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THE EXPONENTIAL STABILIZATION OF THE HIGHER-DIMENSIONAL LINEAR SYSTEM OF THERMOVISCOELASTICITY

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ABSTRACT. – Using multipler techniques and Lyapunov methods, we prove that the energy in the higher-dimensional linear thermoviscoelasticity decays to zero exponentially by introducing a velocity feedback on part of the boundary of a thermoviscoelastic body, which is clamped along the rest of its boundary, to increase the loss of energy. © Elsevier, Paris

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1. Introduction

In this paper, we shall be concerned with the problem of exponential stabilization of the linear thermoviscoelastic model:

$$(1.1)\qquad \begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u & \\ \quad +\mu g * \Delta u + (\lambda+\mu) g * \nabla \mathrm{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,\infty), \\ \theta' - \Delta\theta + \beta \mathrm{div}\ u' = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0, \quad \theta = 0 & \text{on } \Gamma\times(0,\infty), \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x) & \text{in } \Omega, \\ u(x,0) - u(x,-s) = w^0(x,s) & \text{in } \Omega\times(0,\infty), \end{cases}$$

where the sign "$*$" denotes the convolution product in time, which is defined by

$$g * v(t) = \int_{-\infty}^{t} g(t-s)v(x,s)ds.$$

System (1.1) is a model for a linear viscoelastic body Ω of the Boltzmann type with thermal damping. The body Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ (say C^2) and is assumed to be linear, homogeneous, and isotropic. $u(x,t) = (u_1(x,t),\cdots,u_n(x,t))$, $\theta(x,t)$ represent displacement and temperature deviations, respectively, from the natural state of the reference configuration at position x and time t. λ, $\mu > 0$ are Lamé's constants and α, $\beta > 0$ the coupling parameters; $g(t)$ denotes the relaxation function, $w^0(x,s)$ is a specified " history", and $u^0(x)$, $u^1(x)$, $\theta^0(x)$

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are initial data. By $'$ we denote the derivative with respect to the time variable; Δ, ∇, div denote the Laplace, gradient, and divergence operators in the space variables, respectively. We refer to [Nav] for the derivation of model (1.1).

The following basic conditions on the relaxation function $g(t)$ are standard (see [Daf$_2$, Daf$_3$]):

(H_1) $g \in C^1[0,\infty) \cap L^1(0,\infty)$;

(H_2) $g(t) \geq 0$ and $g'(t) \leq 0$ for $t > 0$;

(H_3) $\kappa = 1 - \int_0^\infty g(t)dt > 0$.

Condition (H_3) simply states that the static modulus of elasticity is positive. This restriction is quite natural. In addition, conditions (H_1) and (H_2) imply

$$g(\infty) = \lim_{t\to\infty} g(t) = 0. \tag{1.2}$$

In what follows, we denote by $\|\cdot\|$ the norm of $L^2(\Omega)$. The energy $E(u,\theta,t)$ of (1.1) is defined by

$$\begin{aligned} E(u,\theta,t) = {} & \frac{1}{2}[\|u'(t)\|^2 + \frac{\alpha}{\beta}\|\theta(t)\|^2] \\ & + \frac{\mu\kappa}{2}\|\nabla u(t)\|^2 + \frac{(\lambda+\mu)\kappa}{2}\|\mathrm{div}u(t)\|^2 \\ & + \frac{\mu}{2}\int_{-\infty}^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \\ & + \frac{\lambda+\mu}{2}\int_{-\infty}^t g(t-s)\|\mathrm{div}u(t) - \mathrm{div}u(s)\|^2 ds. \end{aligned} \tag{1.3}$$

Here we have used the notation

$$\|v\|^2 = \sum_{i=1}^n \|v_i\|^2, \qquad \text{for } v = (v_1, \cdots, v_n).$$

By straightforward calculation, we have

$$\begin{aligned} E'(u,\theta,t) = {} & \frac{\mu}{2}\int_{-\infty}^t g'(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \\ & + \frac{\lambda+\mu}{2}\int_{-\infty}^t g'(t-s)\|\mathrm{div}u(t) - \mathrm{div}u(s)\|^2 ds \\ & - \frac{\alpha}{\beta}\|\nabla\theta(t)\|^2. \end{aligned} \tag{1.4}$$

Hence, the energy $E(u,\theta,t)$ decreases on $(0,\infty)$. Indeed, if the relaxation function $g(t)$ satisfies conditions (H_1), (H_2) and (H_3), Navarro [Nav] proved the asymptotic stability for system (1.1), that is:

$$\lim_{t\to\infty} \|u(t)\|_{(H_0^1(\Omega))^n} = \lim_{t\to\infty} \|u'(t)\|_{(L^2(\Omega))^n} = \lim_{t\to\infty} \|\theta(t)\|_{L^2(\Omega)} = 0.$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space.

However, the most interesting question is whether the energy decays exponentially as $t \to \infty$. Namely, are there positive constants M, ω such that

$$E(u,\theta,t) \le Me^{-\omega t}E(u,\theta,0), \qquad \forall t \ge 0 \ ? \tag{1.5}$$

In the case of one space dimension, this question is in part positively answered. Indeed, if $g(t)$ decays exponentially, Liu and Zheng [LZ_2] recently proved that the energy also decays to zero exponentially. However, in the case of higher space dimension, the problem is much more complicated. In order to see such complexity, we look at some special cases: thermoelastic system and viscoelastic system.

When $g \equiv 0$, system (1.1) is reduced to the following thermoelastic system:

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,\infty), \\ \theta' - \Delta\theta + \beta \mathrm{div}\, u' = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0, \quad \theta = 0 & \text{on } \Gamma\times(0,\infty), \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Its energy $E(u,\theta,t)$ is defined by

$$\begin{aligned} E(u,\theta,t) = {} & \frac{1}{2}[\|u'(t)\|^2 + \frac{\alpha}{\beta}\|\theta(t)\|^2] \\ & + \frac{\mu}{2}\|\nabla u(t)\|^2 + \frac{(\lambda+\mu)}{2}\|\mathrm{div} u(t)\|^2. \end{aligned} \tag{1.7}$$

In the one-dimensional space case, it has been shown (see [BLZ, Han, Kim, LZ_1]) that the energy decays exponentially. However, in the higher dimensional space case, it is by now well known (see [Daf_1]) that the energy, in general, does not tend to zero as $t \to \infty$. Indeed, Lebeau and Zuazua [LeZ] recently gave a sufficient and necessary condition ensuring that the energy tends to zero exponentially as $t \to +\infty$ in a bounded multi-dimensional smooth domain Ω. This condition is written in terms of the dynamics of the rays of geometric optics. As a consequence of the result of [LeZ], it follows that when Ω is a bounded smooth convex open set, the energy does not decay exponentially to zero. This is because the total energy is not dissipated completely in the form of thermal energy. Therefore, in order to ensure the exponential stabilization in such case, a boundary velocity feedback was introduced in [Liu] to increase the loss of energy.

When $\alpha = \beta = 0$, system (1.1) is decoupled into the following viscoelastic system:

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u \\ \quad +\mu g * \Delta u + (\lambda+\mu) g * \nabla \mathrm{div} u = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0, & \text{on } \Gamma\times(0,\infty), \\ u(x,0) = u^0(x),\ u'(x,0) = u^1(x) & \text{in } \Omega, \\ u(x,0) - u(x,-s) = w^0(x,s) & \text{in } \Omega\times(0,\infty), \end{cases} \tag{1.8}$$

and the heat equation. The energy $E(u,t)$ of (1.8) is defined by

$$\begin{aligned} E(u,t) = {} & \frac{1}{2}[\|u'(t)\|^2 + \mu\kappa\|\nabla u(t)\|^2 + (\lambda+\mu)\kappa\|\mathrm{div} u(t)\|^2] \\ & + \frac{\mu}{2}\int_{-\infty}^{t} g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \\ & + \frac{\lambda+\mu}{2}\int_{-\infty}^{t} g(t-s)\|\mathrm{div} u(t) - \mathrm{div} u(s)\|^2 ds. \end{aligned} \tag{1.9}$$

It can be easily seen from (1.4) that the asymptotic behaviour of the energy $E(u,t)$ depends significantly on the relaxation function $g(t)$ and the history $w^0(x,s)$ as well. Indeed, if $g(t)$ satisfies (H_1), (H_2) and (H_3) and $w^0 \in L^2(g,(0,\infty),(H_0^1(\Omega))^n)$, Dafermos in his pioneering work [Daf$_2$, Daf$_3$] proved that $E(u,t)$ tends to zero asymptotically, where $L^2(g,(0,\infty),\ (H_0^1(\Omega))^n)$ denotes the "history space" of $(H_0^1(\Omega))^n)$-valued functions w on $(0,\infty)$ for which

$$\begin{aligned}&\|w\|^2_{L^2(g,(0,\infty),(H_0^1(\Omega))^n)}\\&\quad=\frac{\mu}{2}\int_0^\infty g(s)\|\nabla w(s)\|^2ds+\frac{\lambda+\mu}{2}\int_0^\infty g(s)\|\mathrm{div}w(s)\|^2ds<\infty.\end{aligned}$$

Subsequently, extensive attention was paid to the problem of obtaining an explicit decay rate. In this aspect, Day [Day] first obtained a decay rate of t^{-1} in the case of one space dimension by introducing a feedback at one end of an interval. Later, in the case of two space dimension, Leugering [Leu$_1$] established an exponential decay rate by introducing a velocity feedback on part of the boundary of a domain. On the other hand, if $g(t)$ decays exponentially and the initial history w^0 is taken to be zero, Desch and Miller [DM] proved that, in the case of one space dimension, the energy also decays to zero exponentially at a rate no better than $g(t)$ decays.

In view of the above, in order to obtain an explicit decay rate of energy of higher dimensional thermoviscoelastic system (1.1), it may be indispensable to introduce a velocity feedback on part of the boundary of a thermoviscoelastic body. Thus, in this paper, we introduce such a feedback to increase the loss of energy and establish the exponential stabilization. Similar boundary velocity feedbacks were extensively used for the wave equation [Che, KZ, Lag$_1$], elastodynamic systems [AK, Lag$_2$] and viscoelasticity [Leu$_1$, Leu$_2$].

