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Strange eigenmodes and decay of variance in the mixing of diffusive tracers

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Abstract

We prove the existence of asymptotic spatial patterns for diffusive tracers advected by unsteady velocity fields. The asymptotic patterns arise from convergence to a time-dependent inertial manifold in the underlying advection–diffusion equation. For time-periodic velocity fields, we find that the inertial manifold is spanned by a finite number of Floquet solutions, the *strange eigenmodes*, observed first numerically by Pierrehumbert. These strange eigenmodes only admit a regular asymptotic expansion in the diffusivity if the velocity field is completely integrable.

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Keywords: Diffusive mixing; Strange eigenmodes; Intertial manifolds

1. Introduction

In a seminal paper, Pierrehumbert [26] observed intriguing patterns in a diffusive tracer field stirred by the discrete map model of a time-periodic velocity field. The concentration patterns formed within just a few stirring periods, then repeated with exponentially decaying intensity as the tracer field approached the fully mixed state. This exponentially modulated time-periodic behavior and the complex spatial structure prompted Pierrehumbert to call the repeating patterns *strange eigenmodes*.

Several numerical studies have since confirmed that strange eigenmodes develop in periodically stirred two-dimensional diffusive tracer fields (see, e.g. [3,27,30]). Remarkably, Rothstein et al. [29] observed strange eigenmodes experimentally in a periodically driven two-dimensional fluid layer. Both the numerical evidence in [26] and the experimental findings in [29] suggest that strange eigenmodes also appear in flows with aperiodic time-dependence, as evidenced by an asymptotic self-similarity of the tracer probability distribution function (PDF). This makes one suspect that the asymptotics of scalar mixing in large-scale geophysical flows may be universally governed by statistically defined strange eigenmodes (see [16]).

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1.1. Prior work on strange eigenmodes

Pierrehumbert's observation has led to several statistical theories for the asymptotic decay rate of the tracer variance. For chaotic advection, Antonsen et al. [3] showed a relationship between the statistics of finite-time Lyapunov exponents and the decay rate. Fereday et al. [11] and Wonhas and Vassilicos [35] found, however, that the variance remains exponential in examples where the theory in [3] predicts super-exponential decay. The conclusion put forward by Fereday et al. [11] is that Lyapunov exponents alone fail to describe correctly the exponential decay of tracer variance. The same conclusion pertains to the work of Pattanayak [25], who generalizes the idea of equating diffusive effects with straining in the Lagrangian frame (cf. [5,31]) to obtain a variance decay prediction for ergodic fluid mixing.

Working with random but spatially smooth velocity fields, Balkovsky and Fouxon [7] used Lyapunov exponent statistics to obtain tracer PDF with no asymptotic self-similarity. This suggests that strange eigenmodes are absent in the Batchelor regime of turbulence, i.e., in the regime with length scales below the viscous cutoff but above the diffusive cutoff. At the same time, the simulations of Pierrehumbert [27] and Hu and Pierrehumbert [16] showed self-similar tracer PDF in the same regime, giving further basis to the critical view of Fereday et al. [11] on Lyapunov exponent-based theories.

Hu and Pierrehumbert [17] and Fereday and Haynes [12] argue that [3,7] both describe a transient stage of tracer mixing after which strange eigenmodes with self-similar PDF prevail. By contrast, Antonsen and Ott [4] maintain that the mechanism described in [3] remains an important component in tracer evolution for infinite times, and that this mechanism provides a lower bound on the decay of the passive scalar variance in the limit of vanishing diffusivity.

Despite all the work on asymptotic variance decay, there has been little progress in justifying the strange eigenmode view itself. As a notable exception, Antonsen et al. [3] give a heuristic description of the wave-number spectrum of an eigenmode for flows with no mixing barriers. A refined description of the spectrum appears in Fereday et al. [11] for a one-dimensional diffusive baker's map model.

More recently, Sukhatme and Pierrehumbert [30] pointed out a similarity between the differential operators of advection–diffusion and magnetic dynamo theory. For the latter, rigorous results by Childress and Gilbert [8] guarantee periodic eigenmodes if the velocity field is time-periodic (see also [13]). For smooth velocity fields, the dynamo operator only differs from the advection–diffusion operator by a bounded term, thus the existence of eigenmodes in the advection–diffusion equation appears plausible, at least for a smooth, two-dimensional, time-periodic velocity field.

As Sukhatme and Pierrehumbert [30] note, however, the completeness of the eigenmodes is unknown even in dynamo theory, thus recurrent patterns may be unobservable for general initial data. In addition, the dynamo analogy fails for nonsmooth, three-dimensional, or aperiodic velocity fields, leaving the existence of eigenmodes an open question for realistic applications.

Very recently, Pikovsky and Popovych [28] showed that strange eigenmodes may also be viewed as eigenfunctions of an appropriate Frobenius–Perron operator. They computed some of these eigenfunctions numerically for a scalar advected by the standard map. This fresh approach offers an alternative view on scalar mixing, but leaves the questions of completeness and general time-dependence open.

1.2. The main results of this paper

In this paper, we prove the existence of strange eigenmodes for advection–diffusion problems of the form

$$c_t + \nabla c \cdot \mathbf{v} = \kappa \Delta c, \qquad (1)$$

where $c(\mathbf{x}, t)$ denotes the tracer concentration, $\kappa > 0$ is the diffusivity, and $\mathbf{v}(\mathbf{x}, t)$ is a bounded velocity field with general time-dependence. The spatial variable $\mathbf{x}$ is defined over a bounded spatial domain S that is either two- or

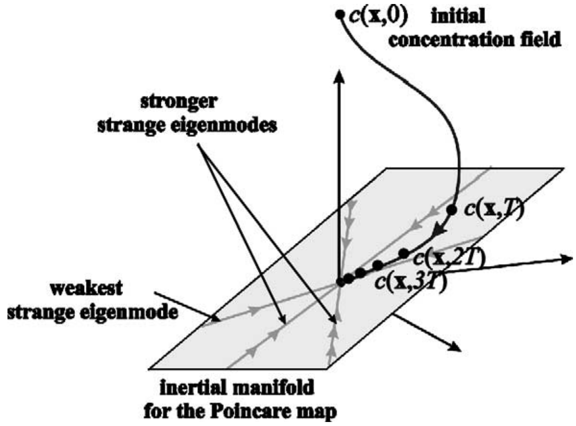

Fig. 1. Evolution of the tracer concentration $c(\mathbf{x}, t)$, and the geometry of the Poincaré map $c(\mathbf{x}, t) \mapsto c(\mathbf{x}, t + T)$.

three-dimensional. We show that Eq. (1) admits a finite-dimensional invariant manifold $\mathcal{M}(t)$ in an appropriate Sobolev space provided that the Laplacian operator Δ satisfies a spectral gap condition on S. On the manifold $\mathcal{M}(t)$, the tracer evolution is governed by a time-dependent system of linear ordinary differential equations, whose general solutions are the generalized strange eigenmodes. The slowest-decaying such eigenmode becomes dominant asymptotically.

