Government, the University of Wollongong Postgraduate Research Award, and the Department of Mathematics Analysis Research Group Scholarship from the Faculty of Informatics at the University of Wollongong.

REFERENCES

1. C. Carthel, R. Glowinski, and J. L. Lions, On exact and approximate boundary controllabilities for the heat equation: A numerical approach, *J. Optim. Theory Appl.* 82, No. 3, (1994), 429–484.
2. R. Dautray and J. L. Lions, Functional and variational methods, *in* "Mathematical Analysis and Numerical Methods for Science and Technology," Vol. 2, Springer-Verlag, Berlin, 1992.
3. R. Dautray and J. L. Lions, Evolution problems, I, *in* "Mathematical Analysis and Numerical Methods for Science and Technology," Vol. 5, Springer-Verlag, Berlin, 1992.
4. K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin, 1985.
5. C. Fabre, J. P. Puel, and E. Zuazua, Approximate controllability of the semilinear heat equation, *Proc. Roy. Soc. Edinburgh Sect. A* 125 (1995), 31–61.
6. R. Glowinski and J. L. Lions, Exact and approximate controllability for distributed parameter systems, *Acta Numerica* (1994), 269–378.
7. J. K. Hale, "Ordinary Differential Equations," Wiley–Interscience, New York, 1969.
8. G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, *Comm. Partial Differential Equations* 20, Nos. 1–2 (1995), 335–356.
9. J. L. Lions, Perturbations, *in* "Contrôlabilité exacte perturbations et stabilisation de systèmes distribués," Vol. 2, Masson, Paris, 1988.
10. J. L. Lions, Exact controllability for distributed systems: Some trends and some problems, *in* "Applied and Industrial Mathematics" (R. Spigler, Ed.), pp. 59–84, Kluwer Academic, Dordrecht/Norwell, MA, 1991.
11. A. Pazy, "Semigroup of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.
12. E. Zeidler, Nonlinear monotone operators, *in* "Nonlinear Functional Analysis and its Applications," II/B, Springer-Verlag, New York, 1990.