In order to design a boundary velocity feedback, we set

$$\Gamma_1=\{x\in\Gamma\ :\ m(x)\cdot\nu(x)\le 0\},\tag{1.10}$$

$$\Gamma_2=\{x\in\Gamma\ :\ m(x)\cdot\nu(x)\ge 0\},\tag{1.11}$$

where

$$m(x)=x-x^0=(x_1-x_1^0,\cdots,x_n-x_n^0)\tag{1.12}$$

for some $x^0\in\mathbb{R}^n$, $\nu=(\nu_1,\cdots,\nu_n)$ denotes the unit normal on Γ directed towards the exterior of Ω and

$$m\cdot\nu=m(x)\cdot\nu(x)=\sum_{i=1}^n(x_i-x_i^0)\nu_i.\tag{1.13}$$

Γ_1 is assumed either to be empty or to have a nonempty interior relative to Γ. Note that assumptions (1.10) and (1.11) imply that the domain Ω is simply connected and

star-shaped with respect to $x^0 \in \Omega$ or $\Omega = \Omega_1 - \overline{\Omega}_2$, both Ω_1 and Ω_2 being star-shaped with respect to x^0.

The boundary velocity feedback can be given as follows

$$(1.14)\qquad \begin{cases} \theta = 0 & \text{on } \Gamma \times (0, \infty), \\ u = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \mu \dfrac{\partial}{\partial \nu}(u - g * u) + (\lambda + \mu)\nu \operatorname{div}(u - g * u) & \\ \qquad + am \cdot \nu(u - g * u) + m \cdot \nu u' = 0 & \text{on } \Gamma_2 \times (0, \infty). \end{cases}$$

where $a = a(x)$ is a given nonnegative function on Γ_2 with

$$(1.15)\qquad a(x) \in C^1(\Gamma_2).$$

It is clear that if $m(x) \cdot \nu(x) \geq \eta$ on Γ_2 for some $\eta > 0$ then $a(x)m(x) \cdot \nu(x)$ can be any nonnegative function as we can take $a(x) = f(x)/(m(x) \cdot \nu(x))$, $f(x)$ being any nonnegative function. Note that there is no velocity feedback on the part of Γ_2 where $m(x) \cdot \nu(x) = 0$.

We will prove (see Theorem 2.1 below) that the energy of the thermoviscoelastic system with boundary velocity feedback (1.14) decays to zero exponentially as $t \to \infty$ if $A = \max_{x \in \Gamma_2} a(x)$ is small enough and the following condition holds

$$(1.16)\qquad \Gamma_1 \neq \emptyset \quad \text{or} \quad a(x) \not\equiv 0.$$

Whether this exponential stabilization holds for the large A is open.

The rest of this paper is organized as follows. The main result of this paper is presented in Section 2. For the sake of completeness, we briefly discuss the semigroup associated with the thermoviscoelastic equations in Section 3. In Section 4, we prove the main result by using multiplier techniques and Lyapunov methods. Finally, in Section 5, we give comments on the case that (1.16) fails.

2. Main Result

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (see [Ada]). For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T]; X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T]; X)$ for $C^0([0,T]; X)$.

Let $a(x)$ be the nonnegative function given in (1.14) satisfying (1.15) and $\nu = (\nu_1, \cdots, \nu_n)$ the unit normal on Γ directed towards the exterior of Ω. Suppose that Γ_1 and Γ_2 are given by (1.10) and (1.11), respectively, and Γ_1 either is empty or has a nonempty interior relative to Γ. Assume that (1.16) holds. We recall that $\|\cdot\|$ denotes the norm of $L^2(\Omega)$.

We further introduce some function spaces. Set

$$(2.1)\qquad H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma_1\}.$$

We define the norm of $H^1_{\Gamma_1}(\Omega)$ by

$$\|u\|_{H^1_{\Gamma_1}(\Omega)} = \left(\frac{\mu}{2}\|\nabla u\|^2 + \frac{\lambda+\mu}{2}\|\mathrm{div} u\|^2 + \frac{1}{2}\int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\right)^{1/2}. \tag{2.2}$$

Under condition (1.16), this norm is equivalent to the usual one induced by $H^1(\Omega)$ (see [Liu]). Let the "history space" $L^2(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n)$ consist of $(H^1_{\Gamma_1}(\Omega))^n)$-valued functions w on $(0,\infty)$ for which

$$\|w\|^2_{L^2(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n)} = \int_0^\infty g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds < \infty. \tag{2.3}$$

Set

$$\mathcal{H} = (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \times L^2(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n), \tag{2.4}$$

with the energy norm

$$\begin{aligned}\|(u,v,\theta,w)\|_{\mathcal{H}} = \Big(&\kappa\|u\|^2_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{1}{2}[\|v\|^2 + \frac{\alpha}{\beta}\|\theta\|^2] \\ &+ \int_0^\infty g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds\Big)^{1/2}.\end{aligned} \tag{2.5}$$

where κ denotes the positive constant in (H_3), that is,

$$\kappa = 1 - \int_0^\infty g(t)dt > 0. \tag{2.6}$$

We further introduce three constants as follows. Set

$$R_0 = \max_{x\in\Omega}|m(x)| = \max_{x\in\Omega}|\sum_{k=1}^n (x_k - x_k^0)^2|^{1/2}. \tag{2.7}$$

where $m(x)$ is given by (1.12). Let γ be the smallest positive constant such that

$$\int_{\Gamma_2}|u|^2 d\Gamma \le \gamma^2\|u\|_{H^1_{\Gamma_1}(\Omega)}. \qquad \forall u \in H^1_{\Gamma_1}(\Omega). \tag{2.8}$$

Let λ_0 be the best constant in Poincaré's inequality [DL, p.125], namely, the smallest positive constant such that

$$\|u\| \le \lambda_0\|u\|_{H^1_{\Gamma_1}(\Omega)}. \qquad \forall u \in H^1_{\Gamma_1}(\Omega). \tag{2.9}$$

For each time t we may regard u and θ as elements of function spaces. Accordingly, we suppress their argument $x \in \mathbb{R}^n$ from the notation.

Consider the thermoviscoelastic system with a boundary velocity feedback

$$
(2.10)\qquad \begin{cases}
u'' - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}u & \\
\quad +\mu g * \Delta u + (\lambda+\mu) g * \nabla\mathrm{div}u + \alpha\nabla\theta = 0 & \text{in } \Omega\times(0,\infty),\\
\theta' - \Delta\theta + \beta\mathrm{div}\ u' = 0 & \text{in } \Omega\times(0,\infty),\\
\theta = 0 & \text{on } \Gamma\times(0,\infty),\\
u = 0 & \text{on } \Gamma_1\times(0,\infty),\\
\mu\dfrac{\partial}{\partial\nu}(u - g*u) + (\lambda+\mu)\nu\mathrm{div}(u - g*u) & \\
\quad +\alpha m\cdot\nu(u - g*u) + m\cdot\nu u' = 0 & \text{on } \Gamma_2\times(0,\infty),\\
u(0) = u^0,\ u'(0) = u^1,\ \theta(0) = \theta^0 & \text{in } \Omega,\\
u(0) - u(-s) = w^0(s) & \text{in } \Omega\times(0,\infty).
\end{cases}
$$

We will prove that problem (2.10) is well-posed in Section 3. In fact, we will prove that system (2.10) generates a strongly continuous semigroup $S(t)$ of contractions on $\mathcal{H}$. Further, in order to ensure that the solution u of (2.10) has sufficient regularity, in this paper, we suppose that

$$
(2.11)\qquad \overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset.
$$

Under this assumption, we have

$$
(2.12)\qquad u \in C([0,\infty),(H^2(\Omega))^n).
$$

This regularity property is needed for the proof of the following theorem.

The thermoviscoelastic energy $E(u,\theta,t)$ of (2.10) is defined by:

$$
\begin{aligned}
(2.13)\qquad E(u,\theta,t) &= \|(u(t),u'(t),\theta(t),u(t)-u(t-s))\|^2_{\mathcal{H}}\\
&= \kappa\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{1}{2}[\|u'(t)\|^2 + \frac{\alpha}{\beta}\|\theta(t)\|^2]\\
&\quad + \int_0^\infty g(s)\|u(t)-u(t-s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds\\
&= \kappa\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{1}{2}[\|u'(t)\|^2 + \frac{\alpha}{\beta}\|\theta(t)\|^2]\\
&\quad + \int_{-\infty}^t g(t-s)\|u(t)-u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds.
\end{aligned}
$$

By a straightforward calculation, we obtain

$$
\begin{aligned}
(2.14)\qquad E'(u,\theta,t) &= \int_{-\infty}^t g'(t-s)\|u(t)-u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds\\
&\quad - \int_{\Gamma_2} m\cdot\nu|u'(t)|^2 d\Gamma - \frac{\alpha}{\beta}\|\nabla\theta(t)\|^2.
\end{aligned}
$$

Therefore, the energy $E(u,\theta,t)$ decreases in $(0,\infty)$, and what is more, we have the following exponential decay rate. This is our main result of this paper.