Under a stronger gap condition, we show that $\mathcal{M}(t)$ is in fact an inertial manifold, i.e., it attracts all square-integrable initial tracer distributions in the Sobolev space $H^1(S)$. Consequently, any initial tracer distribution evolves towards a finite set of strange eigenmodes and becomes indistinguishable from the slowest-decaying such eigenmode over long enough time scales. As we show, the stronger gap condition guaranteeing all this holds for canonical two-dimensional domains such as the square.

If we add a square-integrable source term to the left-hand side of (1), the asymptotic tracer distribution will be determined by a particular solution, not an eigenmode. Still, the convergence of general solutions to this particular solution will be governed by strange eigenmodes.

For time-periodic velocity fields that are continuous-in-time, the dynamics on the inertial manifold admits a classical Floquet decomposition. This decomposition gives a rigorous proof for the time-periodic yet exponentially fading spatial patterns observed by Pierrehumbert for the tracer field. We show this result schematically in Fig. 1.

For incompressible time-periodic velocity fields, we find the weakest Floquet solution to be of the form

$$c_\infty(\mathbf{x}, t) = \exp\left[-\kappa\left(\frac{\|\nabla\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2} + \mathrm{i}\frac{\|\nabla\bar{\varphi}_0\|^2\|\mathrm{Re}\,\bar{\varphi}_0\|^2 - \|\bar{\varphi}_0\|^2\|\mathrm{Re}\,\nabla\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2\int_S \overline{\mathrm{Re}\,\varphi_0\,\mathrm{Im}\,\varphi_0}\,\mathrm{d}V}\right)t\right]\varphi_0(\mathbf{x}, t), \tag{2}$$

where $\varphi_0(\mathbf{x}, t) = \varphi_0(\mathbf{x}, t + T)$ is a complex-valued function depending on κ, overbar denotes temporal averaging over one period of the velocity field, and $\|\cdot\|^2 = \int_S |\cdot|^2\,\mathrm{d}V$ denotes the L^2 norm over S. As we show in an example, (2) also allows for quasiperiodic and subharmonic eigenmodes.

As anticipated by Sukhatme and Pierrehumbert [30], the behavior of φ_0 in the $\kappa \to 0$ limit turns out to be quite subtle. We find that any attempt to approximate φ_0 through a regular Taylor-expansion in κ will invariably fail for nonintegrable velocity fields.

The outline of this paper is as follows. In Section 2, we first give estimates that establish the decay of tracer variance under general conditions. In Section 3, we prove the existence of an inertial manifold with its attendant generalized strange eigenmodes. Section 4 elaborates on the properties of classical strange eigenmodes observed for time-periodic velocity fields, then Section 5 discusses the role of strange eigenmodes in the presence of sources and sinks. We conclude with a summary and a list of open questions in Section 6.

2. General properties of tracer evolution

Consider an unsteady velocity field $\mathbf{v}(\mathbf{x}, t)$ on a bounded two- or three-dimensional spatial domain S. We denote the measure (area or volume) of S by

$$\varkappa = \mathrm{mes}\,(S). \tag{3}$$

We assume that $\mathbf{v}$ is incompressible, square-integrable over S, and satisfies periodic, no-flow, or no-slip boundary conditions on the boundary ∂S.

We denote fluid particle positions at time t by $\mathbf{x}(t; t_0, \mathbf{x}_0)$, referring to the initial positions as $\mathbf{x}_0$ at time t_0. We shall also use the flow map $\mathbf{F}_{t_0}^{t}(\mathbf{x}_0) = \mathbf{x}(t; t_0, \mathbf{x}_0)$, the map that relates initial particle positions at time t_0 to their current positions at time t.

We are interested in the mixing of a diffusive tracer field $c(\mathbf{x}, t)$ under the action of $\mathbf{v}$. The tracer field is assumed to satisfy the advection–diffusion equation

$$c_t + \nabla c \cdot \mathbf{v} = \kappa \Delta c + f(\mathbf{x}, t),$$

where κ is the diffusivity of the tracer, and $f(\mathbf{x}, t)$ denotes a source term.

Below we collect some basic properties of the solutions of the above equation. Specifically, we estimate how the *mean concentration* $\langle c \rangle = (1/\varkappa) \int_S c\, \mathrm{d}V$, the *concentration variance* $\|c\|^2 = \int_S c^2\, \mathrm{d}V$, and the *concentration-gradient variance* $\|\nabla c\|^2 = \int_S |\nabla c|^2\, \mathrm{d}V$ evolve in time.

Some of the estimates below appear to be unknown in the physical advection–diffusion literature, which has traditionally preferred statistical models over analytic estimates. Some other reviewed facts, such as the conservation of mean concentration in the absence of sources, are well-known.

2.1. Evolution of a conserved tracer

In the absence of diffusion, sources, and sinks, c is conserved along fluid trajectories. The governing equation for c then simplifies to the conservation law

$$c_t + \nabla c \cdot \mathbf{v} = 0, \qquad c(\mathbf{x}, t_0) = c_0(\mathbf{x}), \qquad \left.\frac{\partial c}{\partial n}\right|_{\partial S} = 0 \tag{4}$$

with $c_0(\mathbf{x})$ denoting the initial concentration at $t = t_0$, and with $(\partial c/\partial n)|_{\partial S}$ referring to the normal derivative of c along the boundary. The above equation is solved by

$$c(\mathbf{x}, t) = c_0(\mathbf{x}_0(t_0; t, \mathbf{x})), \tag{5}$$

where $\mathbf{x}_0(t_0; t, \mathbf{x})$ denotes the position at time t_0 for the fluid particle that is located at the point $\mathbf{x}$ at time t.

Proposition 1. *The solution $c(\mathbf{x}, t)$ of Eq. (4) has the following properties*:

(i) *The mean concentration is constant in time, i.e.,* $\langle c \rangle = \langle c_0 \rangle$.
(ii) *The concentration variance is constant in time, i.e.,* $\|c\|^2 = \|c_0\|^2$.

(iii) *If* $\mathbf{v}$ *has bounded gradients over S, then the variance of the concentration-gradient satisfies the estimate*

$$\|e^{\lambda_-(t,t_0,\mathbf{x}_0)(t-t_0)}\boldsymbol{\nabla} c_0(\mathbf{x}_0)\|^2 \le \|\boldsymbol{\nabla} c\|^2 \le \|e^{\lambda_+(t,t_0,\mathbf{x}_0)(t-t_0)}\boldsymbol{\nabla} c_0(\mathbf{x}_0)\|^2,$$

where $\lambda_+(t, t_0, \mathbf{x}_0)$ *and* $\lambda_-(t, t_0, \mathbf{x}_0)$ *denote the maximal and minimal finite-time Lyapunov exponent associated with the fluid trajectory starting from* $\mathbf{x}_0$ *at time* t_0. (*For two-dimensional flows, we have* $\lambda_- = -\lambda_+$.)

We prove the above proposition in Appendix A.

Statement (iii) of Proposition 1 gives upper and lower bounds on the well-documented growth of tracer gradients for nondiffusive passive scalars in chaotic or turbulent advection. In particular, the gradient variance does not grow faster than the initial tracer variance weighted with an exponentially growing term, whose exponent is the finite-time Lyapunov exponent distribution for the velocity field $\mathbf{v}$.

2.2. Evolution of a diffusive tracer

Consider now a diffusive tracer $c(\mathbf{x}, t)$ with the corresponding full advection–diffusion equation

$$c_t + \nabla c \cdot \mathbf{v} = \kappa \Delta c + f, \qquad c(\mathbf{x}, t_0) = c_0(\mathbf{x}). \tag{6}$$

In this section, we fix the Neumann boundary condition $(\partial c/\partial n)|_{\partial S} = 0$ for concreteness, but our main results in later sections are equally valid for Dirichlet or spatially periodic boundary conditions.

We define the maximal strain rate $\sigma(t)$ as the maximal eigenvalue of the rate-of-strain-tensor over all $\mathbf{x} \in S$. More specifically, we let

$$\sigma(t) = \max_{\mathbf{x}\in S} \lambda_{\max}(\tfrac{1}{2}(\nabla \mathbf{v}(\mathbf{x}, t) + \nabla \mathbf{v}^{\mathrm{T}}(\mathbf{x}, t))).$$

For two-dimensional flows, incompressibility implies

$$\sigma(t) = \max_{\mathbf{x}\in S} \tfrac{1}{2}(\sqrt{-\det[\nabla \mathbf{v}(\mathbf{x}, t) + \nabla \mathbf{v}^{\mathrm{T}}(\mathbf{x}, t)]}. \tag{7}$$

Proposition 2. *The solution* $c(\mathbf{x}, t)$ *of* Eq. (6) *has the following properties*:

(i) *The mean concentration satisfies* $\langle c\rangle = \langle c_0\rangle + \int_{t_0}^{t}\langle f(\mathbf{x}, \tau)\rangle\,d\tau$. *Therefore,* $\langle c\rangle$ *is conserved in the absence of sinks and sources*.

(ii) *Let* $\epsilon > 0$ *be an arbitrary positive constant, and let* $\mu_1 > 0$ *be the smallest eigenvalue of the Laplacian* $-\Delta$ *on the domain S. The concentration variance then satisfies the upper estimate*

$$\|c - \langle c_0\rangle\|^2 \le \|c_0 - \langle c_0\rangle\|^2\, e^{-2(\kappa\mu_1-\epsilon)(t-t_0)} + \frac{1}{2\epsilon}\int_{t_0}^{t} e^{-2(\kappa\mu_1-\epsilon)(t-\tau)}\|f - \langle f\rangle\|^2\,d\tau. \tag{8}$$

In particular, in the absence of sinks and sources, the concentration variance obeys the estimate

$$\|c - \langle c_0\rangle\|^2 \le \|c_0 - \langle c_0\rangle\|^2\, e^{-2\kappa\mu_1(t-t_0)}. \tag{9}$$

(iii) *If* $\mathbf{v}$ *has bounded gradients over S, then with* ϵ *and* μ_1 *defined above, the variance of the concentration-gradient satisfies the estimate*

$$\|\boldsymbol{\nabla} c\|^2 \le \|\boldsymbol{\nabla} c_0\|^2\, e^{\int_{t_0}^{t} 2(\sigma(\tau)-\kappa\mu_1+\epsilon)\,d\tau} + \frac{1}{2\epsilon}\int_{t_0}^{t} e^{2\int_{\tau}^{t}(\sigma(r)-\kappa\mu_1+\epsilon)\,dr}\|\boldsymbol{\nabla} f\|_2^2\,d\tau. \tag{10}$$

In particular, in the absence of sinks and sources, we have

$$\|\boldsymbol{\nabla} c\|^2 \le \|\boldsymbol{\nabla} c_0\|^2\, e^{2\int_{t_0}^{t}(\sigma(\tau)-\kappa\mu_1)\,d\tau}. \tag{11}$$

We prove the above proposition in Appendix B. Note that if the diffusivity is large enough so that

$$\kappa > \frac{\sigma(t)}{\mu_1}, \tag{12}$$

then, by (11), the norm of the tracer gradient starts decaying exponentially immediately after t_0, regardless of the initial gradient distribution. Thus $\max_{t\geq t_0}[\sigma(t)/\mu_1]$ gives a lower bound on the diffusivity that is needed to completely eliminate the initial growth in the tracer gradient variance caused by advection. Selecting a spatial region S with a smaller μ_1 value is therefore expected to lead to faster mixing for the same diffusivity κ.