THEOREM 2.1. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies (H_1), (H_2) and (H_3) and the following condition:*

(H_4) there exists a constant $K > 0$ such that for $t \in (0, \infty)$

$$-G(t) = \int_t^\infty g(s)ds \leq Kg(t). \tag{2.15}$$

If the function $a(x)$ satisfies

$$2\Big[\frac{2a^2R_0^2}{\mu} + (2-n)a\Big]R_0\gamma^2[1+(1-\kappa)^2] < \kappa, \qquad \text{for } n \leq 2. \tag{2.16}$$

$$a \leq \frac{(n-2)\mu}{2R_0^2}, \qquad \text{for } n \geq 3, \tag{2.17}$$

then there are positive constants M, ω, independent of $(u^0, u^1, \theta^0, w^0)$, such that

$$E(u,\theta,t) \leq ME(u,\theta,0)e^{-\omega t}, \qquad \forall\, t \geq 0, \tag{2.18}$$

for all solutions of (2.10) with $(u^0, u^1, \theta^0, w^0) \in \mathcal{H}$. Further, the positive constants M, ω can be explicitly given by

$$M = (\rho(T))^{-1}, \tag{2.19}$$

$$\omega = -T^{-1}\ln\rho(T), \tag{2.20}$$

$$\rho(T) = \frac{\exp[\delta T/(1+\delta C_1)](1-\delta C_1)\kappa + 2\delta C_5 K}{\kappa(1-\delta C_1)}, \tag{2.21}$$

$$T = 1 + \frac{1+\delta C_1}{\delta}\ln\frac{2(1+\delta C_1)}{(1-\delta C_1)}, \tag{2.22}$$

$$\delta = \min\Big\{\frac{\kappa}{4KC_5 + C_1\kappa}, \frac{1}{C_3}, \frac{1}{C_4}, \frac{2\alpha}{2\beta C_2 + \alpha\lambda_0^2}\Big\}, \tag{2.23}$$

$$C_1 = \max\{c_1, c_2, c_3\}, \tag{2.24}$$

$$c_1 = 2R_0(2-\kappa) + |n-2|(2-\kappa) + 1,$$

$$c_2 = \frac{1}{\kappa}\Big[\frac{2R_0(3-2\kappa)}{\mu} + |n-2|\lambda_0^2\Big(\frac{3}{2} - \kappa\Big) + \frac{\lambda_0^2}{2}\Big],$$

$$c_3 = \frac{4R_0}{\mu} + \lambda_0^2,$$

$$C_2 = \frac{\alpha^2[8R_0^2(2-\kappa)/\mu + \lambda_0^2(2-\kappa)|n-2| + \lambda_0^2]}{4\varepsilon}, \tag{2.25}$$

$$C_3 = \frac{2R_0^2}{\mu} + \frac{\gamma^2 R_0[(2-\kappa)|n-2|+1]}{4\varepsilon} + 1, \tag{2.26}$$

$$C_4 = \frac{g(0)[8R_0^2 + \mu|n-2|\lambda_0^2]}{4\varepsilon\mu}, \tag{2.27}$$

$$C_5 = \begin{cases} 1+\varepsilon+2|n-2|\varepsilon + 2K(A)(1-\kappa)R_0\gamma^2, & n \le 2, \\ 1+\varepsilon+2|n-2|\varepsilon, & n \ge 3, \end{cases} \tag{2.28}$$

$$K(A) = \frac{2A^2R_0^2}{\mu} - (n-2)A, \quad A = \max_{x\in\Gamma_2} a(x), \tag{2.29}$$

$$0 < \varepsilon = \begin{cases} \min\left\{\frac{1}{2(1+|n-2|)}, \frac{\kappa - 2K(A)R_0\gamma^2(1+(1-\kappa)^2)}{3+2|n-2|+(1+2|n-2|)(1-\kappa)}\right\}, & n \le 2, \\ \min\left\{\frac{1}{2(1+|n-2|)}, \frac{\kappa}{3+2|n-2|+(1+2|n-2|)(1-\kappa)}\right\}, & n \ge 3. \end{cases} \tag{2.30}$$

REMARK 2.2. – Note that condition (H_4) implies that the relaxation function $g(t)$ decays exponentially. If the decay rate of g is weaker, we do not know if Theorem 2.1 still holds.

REMARK 2.3. – Conditions (2.16) and (2.17) on the function $a(x)$ imply that a can not be very large. If a is large, whether or not Theorem 2.1 still holds is an open problem. However, for the Lamé system

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \Gamma_1\times(0,\infty), \\ \mu\dfrac{\partial u}{\partial\nu} + (\lambda+\mu)\nu \mathrm{div} u + am\cdot\nu u + m\cdot\nu u' = 0 & \text{on } \Gamma_2\times(0,\infty), \\ u(0) = u^0,\ u'(0) = u^1 & \text{in } \Omega, \end{cases}$$

this problem has been solved (see [Liu]). Thus, it is plausible to conjecture that Theorem 2.1 should hold for large a as the energy of the thermoviscoelastic system may dissipate faster than the Lamé system due to the additional dissipation of thermal energy and the viscous dissipation mechanism.

REMARK 2.4. – Whether or not Theorem 2.1 holds if (2.11) fails is also an open problem. For the wave equation, this is true in the case where $n \le 3$ (see [KZ]) because a key inequality has been established by Grisvard [Gri] for the solution of the wave equation with such boundary singularity. However, the similar inequality of Grisvard for the system of thermoviscoelasticity has not been proved yet in the literature.

In addition, one could try to replace $m(x)$ by a general vector field $l = (l_1, \cdots, l_n)$ as done for the wave equation in [Lag$_1$]. But in the thermoviscoelastic system, l must satisfy the following

$$|\mathrm{div}(u - g*u)|^2 - \mathrm{div}(u - g*u)\sum_{i,k=1}^{n}\frac{\partial l_k}{\partial x_i}\frac{\partial}{\partial x_k}(u_i - g*u_i) \le 0. \tag{2.32}$$

As this inequality holds for all $u \in (H^1_{\Gamma_1}(\Omega))^n$, it turns out that l has to be nearly equal to m.

3. Well-posedness

The treatment of well-posedness of (2.10) is standard. For the sake of completeness, we give a brief discussion by using the theory of semigroups. For detailed discussion about the existence and regularity of a solution of viscoelastic system, we refer to [JR, RB, RL].

In order to use the theory of semigroups, we write (2.10) as follows:

$$
(3.1)\quad \begin{cases}
u'' - \kappa\mu\Delta u - \kappa(\lambda+\mu)\nabla\mathrm{div}u + \alpha\nabla\theta & \\
\quad -\mu\int_0^\infty g(s)\Delta w(t,s)ds & \\
\quad -(\lambda+\mu)\int_0^\infty g(s)\nabla\mathrm{div}w(t,s)ds = 0 & \text{in } \Omega\times(0,\infty), \\
\theta' - \Delta\theta + \beta\mathrm{div}\ u' = 0 & \text{in } \Omega\times(0,\infty), \\
w(t,s) = u(t) - u(t-s) & \text{in } \Omega\times(0,\infty)\times(0,\infty), \\
\theta = 0 & \text{on } \Gamma\times(0,\infty), \\
u = 0 & \text{on } \Gamma_1\times(0,\infty), \\
\kappa\mu\dfrac{\partial u}{\partial\nu} + \kappa(\lambda+\mu)\nu\mathrm{div}u + \kappa am\cdot\nu u & \\
\quad +\mu\int_0^\infty g(s)\frac{\partial w(t,s)}{\partial\nu}ds & \\
\quad +(\lambda+\mu)\nu\int_0^\infty g(s)\mathrm{div}w(t,s)ds & \\
\quad +am\cdot\nu\int_0^\infty g(s)w(t,s)ds + m\cdot\nu u' = 0 & \text{on } \Gamma_2\times(0,\infty), \\
u(0) = u^0,\ u'(0) = u^1,\ \theta(0) = \theta^0 & \text{in } \Omega, \\
w(0,s) = w^0(s) & \text{in } \Omega\times(0,\infty).
\end{cases}
$$

Note that κ is given by (2.6).

We define a linear unbounded operator A on $\mathcal{H}$ by

$$A(u,v,\theta,w) = (v,\ B(u,w) - \alpha\nabla\theta,\ \Delta\theta - \beta\mathrm{div}v,\ v - w_s).$$

where $w_s = \dfrac{\partial w}{\partial s}$ and

$$
\begin{aligned}
B(u,w) = \kappa\mu\Delta u + \kappa(\lambda+\mu)\nabla\mathrm{div}u + \mu\int_0^\infty g(s)\Delta w(s)ds \\
+ (\lambda+\mu)\int_0^\infty g(s)\nabla\mathrm{div}w(s)ds.
\end{aligned}
$$

Set

$$v(x,t) = u'(x,t),\ w(x,t,s) = u(x,t) - u(x,t-s),$$

$$\Phi = (u,v,\theta,w).$$

Then problem (3.1) can be formulated as an abstract Cauchy problem

$$\Phi' = A\Phi.$$

on the Hilbert space $\mathcal{H}$ for an initial condition $\Phi(0) = (u^0, u^1, \theta^0, w^0)$. The domain of A is given by:

$$\begin{aligned} D(A) = \{(u, v, \theta, w) \in \mathcal{H} \ : \ & \theta \in H^2(\Omega) \cap H_0^1(\Omega), \ v \in (H^1_{\Gamma_1}(\Omega))^n, \\ & \kappa u + \int_0^\infty g(s) w(s) ds \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n, \\ & w(s) \in H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n), \ w(0) = 0, \\ & \kappa\mu \frac{\partial u}{\partial \nu} + \kappa(\lambda + \mu)\nu \mathrm{div} u + \kappa a m \cdot \nu u \\ & + \mu \int_0^\infty g(s) \frac{\partial w(s)}{\partial \nu} ds + (\lambda + \mu)\nu \int_0^\infty g(s) \mathrm{div} w(s) ds \\ & + a m \cdot \nu \int_0^\infty g(s) w(s) ds + m \cdot \nu v = 0 \ \text{ on } \Gamma_2\}. \end{aligned}$$

where

$$H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n) = \{w \ : \ w, \ w_s \in L^2(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n)\}.$$

It is clear that $D(A)$ is dense in $\mathcal{H}$.

To prove that A is dissipative, we need the following lemma.