If (12) is not satisfied for a given flow, the estimate (10) *does not* predict decay for the variance of tracer gradients. In that case, the estimate simply serves as an upper bound for the growth rate that $\|\nabla c\|^2$ may exhibit while the tracer is mixed.

Example 1 (Variance decay estimate for a planar rectangular domain). If $S = [0, a] \times [0, b]$ is a two-dimensional rectangular domain with $a \geq b > 0$, then

$$\mu_1 = \frac{\pi^2}{b^2} \tag{13}$$

and hence immediate exponential decay in the tracer variance occurs for

$$\kappa > \frac{b^2\sigma(t)}{\pi^2}.$$

In view of the above discussion, selecting a narrower rectangular domain with smaller b leads to faster mixing for the same diffusivity.

Example 2 (Vorticity evolution in two-dimensional decaying turbulence). As an illustration of Proposition 2, we consider a two-dimensional physical domain S, and assume that the velocity field $\mathbf{v}$ satisfies the incompressible Navier–Stokes equation

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}.$$

Here ρ denotes the density, p is the pressure, ν the kinematic viscosity, and $\mathbf{f}$ denotes the body force distribution in the fluid. Taking the curl of this equation and using incompressibility, we arrive at the scalar vorticity equation

$$\omega_t + \nabla\omega\cdot\mathbf{v} = \nu\Delta\omega + (\nabla\times\mathbf{f})_z$$

with $\omega = (\nabla\times\mathbf{v})_z$ denoting the component of the vorticity that is orthogonal to the plane of S. Since ω satisfies Eq. (6), Proposition 2 applies and gives

$$\langle\omega\rangle = \langle\omega_0\rangle + \int_{t_0}^{t}\langle(\nabla\times\mathbf{f}(\mathbf{x},\tau))_z\rangle\,\mathrm{d}\tau,$$

$$\|\omega-\langle\omega_0\rangle\|^2 \leq \|\omega_0-\langle\omega_0\rangle\|^2\,\mathrm{e}^{-2(\nu\mu_1-\epsilon)(t-t_0)} + \frac{1}{2\epsilon}\int_{t_0}^{t}\mathrm{e}^{-2(\nu\mu_1-\epsilon)(t-\tau)}\|(\nabla\times\mathbf{f}(\mathbf{x},\tau))_z\|^2\,\mathrm{d}\tau,$$

$$\|\nabla\omega\|^2 \leq \|\nabla\omega_0\|^2\,\mathrm{e}^{\int_{t_0}^{t}2(\sigma(\tau)-\nu\mu_1+\epsilon)\,\mathrm{d}\tau} + \frac{1}{2\epsilon}\int_{t_0}^{t}\mathrm{e}^{2\int_{\tau}^{t}(\sigma(r)-\nu\mu_1+\epsilon)\,\mathrm{d}r}\|\nabla(\nabla\times\mathbf{f}(\mathbf{x},\tau))_z\|_2^2\,\mathrm{d}\tau$$

for any constant $\epsilon > 0$.

If the force $\mathbf{f}$ is potential, then $(\nabla \times \mathbf{f})_z \equiv 0$, and hence the above argument yield the vorticity decay estimates

$$\langle\omega\rangle = \langle\omega_0\rangle, \qquad \|\omega - \langle\omega_0\rangle\|^2 \le \|\omega_0 - \langle\omega_0\rangle\|^2 \,\mathrm{e}^{-2\nu\mu_1(t-t_0)}, \qquad \|\nabla\omega\|^2 \le \|\nabla\omega_0\|^2 \,\mathrm{e}^{2\int_{t_0}^{t}(\sigma(\tau)-\nu\mu_1)\,\mathrm{d}\tau}.$$

The first estimate gives an upper bound on the decay of vorticity variance, while the second estimate establishes an upper bound on intermediate growth rates for the vorticity gradient variance. If the viscosity is large enough so that $\nu > \sigma(t)/\mu_1$, then we obtain immediate exponential decay for the vorticity gradient distribution.

Example 3 (Tracer variance decay in a numerical example). We consider the two-dimensional square domain $S = [0, 1] \times [0, 1]$ and fix the diffusivity value $\kappa = 0.001$. Motivated by the example used by Liu et al. [19], Alvarez et al. [2], and Muzzio et al. [24], we select the velocity field

$$v_1(x, y, t) = \begin{cases} \sin(\pi x)\cos(\pi y) & \text{if } n \le t < n + 0.75, \\ -\sin(2\pi x)\cos(\pi y) & \text{if } n + 0.75 \le t < n + 1, \end{cases}$$
$$v_2(x, y, t) = \begin{cases} -\cos(\pi x)\sin(\pi y) & \text{if } n \le t < n + 0.75, \\ 2\cos(2\pi x)\sin(\pi y) & \text{if } n + 0.75 \le t < n + 1 \end{cases} \tag{14}$$

and the initial tracer distribution

$$c_0(x, y) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \quad \text{and} \quad 0 \le y \le 1, \\ 0 & \text{if } \frac{1}{2} < x \le 1 \quad \text{and} \quad 0 \le y \le 1 \end{cases} \tag{15}$$

for the advection–diffusion equation (6) with $f \equiv 0$. We define

$$V(t) = \|c(t) - \langle c_0\rangle\|^2$$

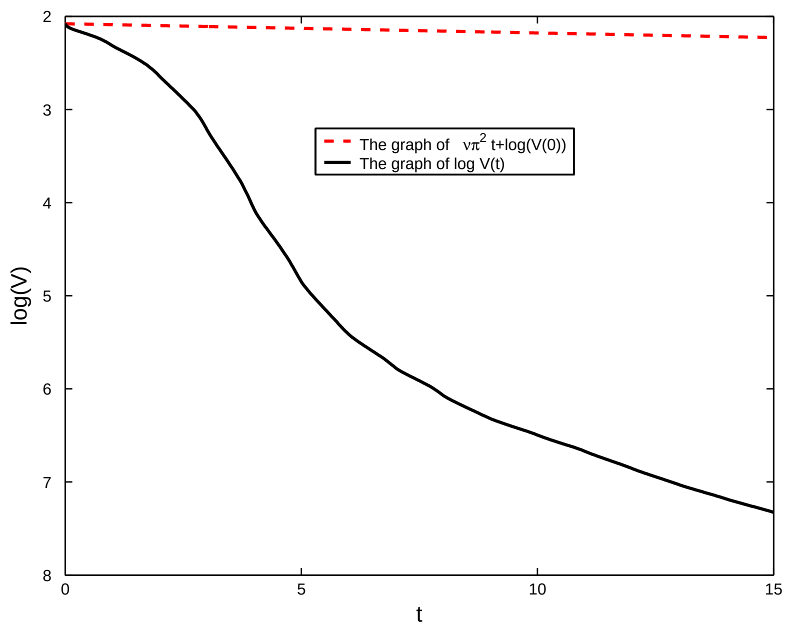

Fig. 2. Decay of concentration variance (solid line), and the universal upper estimate for this decay (dashed line).