LEMMA 3.1. [KKK, p.491] – *If the function $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous and is in $L^1(0, \infty)$, then*

$$\lim_{t \to \infty} f(t) = 0.$$

LEMMA 3.2. – *Suppose that the relaxation function g satisfies (H_1) and (H_2). If $w \in H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n)$ and $w(0) = 0$, then*

$$g'(s) \|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} \in L^1(0, \infty)$$

and

$$\lim_{s \to \infty} g(s) \|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} = 0.$$

Proof. – For $w \in H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n)$ and $w(0) = 0$, we have

$$\lim_{s \to 0} g(s) \|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} = \lim_{s \to 0} g(0) \|w(0)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} = 0.$$

It therefore follows that

$$\begin{aligned} & 2 \int_0^t g(s) (w_s(s), w(s))_{(H^1_{\Gamma_1}(\Omega))^n} ds \\ & = \int_0^t g(s) \frac{\partial}{\partial s} [\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}] ds \\ & = g(t) \|w(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} - \int_0^t g'(s) \|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds, \end{aligned}$$

which implies that for all $t \geq 0$

$$\begin{aligned}
&\int_0^t |g'(s)| \|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\leq 2\Big(\int_0^\infty g(s)\|w_s(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds\Big)^{1/2} \Big(\int_0^\infty g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds\Big)^{1/2}.
\end{aligned}$$

Thus

$$g'(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} \in L^1(0,\infty). \tag{3.2}$$

On the other hand, for any $0 \leq s_1 < s_2 < \infty$, we have

$$\begin{aligned}
&g(s_2)\|w(s_2)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} - g(s_1)\|w(s_1)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} \\
&= \int_{s_1}^{s_2} \frac{d}{ds}[g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}]ds \\
&= \int_{s_1}^{s_2} g'(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds + 2\int_{s_1}^{s_2} g(s)(w_s(s), w(s))_{(H^1_{\Gamma_1}(\Omega))^n} ds,
\end{aligned}$$

which, combining (3.2), implies that $g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}$ is uniformly continuous on $[0,\infty)$. Hence Lemma 3.1 gives

$$\lim_{s\to\infty} g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} = 0. \qquad \square$$

We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^n$.

LEMMA 3.3. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies* (H_1), (H_2) *and* (H_3). *Then the operator A is dissipative and closed.*

Proof. – By a straightforward calculation, it follows from Lemma 3.2 that

$$\begin{aligned}
&\langle A(u,v,\theta,w),(u,v,\theta,w)\rangle_{\mathcal{H}} \\
&= \kappa(v,u)_{(H^1_{\Gamma_1}(\Omega))^n} + \frac{1}{2}(B(u,w) - \alpha\nabla\theta, v) \\
&\quad + \frac{\alpha}{2\beta}(\Delta\theta - \beta \operatorname{div} v, \theta) + (v - w_s, w)_{L^2(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n)} \\
&= -\frac{1}{2}\int_{\Gamma_2} m\cdot\nu|v|^2 d\Gamma - \frac{\alpha}{2\beta}\|\nabla\theta\|^2 - g(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}\Big|_0^\infty \\
&\quad + \int_0^\infty g'(s)\|w(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\leq 0.
\end{aligned}$$

Thus, A is dissipative.

To prove that A is closed, let $(u_n, v_n, \theta_n, w_n) \in D(A)$ be such that

$$(u_n, v_n, \theta_n, w_n) \to (u, v, \theta, w) \quad \text{in } \mathcal{H}$$

and

$$A(u_n, v_n, \theta_n, w_n) \to (\varphi, \psi, \xi, z) \quad \text{in } \mathcal{H}.$$

Then we have

$$u_n \to u \quad \text{in } (H^1_{\Gamma_1}(\Omega))^n, \tag{3.3}$$

$$v_n \to v \quad \text{in } (L^2(\Omega))^n, \tag{3.4}$$

$$\theta_n \to \theta \quad \text{in } L^2(\Omega), \tag{3.5}$$

$$w_n \to w \quad \text{in } L^2(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n), \tag{3.6}$$

and

$$v_n \to \varphi \quad \text{in } (H^1_{\Gamma_1}(\Omega))^n, \tag{3.7}$$

$$B(u_n, w_n) - \alpha \nabla \theta_n \to \psi \quad \text{in } (L^2(\Omega))^n, \tag{3.8}$$

$$\Delta \theta_n - \beta \operatorname{div} v_n \to \xi \quad \text{in } L^2(\Omega), \tag{3.9}$$

$$v_n - w_{ns} \to z \quad \text{in } L^2(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n). \tag{3.10}$$

By (3.4) and (3.7), we deduce

$$v_n \to v \quad \text{in } (H^1_{\Gamma_1}(\Omega))^n \tag{3.11}$$

and

$$v = \varphi \in (H^1_{\Gamma_1}(\Omega))^n. \tag{3.12}$$

By (3.9) and (3.11), we deduce

$$\Delta \theta_n \to \beta \operatorname{div} v + \xi \quad \text{in } L^2(\Omega), \tag{3.13}$$

and consequently, it follows from (3.5) that

$$\theta_n \to \theta \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega), \tag{3.14}$$

since Δ is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$. It therefore follows from (3.9) and (3.14) that

$$\xi = \Delta\theta - \beta \mathrm{div} v, \quad \theta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3.15}$$

By (3.6), (3.10) and (3.11), we deduce

$$w_n \to w \quad \text{in } H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n) \tag{3.16}$$

and

$$z = v - w_s, \quad w \in H^1(g, (0, \infty), (H^1_{\Gamma_1}(\Omega))^n), \quad w(0) = 0. \tag{3.17}$$

In addition, it follows from (3.3), (3.6) and (3.14) that

$$B(u_n, w_n) - \alpha\nabla\theta_n \to B(u, w) - \alpha\nabla\theta \tag{3.18}$$

in the sense of distribution. It therefore follows from (3.8) and (3.18) that

$$\psi = B(u, w) - \alpha\nabla\theta, \quad B(u, w) \in (L^2(\Omega))^n. \tag{3.19}$$

and consequently,

$$\kappa u + \int_0^\infty g(s) w(s) ds \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n. \tag{3.20}$$

since $\mu\Delta + (\lambda + \mu)\nabla \mathrm{div}$ is an isomorphism from $H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ onto $L^2(\Omega)$. Moreover, by (3.11), (3.20) and the trace theorem (see [LM], Chap. 1), we deduce that:

$$\begin{aligned} &\kappa\mu \frac{\partial u}{\partial \nu} + \kappa(\lambda + \mu)\nu \mathrm{div} u + \kappa a m \cdot \nu u \\ &+ \mu \int_0^\infty g(s) \frac{\partial w(s)}{\partial \nu} ds + (\lambda + \mu)\nu \int_0^\infty g(s) \mathrm{div} w(s) ds \\ &+ a m \cdot \nu \int_0^\infty g(s) w(s) ds + m \cdot \nu v = 0 \quad \text{on } \Gamma_2. \end{aligned}$$

Thus, by (3.12), (3.15), (3.17), (3.19) and (3.20), we deduce

$$A(u, v, \theta, w) = (\varphi, \psi, \xi, z), \quad (u, v, \theta, w) \in D(A).$$

Hence, A is closed. □

LEMMA 3.4. – *The adjoint operator A^* of A is also dissipative.*

Proof. – By a straightforward calculation, we can obtain

$$A^*(\phi, \psi, \xi, z) = (-\psi, B_1(\phi, z) + \alpha\nabla\xi, \Delta\xi + \beta \mathrm{div}\psi, -\psi + z_s + \frac{g'(s)}{g(s)} z),$$

where

$$B_1(\phi, z) = -\kappa\mu\Delta\phi - \kappa(\lambda+\mu)\nabla\mathrm{div}\phi - \mu\int_0^\infty g(s)\Delta z(s)ds$$
$$-(\lambda+\mu)\int_0^\infty g(s)\nabla\mathrm{div}z(s)ds.$$

The domain of A^* is given by

$$D(A^*) = \{(\phi,\psi,\xi,z)\in\mathcal{H} \ : \ \xi\in H^2(\Omega)\cap H_0^1(\Omega),\ \psi\in(H^1_{\Gamma_1}(\Omega))^n,$$
$$\kappa\phi + \int_0^\infty g(s)z(s)ds \in (H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega))^n,$$
$$z(s)\in H^1(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n),\ \frac{g'(s)}{g(s)}z(s)\in L^2(g,(0,\infty),(H^1_{\Gamma_1}(\Omega))^n),$$
$$\kappa\mu\frac{\partial\phi}{\partial\nu} + \kappa(\lambda+\mu)\nu\mathrm{div}\phi + \kappa am\cdot\nu\phi$$
$$+\mu\int_0^\infty g(s)\frac{\partial z(s)}{\partial\nu}ds + (\lambda+\mu)\nu\int_0^\infty g(s)\mathrm{div}z(s)ds$$
$$am\cdot\nu\int_0^\infty g(s)z(s)ds - m\cdot\nu\psi = 0 \ \text{ on } \Gamma_2\}.$$

By a straightforward calculation, it follows from Lemma 3.2 that

$$\begin{aligned}
&\langle A^*(\phi,\psi,\xi,z),(\phi,\psi,\xi,z)\rangle_{\mathcal{H}}\\
&= -\frac{1}{2}\int_{\Gamma_2} m\cdot\nu|\psi|^2 d\Gamma - \frac{\alpha}{2\beta}\|\nabla\xi\|^2 + \frac{1}{2}g(s)\|z(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}\Big|_0^\infty\\
&\quad + \frac{1}{2}\int_0^t g'(s)\|z(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds\\
&\le 0.
\end{aligned}$$

Thus, A^* is dissipative. □

From Lemmas 3.3. and 3.4 and Corollary 4.4 in [Paz, p.15], we conclude that A generates a strongly continuous semigroup of contractions on $\mathcal{H}$. Now an application of the theory of semigroups [Paz, Chap.4] gives:

THEOREM 3.5. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies* (H_1), (H_2) *and* (H_3). *Then*

(i) *for every initial condition $(u^0,u^1,\theta^0,w^0)\in\mathcal{H}$, problem (3.1) has a unique mild solution satisfying*

$$(u,v,\theta,w)\in C([0,\infty);\mathcal{H}).$$

Moreover, we have for every $t\in[0,\infty)$

$$\|(u(t),u'(t),\theta(t),w(t))\|_{\mathcal{H}} \le \|(u^0,u^1,\theta^0,w^0)\|_{\mathcal{H}}. \tag{3.21}$$

(i) *For every initial condition $(u^0,u^1,\theta^0,w^0)\in D(A)$, problem (3.1) has a unique classical solution satisfying*

$$(u,v,\theta,w)\in C([0,\infty);D(A)).$$

4. Proof of Main Result

The idea of the proof of Theorem 2.1 is simple. It suffices to show that there exist positive constants $T > 0$ and $0 < \rho < 1$ such that

$$E(u,\theta,t) \leq \rho E(u,\theta,0), \qquad \forall\, t \geq T. \tag{4.1}$$

However, the verification of (4.1) is generally not easy. Here we borrow the idea of the Lyapunov method to prove it. It is well known that the Lyapunov method is quite useful tool to treat the problem of stability for nonlinear dynamical systems and evolution equations (see [KKK, Wal]). The key part of this method is the construction of a useful Lyapunov functional. Finding a Lyapunov function is easy, but finding a useful one not. The useful property of any Lyapunov functional is that its value can be shown to be nonincreasing along trajectories of solutions of a system and this property leads to many interesting conclusions. Therefore, we need to carefully construct a Lyapunov functional. In our situation, the Lyapunov functional we are going to construct actually is a generalized energy functional which is closely related to the energy functional $E(u,\theta,t)$. Such similar Lyapunov functional was constructed for the wave equation (see [Che, Lag_1, Zau]), thermoelastic plate models (see [MZ_1, MZ_2]) and others. As we will see, our Lyapunov functional does not exactly possess the properties that the Lyapunov functional defined for the nonlinear dynamical systems (see [KKK, Wal]) has. Therefore, we may call our Lyapunov functional a generalized Lyapunov functional.