and recall that estimate (9) and formula (13) with $b = 1$ together yield

$$V(t) \leq \mathrm{e}^{-\kappa\pi^2 t} V(0). \tag{16}$$

In Fig. 2, the dashed curve shows $\log \mathrm{e}^{-\kappa\pi^2 t} V(0)$, the logarithm of the right-hand side of (16), and the dash-dotted curve shows $\log V(t)$. The figure illustrates the universal validity of estimate (9), but also shows that the actual decay rate of the concentration variance may be much faster than the rate on the right-hand side of (9).

Parallel to the exponential decay of variance, the concentration converges at an exponential speed to a time-periodic pattern, the strange eigenmode described by Pierrehumbert [26]. Shown in Fig. 3, the pattern quickly develops within a few periods, then fades away gradually as the concentration field converges to the fully mixed state $c(\mathbf{x}, t) \equiv \langle c_0(\mathbf{x}) \rangle$.

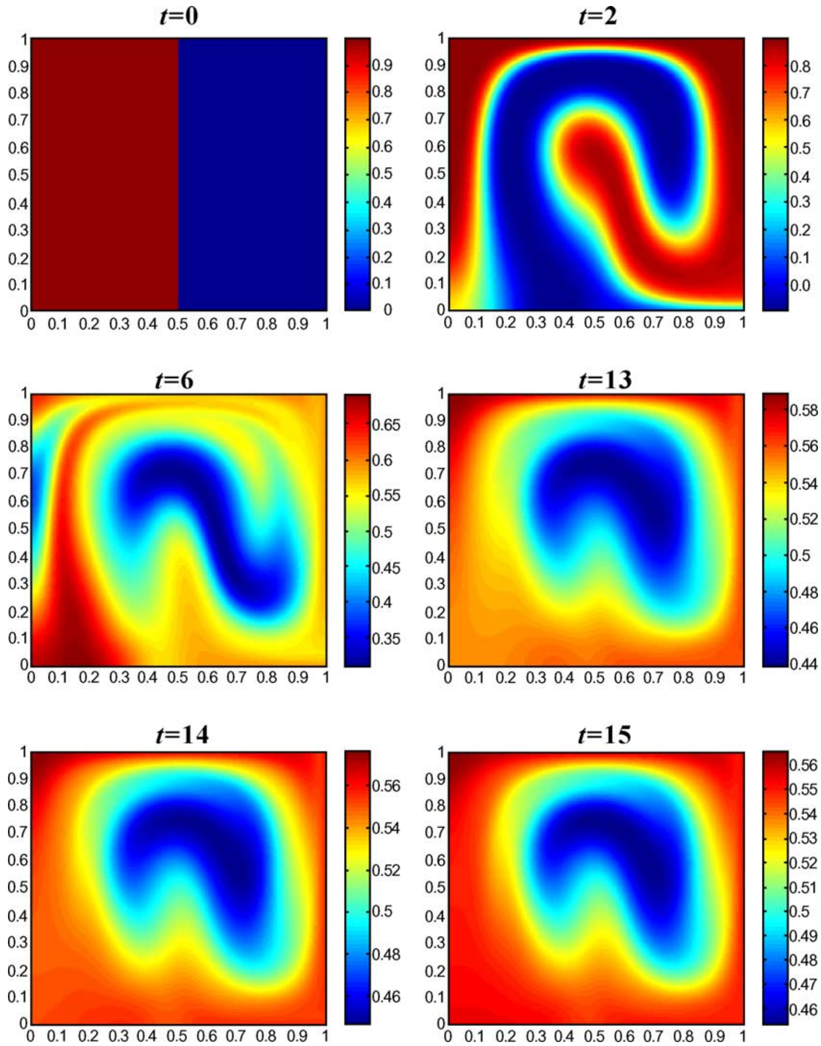

Fig. 3. Convergence of the concentration to a strange eigenmode. The figures show snapshots of the concentration field $c(\mathbf{x}, t)$ at multiples of $T = 1$, the time period of the velocity field.

The emergence of strange eigenmodes suggests a deeper mechanism behind the exponential decay of tracer variance, one that cannot be captured by the elementary L^2 estimates of Proposition 2. In the following sections, we describe this deeper mechanism using the phase–space geometry of the advection–diffusion equation.

3. Inertial manifold and generalized strange eigenmodes

Here we show that under certain conditions, the advection–diffusion equation (6) admits a finite-dimensional invariant manifold that inherits any special time-dependence, periodic or quasiperiodic, that $\mathbf{v}(\mathbf{x}, t)$ may have. Under further conditions, this finite-dimensional invariant manifold turns out to be an *inertial manifold* that attracts all solutions of (6). In that case, the asymptotics of $c(\mathbf{x}, t)$ will always be governed by a finite-dimensional system of time-dependent linear differential equations, obtained by reducing (6) to the inertial manifold. An independent set of solutions to this equation can be thought of as a set of generalized strange eigenmodes.

3.1. Invariant and inertial manifolds

To formulate our main result, we first introduce the negative Laplacian operator

$$A = -\Delta, \tag{17}$$

defined on mean-zero concentration fields that satisfy the boundary condition in (6), and admit two square-integrable derivatives in the interior of the physical domain S. Specifically, we define A on the function space

$$D(A) = \left\{ c \in H^2(\tilde{S}) : \langle c \rangle = 0, \qquad \left.\frac{\partial c}{\partial n}\right|_{\partial S} = 0 \right\} \tag{18}$$

with

$$\tilde{S} = S - \partial S, \tag{19}$$

denoting the interior of the physical domain S, and with $H^2(\tilde{S})$ denoting the Sobolev space of square-integrable functions with two square-integrable derivatives over $\tilde{S}$ (see [1]). In (18) we fixed Neumann boundary conditions for c, but our forthcoming analysis is equally valid for periodic or Dirichlet boundary conditions.

By classic results, A is a self-adjoint operator with positive real eigenvalues

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots \tag{20}$$

and with corresponding real eigenfunctions $e_1(\mathbf{x}), \dots, e_n(\mathbf{x}), \dots$

We assume that the velocity field $\mathbf{v}(\mathbf{x}, t)$ appearing in (6) is uniformly bounded over S, i.e., there exists a constant $v_0 > 0$ such that

$$|\mathbf{v}(\mathbf{x}, t)| \le v_0 \tag{21}$$

for all times in S. We make no assumption about the incompressibility of $\mathbf{v}$ for the following result.

Theorem 1. *Assume that $f(\mathbf{x}, t) \equiv 0$ in the advection–diffusion equation (6).*

(i) *If, for a positive integer $N > 0$, the eigenvalues of the operator $-\Delta$ satisfy the gap condition*

$$\mu_{N+1} - \mu_N > \frac{2^8\pi}{e}\frac{v_0^2}{\kappa}, \tag{22}$$

then Eq. (6) *admits an N-dimensional invariant manifold* $\mathcal{M}(t)$ *in the function space* $H^1(S)$. *The manifold* $\mathcal{M}(t)$ *depends continuously on t. Furthermore, if* $\mathbf{v}(\mathbf{x}, t)$ *is periodic or quasiperiodic in time, then so is* $\mathcal{M}(t)$.

(ii) *If the eigenvalues of the operator* $-\Delta$ *satisfy the stronger gap condition*

$$0 \leq \limsup_{n\to\infty} \frac{32\sqrt{\pi}v_0}{\sqrt{e\kappa}\sqrt{\mu_{n+1}-\mu_n} - 16\sqrt{\pi}v_0} < 1, \tag{23}$$

then Eq. (6) *admits an inertial manifold (a finite-dimensional attracting invariant manifold)* $\mathcal{M}(t)$ *in the function space* $H^1(S)$. *The manifold* $\mathcal{M}(t)$ *depends continuously on t. Furthermore, if* $\mathbf{v}(\mathbf{x}, t)$ *is periodic or quasiperiodic in time, then so is* $\mathcal{M}(t)$.

We prove a stronger result in Appendix C from which the above theorem follows. The stronger result establishes the existence of $\mathcal{M}(t)$ for an abstract parabolic equation in the cruder function space $H^{2\alpha}$ for any $\alpha \in (0, 1)$. Setting $\alpha = 1/2$ in this abstract result yields the condition (23) of Theorem 1.

By the smoothing property of the parabolic advection–diffusion equation (see [15]), *any* square-integrable initial concentration $c_0(\mathbf{x}) = c(\mathbf{x}, t_0)$ becomes a function in $H^1(S)$ *immediately* after the initial time t_0, thus Theorem 1 is strong enough to apply for any realistic choice of the initial tracer distribution.

Notice that for a fixed velocity field $\mathbf{v}(\mathbf{x}, t)$ and a fixed diffusivity κ, the gap conditions (22) and (23) will automatically hold if

$$\limsup_{n\to\infty} (\mu_{n+1} - \mu_n) = \infty. \tag{24}$$

This last condition is true, for instance, for any two-dimensional rectangular domain $S = [0, 2\pi/a] \times [0, 2\pi/b]$ with $(a/b)^2$ rational, including the case of a square domain (see [20]). As a consequence, we obtain the following specific result.

Theorem 2. *Let S be a two-dimensional rectangular domain* $S = [0, 2\pi/a] \times [0, 2\pi/b]$ *with* $(a/b)^2$ *rational. Then there exists an integer* $N > 0$ *such that the advection–diffusion equation* (6) *admits an N-dimensional time-dependent inertial manifold* $\mathcal{M}(t)$ *in the function space* $H^1(S)$. *The manifold* $\mathcal{M}(t)$ *depends continuously on t. Furthermore, if* $\mathbf{v}(\mathbf{x}, t)$ *is periodic or quasiperiodic in time, then so is* $\mathcal{M}(t)$.

For three-dimensional flows, condition (23) becomes more restrictive, and may only hold for larger values of κ, as the example below shows.

Example 4 (Gap conditions for a three-dimensional domain). Consider the cubic domain $S = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. For this physical domain, the gap between adjacent eigenvalues of $-\Delta$ equals 0, 1, 2, or 3 (cf. [20]). Therefore, a finite-dimensional invariant manifold exists by (i) of Theorem 1 if

$$\kappa > \frac{2^8\pi v_0^2}{3e}.$$

Furthermore, we have

$$\limsup_{n\to\infty} \frac{32\sqrt{\pi}v_0}{\sqrt{e\kappa}\sqrt{\mu_{n+1}-\mu_n} - 16\sqrt{\pi}v_0} = \frac{32\sqrt{\pi}v_0}{\sqrt{e\kappa}\sqrt{3} - 16\sqrt{\pi}v_0},$$

thus an inertial manifold exists by (ii) of Theorem 1 if

$$\kappa > \frac{3 \times 2^8\pi v_0^2}{e}.$$

3.2. Generalized strange eigenmodes

Under the appropriate spectral gap condition, Theorem 1 guarantees the existence of a continuous-in-time inertial manifold $\mathcal{M}(t)$ for any bounded velocity field $\mathbf{v}$. Because we do not assume differentiability for $\mathbf{v}$, the manifold $\mathcal{M}(t)$ will exist in any physically relevant velocity field, such as a given realization of a turbulent velocity field, provided that the spectral gap condition (23) holds. Accordingly, the asymptotic tracer concentration is always governed by a finite-dimensional linear system of time-dependent ODEs.