In what follows, we assume the summation convention for repeated indices.

Let u, θ be the solution of (2.10) and δ any positive number. We construct a Lyapunov functional V by

$$V(u,\theta,t) = E(u,\theta,t) + \delta F(u,\theta,t), \tag{4.2}$$

where

$$F(u,\theta,t) = \int_\Omega [2u_i' m\cdot\nabla(u_i - g*u_i) + (n-2)u_i'(u_i - g*u_i) + u_i u_i']dx, \tag{4.3}$$

As we have assumed the summation convention, equality (4.3) means that

$$F(u,\theta,t) = \sum_{i=1}^n \int_\Omega [2u_i' m\cdot\nabla(u_i - g*u_i) + (n-2)u_i'(u_i - g*u_i) + u_i u_i']dx.$$

Let us recall some notation. By $\|\cdot\|$ we denote the norm of $L^2(\Omega)$ or $(L^2(\Omega))^n$. The constants κ, R_0, γ and λ_0 are given by (2.6), (2.7), (2.8) and (2.9), respectively.

In order to show that V is a generalized Lyapunov functional, we need to estimate $V'(u,\theta,t)$. We begin with the following lemma:

LEMMA 4.1. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies* (H_1), (H_2) *and* (H_3). *Then we have*

$$(1-\delta C_1)E(u,\theta,t) \leq V(u,\theta,t) \leq (1+\delta C_1)E(u,\theta,t), \tag{4.4}$$

for all solutions u, θ *of* (2.10), *where the positive constant* C_1, *independent of* u, θ, *is given by* (2.24).

Proof. – It is easy to see that

$$
\begin{aligned}
(4.5)\qquad & |2\int_\Omega u_i'(m\cdot\nabla(u_i - g*u_i))dx| \\
&= |2\int_\Omega u_i' m\cdot\nabla u_i dx - 2\int_{-\infty}^t g(t-s)ds\int_\Omega u_i'(t)m\cdot\nabla u_i(s)dx| \\
&\le R_0[\|u_i'(t)\|^2 + \|\nabla u_i(t)\|^2] \\
&\quad + R_0\int_{-\infty}^t g(t-s)[\|u_i'(t)\|^2 + \|\nabla u_i(s)\|^2]ds \\
&= R_0(2-\kappa)\|u_i'(t)\|^2 + R_0\|\nabla u_i(t)\|^2 \\
&\quad + R_0\int_{-\infty}^t g(t-s)\|\nabla u_i(t) - \nabla u_i(s) - \nabla u_i(t)\|^2 ds \\
&\le R_0(2-\kappa)\|u_i'(t)\|^2 + R_0(3-2\kappa)\|\nabla u_i(t)\|^2 \\
&\quad + 2R_0\int_{-\infty}^t g(t-s)\|\nabla u_i(t) - \nabla u_i(s)\|^2 ds.
\end{aligned}
$$

Using (2.9), we obtain

$$
\begin{aligned}
(4.6)\qquad & |\int_\Omega u_i'(u_i - g*u_i)dx| \\
&= |\int_\Omega u_i' u_i dx - \int_{-\infty}^t g(t-s)ds\int_\Omega u_i'(t)u_i(s)dx| \\
&\le \frac{1}{2}\|u_i'(t)\|^2 + \frac{\lambda_0^2}{2}\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + \frac{1}{2}\int_{-\infty}^t g(t-s)[\|u_i'(t)\|^2 + \lambda_0^2\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}]ds \\
&\le \frac{2-\kappa}{2}\|u_i'(t)\|^2 + \lambda_0^2(\frac{3}{2}-\kappa)\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + \lambda_0^2\int_{-\infty}^t g(t-s)\|u_i(t) - u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds,
\end{aligned}
$$

and

$$
(4.7)\qquad |\int_\Omega u_i' u_i dx| \le \frac{1}{2}\|u_i'(t)\|^2 + \frac{\lambda_0^2}{2}\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)}.
$$

Noting (2.24), we deduce from (4.5), (4.6) and (4.7) that

$$
|F(u,\theta,t)| \le C_1 E(u,\theta,t),
$$

which implies (4.4). □

LEMMA 4.2. – *Let* $g \in L^1(0,\infty)$. *Set*

$$\begin{aligned} &H(u,\theta,t) \\ &= \int_\Omega [\frac{1}{2}|u_i'|^2 + \mu\nabla u_i \nabla(u_i - g * u_i) + (\lambda+\mu)\mathrm{div}u\mathrm{div}(u - g * u)]dx \\ &\quad + \int_{\Gamma_2} am \cdot \nu u_i(u_i - g * u_i)d\Gamma, \end{aligned}$$

and

$$G(t) = -\int_t^\infty g(s)ds. \tag{4.8}$$

Then we have

$$\begin{aligned} H(u,\theta,t) = E(u,\theta,t) &- \frac{\alpha}{2\beta}\|\theta(t)\|^2 + \kappa\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} \\ &- \frac{d}{dt}\int_{-\infty}^t G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\int_{-\infty}^t g(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds \\ &= \frac{d}{dt}\int_{-\infty}^t G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}ds + (1-\kappa)\|u(t)\|^2_{(H^1_{\Gamma_1}(\Omega))^n}. \end{aligned} \tag{4.10}$$

Proof. – By straightforward calculation, we obtain

$$\begin{aligned} &\int_{-\infty}^t g(t-s)\|\nabla u_i(t) - \nabla u_i(s)\|^2 ds \\ &= (1-\kappa)\|\nabla u_i(t)\|^2 + \int_{-\infty}^t g(t-s)\|\nabla u_i(s)\|^2 ds \\ &\quad - 2\int_{-\infty}^t\int_\Omega g(t-s)\nabla u_i(t)\nabla u_i(s)dxds \\ &= -(1+\kappa)\|\nabla u_i(t)\|^2 + \int_{-\infty}^t g(t-s)\|\nabla u_i(s)\|^2 ds \\ &\quad + 2\int_\Omega \nabla u_i(t)\nabla(u_i(t) - g * u_i(t))dx, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\int_{-\infty}^t g(t-s)\|\nabla u_i(s)\|^2 ds \\ &= \frac{d}{dt}\int_{-\infty}^t G(t-s)\|\nabla u_i(s)\|^2 ds + (1-\kappa)\|\nabla u_i(t)\|^2. \end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}
&\int_{-\infty}^{t} g(t-s)\|\mathrm{div}u(t)-\mathrm{div}u(s)\|^2 ds \\
&= -(1+\kappa)\|\mathrm{div}u(t)\|^2 + \int_{-\infty}^{t} g(t-s)\|\mathrm{div}u(s)\|^2 ds \\
&\quad + 2\int_{\Omega} \mathrm{div}u(t)\mathrm{div}(u(t)-g*u(t))dx,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\int_{-\infty}^{t} g(t-s)\|\mathrm{div}u(s)\|^2 ds \\
&= \frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|\mathrm{div}u(s)\|^2 ds + (1-\kappa)\|\mathrm{div}u(t)\|^2,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_{-\infty}^{t} g(t-s)\int_{\Gamma_2} am\cdot\nu|u_i(t)-u_i(s)|^2 d\Gamma ds \\
&= -(1+\kappa)\int_{\Gamma_2} am\cdot\nu|u_i(t)|^2 d\Gamma + \int_{-\infty}^{t} g(t-s)\int_{\Gamma_2} am\cdot\nu|u_i(s)|^2 d\Gamma ds \\
&\quad + 2\int_{\Gamma_2} am\cdot\nu u_i(t)(u_i(t)-g*u_i(t))d\Gamma,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\int_{-\infty}^{t} g(t-s)\int_{\Gamma_2} am\cdot\nu|u_i(s)|^2 d\Gamma ds \\
&= \frac{d}{dt}\int_{-\infty}^{t} G(t-s)\int_{\Gamma_2} am\cdot\nu|u_i(s)|^2 d\Gamma ds + (1-\kappa)\int_{\Gamma_2} am\cdot\nu|u_i(t)|^2 d\Gamma.
\end{aligned} \tag{4.16}$$

Hence, (4.9) and (4.10) follow from (4.11)-(4.16). □

LEMMA 4.3. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies* (H_1), (H_2) *and* (H_3). *Let u, θ be the solution of* (2.10). *If the function $a(x)$ satisfies* (2.16) *and* (2.17), *then we have*

$$\begin{aligned}
&F'(u,\theta,t) \\
&\le -E(u,\theta,t) + \Big[C_2 + \frac{\alpha\lambda_0^2}{2\beta}\Big]\|\nabla\theta(t)\|^2 + C_3\int_{\Gamma_2} m\cdot\nu|u'|^2 d\Gamma \\
&\quad + C_4\int_{-\infty}^{t} |g'(t-s)|\|u(t)-u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + C_5\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds,
\end{aligned} \tag{4.17}$$

where the positive constants C_2, C_3, C_4, C_5 are given by (2.25)-(2.28), *respectively.*

Proof. – By (4.3), we have

$$
\begin{aligned}
(4.18)\qquad & F'(u,\theta,t)\\
&= \int_\Omega 2u_i'' m\cdot\nabla(u_i - g*u_i)dx + \int_\Omega 2u_i' m\cdot\nabla(u_i-g*u_i)'dx\\
&\quad + \int_\Omega (n-2)(u_i-g*u_i)u_i''dx + \int_\Omega (n-2)u_i'(u_i-g*u_i)'dx\\
&\quad + \int_\Omega u_i u_i'' dx + \|u_i'\|^2.
\end{aligned}
$$

We now estimate every integral in (4.18) as follows. Since $u - g*u = 0$ on Γ_1, we have

$$
(4.19)\qquad \frac{\partial}{\partial x_k}(u_i - g*u_i) = \frac{\partial}{\partial\nu}(u_i - g*u_i)\nu_k \qquad \text{on} \quad \Gamma_1.
$$

Thus, we obtain

$$
\begin{aligned}
(4.20)\quad & 2\int_\Omega u_i'' m\cdot\nabla(u_i-g*u_i)dx\\
&= 2\int_\Omega \Big[\mu\Delta(u_i-g*u_i) + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}(u-g*u)) - \alpha\frac{\partial\theta}{\partial x_i}\Big]\\
&\quad \times m\cdot\nabla(u_i-g*u_i)dx\\
&= \mu\int_\Gamma \Big[2\frac{\partial}{\partial\nu}(u_i-g*u_i)m\cdot\nabla(u_i-g*u_i) - m\cdot\nu|\nabla(u_i-g*u_i)|^2\Big]d\Gamma\\
&\quad + (n-2)\mu\|\nabla(u_i-g*u_i)\|^2\\
&\quad + (\lambda+\mu)\int_\Gamma 2\mathrm{div}(u-g*u)\ m_k\nu_i\frac{\partial}{\partial x_k}(u_i-g*u_i)d\Gamma\\
&\quad - (\lambda+\mu)\int_\Gamma m\cdot\nu|\mathrm{div}(u-g*u)|^2 d\Gamma\\
&\quad + (n-2)(\lambda+\mu)\|\mathrm{div}(u-g*u)\|^2 - 2\alpha\int_\Omega (m\cdot\nabla(u_i-g*u_i))\frac{\partial\theta}{\partial x_i}dx\\
&= \int_{\Gamma_1} m\cdot\nu\Big[\mu|\frac{\partial}{\partial\nu}(u_i-g*u_i)|^2 + (\lambda+\mu)|\mathrm{div}(u-g*u)|^2\Big]d\Gamma \quad (=I_1)\\
&\quad + 2\int_{\Gamma_2}\Big[\mu\frac{\partial}{\partial\nu}(u_i-g*u_i) + (\lambda+\mu)\nu_i\mathrm{div}(u-g*u)\Big]m\cdot\nabla(u_i-g*u_i)d\Gamma\\
&\quad - \int_{\Gamma_2} m\cdot\nu[\mu|\nabla(u_i-g*u_i)|^2 + (\lambda+\mu)|\mathrm{div}(u-g*u)|^2]d\Gamma \quad (=I_3)\\
&\quad + (n-2)[\mu\|\nabla(u_i-g*u_i)\|^2 + (\lambda+\mu)\|\mathrm{div}(u-g*u)\|^2] \quad (=I_4)\\
&\quad - 2\alpha\int_\Omega (m\cdot\nabla(u_i-g*u_i))\frac{\partial\theta}{\partial x_i}dx \quad (=I_5)\\
&= I_1 + I_3 + I_4 + I_5\\
&\quad - 2\int_{\Gamma_2}[am\cdot\nu(u_i-g*u_i) + m\cdot\nu u_i']m\cdot\nabla(u_i-g*u_i)d\Gamma \quad (=I_2).
\end{aligned}
$$

Since $m \cdot \nu \leq 0$ on Γ_1, we have

$$I_1 \leq 0. \tag{4.21}$$

Since

$$I_2 \leq \int_{\Gamma_2} m \cdot \nu \Big[\frac{2a^2R_0^2}{\mu}|u_i - g * u_i|^2 + \frac{2R_0^2}{\mu}|u_i'|^2 + \mu|\nabla(u_i - g * u_i)|^2\Big] d\Gamma, \tag{4.22}$$

we have

$$I_2 + I_3 \leq \int_{\Gamma_2} m \cdot \nu \Big[\frac{2a^2R_0^2}{\mu}|u_i - g * u_i|^2 + \frac{2R_0^2}{\mu}|u_i'|^2\Big] d\Gamma. \tag{4.23}$$

In addition, we have

$$\begin{aligned} I_5 \leq{}& \frac{\varepsilon\mu}{2}\|\nabla u_i(t)\|^2 + \frac{2\alpha^2R_0^2}{\mu\varepsilon}\|\nabla\theta(t)\|^2 \\ &+ \int_{-\infty}^t g(t-s)[\frac{\mu\varepsilon}{2}\|\nabla u_i(s)\|^2 + \frac{2\alpha^2R_0^2}{\mu\varepsilon}\|\nabla\theta(t)\|^2]ds \\ \leq{}& \varepsilon\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{2\alpha^2R_0^2(2-\kappa)}{\mu\varepsilon}\|\nabla\theta(t)\|^2 \\ &+ \varepsilon\int_{-\infty}^t g(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds. \end{aligned} \tag{4.24}$$

It therefore follows from (4.20)-(4.24) that

$$\begin{aligned} &2\int_\Omega u_i'' m \cdot \nabla(u_i - g * u_i)dx \\ &\leq \int_{\Gamma_2} m \cdot \nu \Big[\frac{2a^2R_0^2}{\mu}|u_i - g * u_i|^2 + \frac{2R_0^2}{\mu}|u_i'|^2\Big] d\Gamma \\ &\quad + \varepsilon\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{2\alpha^2R_0^2(2-\kappa)}{\mu\varepsilon}\|\nabla\theta(t)\|^2 \\ &\quad + \varepsilon\int_{-\infty}^t g(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds \\ &\quad + (n-2)[\mu\|\nabla(u_i(t) - g * u_i(t))\|^2 + (\lambda+\mu)\|\mathrm{div}(u(t) - g * u(t))\|^2]. \end{aligned} \tag{4.25}$$

Since by (1.2)

$$-g(0) = \int_0^\infty g'(s)ds = \int_{-\infty}^t g'(t-s)ds, \tag{4.26}$$

we have

$$
\begin{aligned}
(4.27)\quad & 2\int_\Omega u_i' m\cdot\nabla(u_i - g*u_i)' dx \\
&= 2\int_\Omega u_i' m_k\Big[\frac{\partial u_i'}{\partial x_k} - g(0)\frac{\partial u_i}{\partial x_k} - \int_{-\infty}^t g'(t-s)\frac{\partial u_i(s)}{\partial x_k}ds\Big]dx \\
&= 2\int_\Omega u_i' m_k\Big[\int_{-\infty}^t g'(t-s)\Big[\frac{\partial u_i(t)}{\partial x_k} - \frac{\partial u_i(s)}{\partial x_k}\Big]ds\Big]dx \\
&\quad - n\|u_i'(t)\|^2 + \int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma \\
&\le -n\|u_i'(t)\|^2 + \int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma \\
&\quad + \int_{-\infty}^t |g'(t-s)|\Big[\frac{\varepsilon}{g(0)}\|u_i'(t)\|^2 + \frac{g(0)R_0^2}{\varepsilon}\|\nabla u_i(t) - \nabla u_i(s)\|^2\Big]ds \\
&= (\varepsilon - n)\|u_i'(t)\|^2 + \int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma \\
&\quad + \frac{2g(0)R_0^2}{\mu\varepsilon}\int_{-\infty}^t |g'(t-s)|\|u_i(t) - u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds.
\end{aligned}
$$

Using (2.8) and (2.9), we deduce

$$
\begin{aligned}
(4.28)\quad & (n-2)\int_\Omega (u_i - g*u_i)u_i'' dx \\
&= (n-2)\int_\Omega (u_i - g*u_i)\Big[\mu\Delta(u_i - g*u_i) \\
&\quad + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}(u - g*u)) - \alpha\frac{\partial\theta}{\partial x_i}\Big]dx \\
&= (n-2)\int_{\Gamma_2}\Big[\mu\frac{\partial}{\partial\nu}(u_i - g*u_i) + (\lambda+\mu)\nu_i\mathrm{div}(u - g*u)\Big](u_i - g*u_i)d\Gamma \\
&\quad - (n-2)[\mu\|\nabla(u_i(t) - g*u_i(t))\|^2 + (\lambda+\mu)\|\mathrm{div}(u(t) - g*u(t))\|^2] \\
&\quad - (n-2)\alpha\int_\Omega (u_i - g*u_i)\frac{\partial\theta}{\partial x_i}dx \\
&= -(n-2)\int_{\Gamma_2}[am\cdot\nu(u_i - g*u_i) + m\cdot\nu u_i'](u_i - g*u_i)d\Gamma \\
&\quad - (n-2)[\mu\|\nabla(u_i(t) - g*u_i(t))\|^2 + (\lambda+\mu)\|\mathrm{div}(u(t) - g*u(t))\|^2] \\
&\quad - (n-2)\alpha\int_\Omega \frac{\partial\theta}{\partial x_i}\Big[u_i - \int_{-\infty}^t g(t-s)u_i(s)ds\Big]dx \\
&\le -(n-2)\int_{\Gamma_2} am\cdot\nu|u_i - g*u_i|^2 d\Gamma \\
&\quad + |n-2|\int_{\Gamma_2} m\cdot\nu\Big[\frac{R_0\gamma^2}{4\varepsilon}|u_i'|^2 + \frac{\varepsilon}{R_0\gamma^2}|u_i|^2\Big]d\Gamma \\
&\quad + |n-2|\int_{-\infty}^t g(t-s)ds\int_{\Gamma_2} m\cdot\nu\Big[\frac{R_0\gamma^2}{4\varepsilon}|u_i'(t)|^2 + \frac{\varepsilon}{R_0\gamma^2}|u_i(s)|^2\Big]d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&- (n-2)[\mu\|\nabla(u_i(t) - g * u_i(t))\|^2 + (\lambda+\mu)\|\mathrm{div}(u(t) - g * u(t))\|^2] \\
&+ |n-2|\frac{\alpha^2\lambda_0^2}{4\varepsilon}\|\nabla\theta(t)\|^2 + \varepsilon|n-2|\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&+ |n-2|\int_{-\infty}^t g(t-s)\Big[\frac{\alpha^2\lambda_0^2}{4\varepsilon}\|\nabla\theta(t)\|^2 + \frac{\varepsilon}{\lambda_0^2}\|u_i(s)\|^2\Big]ds \\
\le\ & -(n-2)\int_{\Gamma_2} am\cdot\nu|u_i - g * u_i|^2 d\Gamma \\
&+ \frac{|n-2|\gamma^2 R_0(2-\kappa)}{4\varepsilon}\int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma + 2\varepsilon|n-2|\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&+ \frac{|n-2|\alpha^2\lambda_0^2(2-\kappa)}{4\varepsilon}\|\nabla\theta(t)\|^2 \\
&+ 2\varepsilon|n-2|\int_{-\infty}^t g(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds \\
&- (n-2)[\mu\|\nabla(u_i(t) - g * u_i(t))\|^2 + (\lambda+\mu)\|\mathrm{div}(u(t) - g * u(t))\|^2].
\end{aligned}
$$