As a consequence, the asymptotic tracer concentration on the inertial manifold is of the form

$$c_\infty(\mathbf{x}, t) = \langle c_0 \rangle + \sum_{k=1}^{N} \gamma_k(t)\varphi_k(\mathbf{x}, t), \qquad \mathcal{M}(t) = \mathrm{span}\{\varphi_k(\mathbf{x}, t)\}_{k=1}^{N}, \tag{25}$$

where $\gamma(t) = [\gamma_1(t), \dots, \gamma_n(t)]$ is the solution of a finite-dimensional set of linear ODEs with coefficient matrix

$$\mathbf{M}(t) = (\kappa\Delta - \mathbf{v}\cdot\nabla)|_{\mathcal{M}(t)}. \tag{26}$$

One may refer to the basis functions $\varphi_k(\mathbf{x}, t)$ on $\mathcal{M}(t)$ as *generalized strange eigenmodes*, although these eigenmodes have a general time-dependence that makes their visualization difficult. They may evolve without recurrent spatial features, remaining invisible under iterations of the time-T map $c(\mathbf{x}, t) \mapsto c(\mathbf{x}, t+T)$ for any choice of T. By the universal estimate (9), however, all eigenmodes decay at least exponentially, thus the one showing the slowest decay will dominate asymptotically.

The emergence of a dominant eigenmodes in aperiodic flows is consistent with the observations of Hu and Pierrehumbert [16] and Sukhatme and Pierrehumbert [30], who find that the tracer PDF approaches a self-similar form when normalized by the tracer variance. But self-similarity of the PDF does *not* follow automatically from the result (25): the general time-dependence of γ_k and φ_k disallows here the argument that we use in Section 4.4 to establish self-similarity in the time-periodic case.

4. Strange eigenmodes for time-periodic flows

We now describe the tracer dynamics on the invariant manifold $\mathcal{M}(t)$ for the case of time-periodic flows. As it turns out, asymptotic tracer patterns are generated by a finite number of Floquet solutions on $\mathcal{M}(t)$.

Infinite-dimensional Floquet theory is well-developed for parabolic partial differential equations (see, e.g. [9,10,18]). Surprisingly, however, none of the available results apply to the advection–diffusion equation (6), as we discuss in Appendix D.1. This is why we restrict (6) first to its inertial manifold, then apply classical Floquet theory as a second step.

4.1. Classical Floquet theory

For what follows, we first review the elements of classical Floquet theory for finite-dimensional, time-periodic, linear systems of differential equations. Consider the linear system

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}, \qquad \mathbf{A}(t) = \mathbf{A}(t+T), \tag{27}$$

where $\mathbf{y}$ is an n-dimensional vector and $\mathbf{A}(t)$ is an $n \times n$ matrix depending continuously on t. Floquet theory guarantees that any fundamental matrix solution $\mathbf{\Phi}(t)$ of (27) can be written as a product

$$\mathbf{\Phi}(t) = \mathbf{P}(t)\,\mathrm{e}^{\mathbf{B}t}, \tag{28}$$

where $\mathbf{P}(t)$ is T-periodic $n \times n$ matrix, and $\mathbf{B}$ is a constant $n \times n$ matrix (see, e.g. [14,33]).

As a consequence of (28), any solution of (27) is a linear combination of Floquet solutions of the form

$$\mathbf{y}(t) = \mathrm{e}^{\lambda t}[\boldsymbol{\varphi}_0(t) + t\boldsymbol{\varphi}_1(t) + t^2\boldsymbol{\varphi}_2(t) + \cdots + t^{l-1}\boldsymbol{\varphi}_{l-1}(t)],$$

where the complex constant λ is an eigenvalue of geometric multiplicity l for the matrix $\mathbf{B}$, and $\boldsymbol{\varphi}_k(t)$ are time-periodic complex functions.

In the generic case, $\mathbf{B}$ is semisimple (has N linearly independent eigenvectors), giving rise to a *simple Floquet solution*

$$\mathbf{y}(t) = \mathrm{e}^{\lambda t}\boldsymbol{\varphi}_0(t).$$

If λ is a real eigenvalue, then the function $\boldsymbol{\varphi}_0(t)$ is also real. If λ is complex, then the corresponding simple Floquet solution has two frequencies, $2\pi/T$ and $\mathrm{Im}\,\lambda$.

4.2. Floquet theory for the advection–diffusion equation

We now assume that the velocity field $\mathbf{v}$ in the advection–diffusion equation (6) is time-periodic with period T. By Theorem 1, the time-periodicity of $\mathbf{v}(\mathbf{x}, t)$ implies the time-periodicity of the manifold $\mathcal{M}(t)$, thus the advection–diffusion equation (6) reduced to $\mathcal{M}(t)$ is of the type (27). Applying classical Floquet theory to this reduced system, we obtain the following result.

Theorem 3. *Assume that the velocity field* $\mathbf{v}(\mathbf{x}, t)$ *is time-periodic and continuous in t. Assume further that* $f(\mathbf{x}, t) \equiv 0$ *in Eq. (6), and the gap condition (22) is satisfied. Then any concentration field* $c(\mathbf{x}, t)$ *contained in the invariant manifold* $\mathcal{M}(t)$ *can be written in the form*

$$c(\mathbf{x}, t) = \langle c_0\rangle + \sum_{k=0}^{N-1} \mathrm{e}^{-\lambda_k t}[\varphi_k^0(\mathbf{x}, t) + t\varphi_k^1(\mathbf{x}, t) + \cdots + t^{l(k)}\varphi_k^{l(k)}(\mathbf{x}, t)],$$

where $l(k) \geq 0$ *are nonnegative integers, and the constants* $\lambda_k \in \mathbb{C}$ satisfy

$$\mathrm{Re}\,\lambda_0 \leq \mathrm{Re}\,\lambda_1 \leq \cdots \leq \mathrm{Re}\,\lambda_{N-1}.$$

This result is a direct application of classical Floquet theory to the finite-dimensional linear system on $\mathcal{M}(t)$. The initial mean concentration $\langle c_0\rangle$ appears in the above result because we originally prove the existence of $\mathcal{M}(t)$ for the field $c(\mathbf{x}, t) - \langle c_0\rangle$.

The continuous dependence of $\mathcal{M}(t)$ on t (guaranteed by Theorem 1) and the continuity of $\mathbf{v}(\mathbf{x}, t)$ in t (assumed in Theorem 3) are both essential: they ensure continuity for the reduced linear operator $\mathbf{M}(t)$ in (26), and hence make classical Floquet theory applicable on $\mathcal{M}(t)$.