Using (2.9) and (4.26), we obtain

$$
\begin{aligned}
(4.29)\qquad & (n-2)\int_\Omega u_i'(u_i - g * u_i)' dx \\
&= (n-2)\int_\Omega u_i'[u_i' - g(0)u_i - \int_{-\infty}^t g'(t-s)u_i(s)ds]dx \\
&= (n-2)\|u_i'(t)\|^2 + (n-2)\int_\Omega u_i'\int_{-\infty}^t g'(t-s)[u_i(t) - u_i(s)]dsdx \\
&\le (n-2+|n-2|\varepsilon)\|u_i'(t)\|^2 \\
&\quad + \frac{|n-2|g(0)\lambda_0^2}{4\varepsilon}\int_{-\infty}^t |g'(t-s)|\|u_i(t) - u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds.
\end{aligned}
$$

Similar to (4.28), we deduce

$$
\begin{aligned}
(4.30)\qquad & \int_\Omega u_i u_i'' dx \\
&= \int_\Omega u_i\Big[\mu\Delta(u_i - g * u_i) + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}(u - g * u)) - \alpha\frac{\partial\theta}{\partial x_i}\Big]dx \\
&= \int_{\Gamma_2} u_i\Big[\mu\frac{\partial}{\partial\nu}(u_i - g * u_i) + (\lambda+\mu)\nu_i\mathrm{div}(u - g * u)\Big]d\Gamma \\
&\quad - \int_\Omega[\mu\nabla u_i\nabla(u_i - g * u_i) + (\lambda+\mu)\mathrm{div}u\,\mathrm{div}(u - g * u)]dx \\
&\quad - \alpha\int_\Omega u_i\frac{\partial\theta}{\partial x_i}dx \\
&= -\int_{\Gamma_2} u_i[am\cdot\nu(u_i - g * u_i) + m\cdot\nu u_i']d\Gamma \\
&\quad - \int_\Omega[\mu\nabla u_i\nabla(u_i - g * u_i) + (\lambda+\mu)\mathrm{div}u\,\mathrm{div}(u - g * u)]dx
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha \int_\Omega u_i \frac{\partial \theta}{\partial x_i} dx \\
\leq &- \int_{\Gamma_2} am \cdot \nu u_i (u_i - g * u_i) d\Gamma \\
&+ \frac{R_0 \gamma^2}{4\varepsilon} \int_{\Gamma_2} m \cdot \nu |u_i'|^2 d\Gamma + 2\varepsilon \|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{\alpha^2 \lambda_0^2}{4\varepsilon} \|\nabla \theta(t)\|^2 \\
&- \int_\Omega [\mu \nabla u_i \nabla (u_i - g * u_i) + (\lambda + \mu) \mathrm{div} u \mathrm{div}(u - g * u)] dx.
\end{aligned}
$$

Noting definitions (2.25)-(2.27) of C_2, C_3, C_4, it therefore follows from (4.18) and (4.25)-(4.30) that:

$$
\begin{aligned}
(4.31) \quad & F'(u, \theta, t) \\
\leq &- \int_\Omega [\mu \nabla u_i \nabla (u_i - g * u_i) + (\lambda + \mu) \mathrm{div} u \mathrm{div}(u - g * u)] dx \\
&- \int_{\Gamma_2} am \cdot \nu u_i (u_i - g * u_i) d\Gamma + [\varepsilon(1 + |n-2|) - 1] \|u_i'(t)\|^2 \\
&+ \varepsilon(3 + 2|n-2|) \|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} + C_2 \|\nabla \theta(t)\|^2 \\
&+ C_3 \int_{\Gamma_2} m \cdot \nu |u_i'|^2 d\Gamma \\
&+ C_4 \int_{-\infty}^t |g'(t-s)| \|u_i(t) - u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&+ (\varepsilon + 2|n-2|\varepsilon) \int_{-\infty}^t g(t-s) \|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&+ \int_{\Gamma_2} m \cdot \nu \Big[\frac{2a^2 R_0^2}{\mu} - (n-2)a \Big] |u_i - g * u_i|^2 d\Gamma \\
\leq &- E(u, \theta, t) + \frac{\alpha}{2\beta} \|\theta(t)\|^2 - \kappa \|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \quad \text{(use (4.9))} \\
&+ \frac{d}{dt} \int_{-\infty}^t G(t-s) \|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&+ \Big[\varepsilon(1 + |n-2|) - \frac{1}{2} \Big] \|u_i'(t)\|^2 + \varepsilon(3 + 2|n-2|) \|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&+ C_2 \|\nabla \theta(t)\|^2 + C_3 \int_{\Gamma_2} m \cdot \nu |u_i'|^2 d\Gamma \\
&+ C_4 \int_{-\infty}^t |g'(t-s)| \|u_i(t) - u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&+ (\varepsilon + 2|n-2|\varepsilon) \frac{d}{dt} \int_{-\infty}^t G(t-s) \|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \quad \text{(use (4.10))} \\
&+ (\varepsilon + 2|n-2|\varepsilon)(1 - \kappa) \|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&+ \int_{\Gamma_2} m \cdot \nu \Big[\frac{2a^2 R_0^2}{\mu} - (n-2)a \Big] |u_i - g * u_i|^2 d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&\leq -E(u,\theta,t) + \Big[C_2 + \frac{\alpha\lambda_0^2}{2\beta}\Big]\|\nabla\theta(t)\|^2 + C_3\int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma \\
&\quad + C_4\int_{-\infty}^{t} |g'(t-s)|\|u_i(t)-u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds \\
&\quad + [1+(\varepsilon+2|n-2|\varepsilon)]\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds \\
&\quad + \Big[\varepsilon(1+|n-2|)-\frac{1}{2}\Big]\|u_i'(t)\|^2 \quad (=f_1) \\
&\quad + [\varepsilon(3+2|n-2|)+(\varepsilon+2|n-2|\varepsilon)(1-\kappa)-\kappa]\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \quad (=f_2) \\
&\quad + \int_{\Gamma_2} m\cdot\nu\Big[\frac{2a^2R_0^2}{\mu}-(n-2)a\Big]|u_i-g*u_i|^2 d\Gamma \quad (=f_3).
\end{aligned}
$$

If $n \geq 3$, then, by (2.17), we deduce that $f_3 \leq 0$. In addition, by definition (2.30) of ε, we have $f_1 \leq 0$ and $f_2 \leq 0$. Hence, noting definition (2.28) of C_5, (4.17) follows from (4.31).

If $n \leq 2$, then we estimate f_3 as follows. Let $K(A)$ be given by (2.29). Using (2.8), we deduce

$$
\begin{aligned}
(4.32)\qquad f_3 &\leq 2K(A)\int_{\Gamma_2} m\cdot\nu[|u_i|^2+|g*u_i|^2]d\Gamma \\
&\leq 2K(A)R_0\gamma^2\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + 2K(A)\int_{\Gamma_2} m\cdot\nu\Big(\int_{-\infty}^{t} g(t-s)u_i(s)ds\Big)^2 d\Gamma \\
&\leq 2K(A)R_0\gamma^2\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + 2K(A)\int_{\Gamma_2} m\cdot\nu\int_{-\infty}^{t} g(t-s)ds\int_{-\infty}^{t} g(t-s)|u_i(s)|^2 ds d\Gamma \\
&\leq 2K(A)R_0\gamma^2\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + 2K(A)(1-\kappa)\int_{\Gamma_2} m\cdot\nu\int_{-\infty}^{t} g(t-s)|u_i(s)|^2 ds d\Gamma \\
&\leq 2K(A)R_0\gamma^2\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \\
&\quad + 2K(A)(1-\kappa)R_0\gamma^2\int_{-\infty}^{t} g(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds \\
&= 2K(A)R_0\gamma^2[1+(1-\kappa)^2]\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \quad (\text{use } (4.10)) \\
&\quad + 2K(A)(1-\kappa)R_0\gamma^2\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)}ds.
\end{aligned}
$$

Set

$$
(4.33)\quad C_6 = \varepsilon(3+2|n-2|)+(\varepsilon+2|n-2|\varepsilon)(1-\kappa)-\kappa+2K(A)R_0\gamma^2[1+(1-\kappa)^2].
$$

It therefore follows from (2.28), (4.31) and (4.32) that:

$$
\begin{aligned}
(4.34)\qquad & F'(u,\theta,t) \\
&\le -E(u,\theta,t) + \Big[C_2 + \frac{\alpha\lambda_0^2}{2\beta}\Big]\|\nabla\theta(t)\|^2 + C_3\int_{\Gamma_2} m\cdot\nu|u_i'|^2 d\Gamma \\
&\quad + C_4\int_{-\infty}^{t} |g'(t-s)|\|u_i(t)-u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&\quad + C_5\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u_i(s)\|^2_{H^1_{\Gamma_1}(\Omega)} ds \\
&\quad + [\varepsilon(1+|n-2|) - \frac{1}{2}]\|u_i'(t)\|^2 \quad (=f_1) \\
&\quad + C_6\|u_i(t)\|^2_{H^1_{\Gamma_1}(\Omega)} \quad (=f_2),
\end{aligned}
$$

which implies (4.17) in view of definition (2.30) of ε. □

LEMMA 4.4. – *Let Γ_1 and Γ_2 be given by* (1.10) *and* (1.11), *respectively, satisfying* (2.11). *Let* (1.16) *hold. Suppose that the relaxation function g satisfies* (H_1), (H_2) *and* (H_3). *Assume that the function $a(x)$ satisfies* (2.16) *and* (2.17). *Let u, θ be the solution of* (2.10). *If*

$$
(4.35)\qquad 0 < \delta \le \min\Big\{\frac{1}{C_3},\ \frac{1}{C_4},\ \frac{2\alpha}{2\beta C_2 + \alpha\lambda_0^2}\Big\},
$$

then we have

$$
(4.36)\qquad \tau V(u,\theta,t) \le -V'(u,\theta,t) + \delta C_5\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds,
$$

where

$$
(4.37)\qquad \tau = \frac{\delta}{1+\delta C_1},
$$

and the constant C_1, C_2, C_3, C_5 are given by (2.24)-(2.28), *respectively.*

Proof. – It follows from (2.14), (4.17) and (H_2) that:

$$
\begin{aligned}
V'(u,\theta,t) &= E'(u,\theta,t) + \delta F'(u,\theta,t) \\
&\le -\delta E(u,\theta,t) + \Big[\delta\Big(C_2 + \frac{\alpha\lambda_0^2}{2\beta}\Big) - \frac{\alpha}{\beta}\Big]\|\nabla\theta(t)\|^2 \\
&\quad + (\delta C_3 - 1)\int_{\Gamma_2} m\cdot\nu|u'|^2 d\Gamma \\
&\quad + (\delta C_4 - 1)\int_{-\infty}^{t} |g'(t-s)|\|u(t)-u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + \delta C_5\frac{d}{dt}\int_{-\infty}^{t} G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds,
\end{aligned}
$$

which, combining (4.4) and (4.35), implies (4.36). □

It is well known (see [KKK, Wal]) that, for a dynamical system $S(t)_{t\geq 0}$ on a metric space X, a Lyapunov functional usually has the following property

$$V'(S(t)x) \leq -W(S(t)x) \leq 0, \qquad \forall\, x \in X, \tag{4.38}$$

where W is a given function. Because in (4.36) there is an additional term

$$\delta C_5 \frac{d}{dt} \int_{-\infty}^{t} G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds,$$

the functional V does not satisfies (4.38). Thus, it is referred to as a generalized Lyapunov functional.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. – Multiplying both sides of (4.36) by $e^{\tau t}$ and integrating from 0 to t, we obtain

$$\begin{aligned}
(4.39)\quad & V(u,\theta,t)e^{\tau t} - V(u,\theta,0) \\
&\leq \delta C_5 \int_0^t e^{\tau r} dr \frac{d}{dr} \int_{-\infty}^{r} G(r-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&= \delta C_5 e^{\tau t} \int_{-\infty}^{t} G(t-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad - \delta C_5 \int_{-\infty}^{0} G(-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad - \delta C_5 \tau \int_0^t e^{\tau r} dr \int_{-\infty}^{r} G(r-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\text{(use } (H_4) \text{ and note } G(t-s) \leq 0) \\
&\leq \delta C_5 K \int_{-\infty}^{0} g(-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + \delta C_5 \tau K \int_0^t e^{\tau r} dr \int_{-\infty}^{r} g(r-s)\|u(s)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&= \delta C_5 K \int_{-\infty}^{0} g(-s)\|u(s) - u(0) + u(0)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + \delta C_5 \tau K \int_0^t e^{\tau r} dr \int_{-\infty}^{r} g(r-s)\|u(s) - u(r) + u(r)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\leq 2\delta C_5 K \int_{-\infty}^{0} g(-s)\|u(0)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + 2\delta C_5 K \int_{-\infty}^{0} g(-s)\|u(s) - u(0)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + 2\delta C_5 \tau K \int_0^t e^{\tau r} dr \int_{-\infty}^{r} g(r-s)\|u(r)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\
&\quad + 2\delta C_5 \tau K \int_0^t e^{\tau r} dr \int_{-\infty}^{r} g(r-s)\|u(s) - u(r)\|^2_{(H^1_{\Gamma_1}(\Omega))^n} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\delta C_5 K \frac{1-\kappa}{\kappa} E(u,\theta,0) + 2\delta C_5 K E(u,\theta,0) \\
&\quad + 2\delta C_5 K \frac{1-\kappa}{\kappa}(e^{\tau t}-1)E(u,\theta,0) \\
&\quad + 2\delta C_5 K (e^{\tau t}-1)E(u,\theta,0) \\
&= \frac{2}{\kappa}\delta C_5 K e^{\tau t} E(u,\theta,0).
\end{aligned}
$$

It therefore follows from (4.4) that

$$
\begin{aligned}
&(1-\delta C_1)E(u,\theta,t) \\
&\leq e^{-\tau t}(1+\delta C_1)E(u,\theta,0) + \frac{2}{\kappa}\delta C_5 K E(u,\theta,0)
\end{aligned}
$$

and then

$$E(u,\theta,t) \leq \rho E(u,\theta,0), \tag{4.40}$$

where

$$\rho(t) = \frac{e^{-\tau t}(1+\delta C_1)\kappa + 2\delta C_5 K}{\kappa(1-\delta C_1)}. \tag{4.41}$$

If δ is small enough and t is large enough so that

$$\frac{2\delta C_5 K}{\kappa(1-\delta C_1)} = \frac{1}{2} \tag{4.42}$$

and

$$\frac{e^{-\tau t}(1+\delta C_1)}{(1-\delta C_1)} < \frac{1}{2}, \tag{4.43}$$

then $\rho < 1$. This holds if $t = T$ and δ are given by (2.22) and (2.23), respectively. Let $S(t)$ be the semigroup generated by system (2.10). Then (4.40) implies that

$$\|S(t)\| \leq \rho. \tag{4.44}$$

Let $t = nT + s$, $0 \leq s < T$. Then

$$\|S(t)\| \leq \|S(s)\|\|S(nT)\| \leq \|S(T)\|^n\| \leq (\rho(T))^n,$$

which implies that

$$\|S(t)\| \leq (\rho(T))^{-1} e^{-\omega t},$$

where

$$\omega = -T^{-1}\ln\rho(T) > 0. \tag{4.45}$$

Hence, the proof of Theorem 2.1 is complete. □

5. Further Comments

If $\Gamma_1 = \emptyset$ and $a(x) \equiv 0$, we first note that system (2.10) does not generate a semigroup in the following space with zero average

$$\mathcal{H}_0 = \Big\{(u, v, \theta, w) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \times L^2(g, (0, \infty), (H^1(\Omega))^n) \ : \\ \int_\Omega u(x)dx = \int_\Omega v(x)dx = \int_\Omega w(x, s)dx = 0\Big\}.$$

To see this, we define the function

$$f(t) = \int_\Omega u(t)dx.$$

Since

$$\begin{aligned} f''(t) &= \int_\Omega u''(t)dx \\ &= \int_\Omega [\mu\Delta(u - g * u) + (\lambda + \mu)\nabla \mathrm{div}\ (u - g * u) - \alpha\nabla\theta]dx \\ &= \int_\Gamma [\mu\frac{\partial(u - g * u)}{\partial\nu} + (\lambda + \mu)\mathrm{div}(u - g * u)\nu - \alpha\theta\nu]d\Gamma \\ &= -\int_\Gamma m \cdot \nu u'(t)d\Gamma, \end{aligned}$$

$f(t)$ and $f'(t)$ may not be constants along the solution trajectories of (2.10). Thus, $\mathcal{H}_0$ may not be invariant under the flow given by (2.10). Consequently, system (2.10) is unlikely to generate a semigroup.

For the system of thermoelasticity, we have found a space invariant under the flow given by the system. This space is given by

$$\mathcal{W} = V \times L^2(\Omega),$$

where

$$V = \{(u, v) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \ : \ \int_\Gamma m \cdot \nu u d\Gamma + \int_\Omega v dx = 0\}.$$

However, for the system of thermoviscoelasticity, it seems difficult to find such an invariant subspace. We can consider the following subspace:

$$\mathcal{H} = \Big\{(u, v, \theta, w) \in (H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \times L^2(g, (0, \infty), (H^1(\Omega))^n) \ : \\ \int_\Gamma m \cdot \nu u(x)d\Gamma + \int_0^\infty g(s)ds \int_\Gamma w(x, s)d\Gamma dx + \int_\Omega v(x)dx = 0\Big\}.$$

This space does be invariant under the flow given by (2.10) if the boundary condition on Γ_2 of (2.10) is replaced by

$$\mu\frac{\partial}{\partial\nu}(u - g * u) + (\lambda + \mu)\nu\mathrm{div}(u - g * u) \\ + \int_0^\infty g(s)w'(t, s)ds + m \cdot \nu u' = 0.$$

But bad things happen. The energy $E(u,\theta,t)$ may not decrease as

$$\begin{aligned} E'(u,\theta,t) = & \frac{\mu}{2}\int_{-\infty}^{t} g'(t-s)\|\nabla u(t)-\nabla u(s)\|^2 ds \\ & + \frac{\lambda+\mu}{2}\int_{-\infty}^{t} g'(t-s)\|\mathrm{div} u(t)-\mathrm{div} u(s)\|^2 ds \\ & - \int_{\Gamma_2} m\cdot\nu|u'(t)|^2 d\Gamma - \int_{\Gamma_2}\int_0^{\infty} g(s)w'(t,s)u'(t)dsd\Gamma \\ & - \frac{\alpha}{\beta}\|\nabla\theta(t)\|^2 \end{aligned}$$

may not always less than zero. Further, we do not know whether or not the energy norm on $\mathcal{H}$ is equivalent to the usual one induced by $(H^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) \times L^2(g,(0,\infty),(H^1(\Omega))^n)$ and whether or not system (2.10) with this boundary condition generates a semigroup.

In conclusion, the case that $\Gamma_1 = \emptyset$ and $a(x) \equiv 0$ is open.

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