Typically, the Floquet matrix $\mathbf{B}$ associated with the reduced advection–diffusion equation is semisimple, i.e., $\mathbf{B}$ has N linearly independent eigenvectors. This is the case for spatial domains without any particular symmetry, or for symmetric domains on which the Laplacian Δ still has a complete set of eigenfunctions. Examples of such symmetric domains include the square and the circle. We will refer to the case of a semisimple $\mathbf{B}$ as the *generic case*.

In the generic case, we have $l(k) = 0$ for all k in the above theorem. As a result, any solution on the inertial manifold can be written as the sum of N simple Floquet solutions

$$c(\mathbf{x}, t) = \langle c_0\rangle + \sum_{k=1}^{N} \mathrm{e}^{-\lambda_k t}\varphi_k(\mathbf{x}, t).$$

For large enough times, therefore, the slowest-decaying Floquet solution will determine the asymptotic behavior of c on the manifold $\mathcal{M}(t)$:

$$c(\mathbf{x},t) \approx \langle c_0\rangle + c_\infty(\mathbf{x},t) = \langle c_0\rangle + \mathrm{e}^{-\lambda_0 t}\varphi_0(\mathbf{x},t), \quad \text{as } t\to\infty. \tag{29}$$

Such an asymptotic factorization of $c(\mathbf{x},t)$ agrees with the numerical observations of Pierrehumbert [26] and the experimental results of Rothstein et al. [29]. Following Pierrehumbert, we shall refer to the function $\varphi_0(\mathbf{x},t)$ as a *strange eigenmode*. Over intermediate time scales, numerical or experimental observations may differ from (29) if some of the exponents $\lambda_1,\dots,\lambda_{N-1}$ are close to λ_0. In such a case, two or more eigenmodes will contribute significantly to $c(\mathbf{x},t)$ for long periods of time, and the asymptotic formula (29) is only observed afterwards.

4.3. Convergence to strange eigenmodes

Theorem 3 establishes the existence of strange eigenmodes on the invariant manifold $\mathcal{M}(t)$, but it does *not* guarantee that an *arbitrary* solution of the advection–diffusion equation (6) will converge to strange eigenmodes. Such convergence only follows if $\mathcal{M}(t)$ is in fact an inertial manifold, which requires the stronger gap condition (23) to hold. Under a yet stronger gap condition, we shall even show that the rate of convergence to $\mathcal{M}(t)$ is faster than the decay rate within $\mathcal{M}(t)$, and hence strange eigenmodes prevail very early in the tracer evolution. Specifically, we prove the following result in Appendix D.

Theorem 4. *Assume that the velocity field* $\mathbf{v}(\mathbf{x},t)$ *is time-periodic,* $f(\mathbf{x},t)\equiv 0$, *and the gap condition* (23) *is satisfied. Then the following hold*:

(i) *The advection–diffusion equation* (6) *admits a complete set of Floquet solutions, i.e., for arbitrary* $\epsilon>0$ *there exist an integer* $N>0$ *such that any concentration field* $c(\mathbf{x},t)$ *can be written in the form*

$$c(\mathbf{x},t) = \langle c_0\rangle + \sum_{k=0}^{N-1}\mathrm{e}^{-\lambda_k t}[\varphi_k^0(\mathbf{x},t) + t\varphi_k^1(\mathbf{x},t) + \cdots + t^{l(k)}\varphi_k^{l(k)}(\mathbf{x},t)] + \hat{c}(\mathbf{x},t), \tag{30}$$

$$\|\hat{c}\| + \|\nabla\hat{c}\| \le \epsilon\,\mathrm{e}^{-\rho t},$$

where λ_k *are complex constants,* $l(k)\ge 0$ *are nonnegative integers,* K *is a positive constant, and*

$$\rho = \kappa\mu_N - 4\pi^2 v_0^2. \tag{31}$$

(ii) *If the stronger gap condition*

$$\limsup_{n\to\infty}\left(\mu_{n+1}-\mu_n-\frac{v_0}{\kappa}\sqrt{\mu_n}\right) = \infty, \tag{32}$$

holds, then $\|\hat{c}\| + \|\nabla\hat{c}\|$ *decays to zero faster than* $\|c-\hat{c}\| + \|\nabla(c-\hat{c})\|$ *does*.

As in Theorem 3, formula (30) will have $l(k)=0$ for all k in the generic case. Thus, under condition (23), the concentration field is the sum of N simple Floquet solutions plus an arbitrarily small $\mathcal{O}(\mathrm{e}^{-\rho t})$ term:

$$c(\mathbf{x},t) = \langle c_0\rangle + \sum_{k=0}^{N-1}\mathrm{e}^{-\lambda_k t}\varphi_k(\mathbf{x},t) + \mathcal{O}(\mathrm{e}^{-\rho t}). \tag{33}$$

In statement (i) of Theorem 3, $\rho\le\lambda_k$ is possible, thus the decay of $\hat{c}(\mathbf{x},t)$, the noneigenmode contribution, may be slower than the decay of some Floquet modes. Still, for large enough N, the exponent $-\rho$ tends to $-\infty$ by (31), thus $\hat{c}$ will decay faster than the first few dominant Floquet modes.

If the stronger gap condition (32) also holds, then statement (ii) of the theorem guarantees a decay for $\hat{c}$ that is faster than the decay of *all* of the N Floquet modes. In that case, not only is the initial error between the concentration and N strange eigenmodes arbitrarily small, but this error also decays faster than the strange eigenmodes do. As a consequence, the strange eigenmodes already become visible on short time scales.

Example 5 (The strong gap condition (32) for a square domain). Let us consider $S = [0, \pi] \times [0, \pi]$, a square domain on the two-dimensional plane, and let $0 < \mu_1 \leq \mu_2 \leq \cdots$ denote the eigenvalues of the operator $A = -\Delta$ on S. We recall that

$$\lim_{n\to\infty} \mu_n = \infty. \tag{34}$$

We define the quantity

$$\text{ratio}(n) = \frac{\mu_{n+1} - \mu_n}{\sqrt{\mu_n}}$$

and show its numerically computed distribution as a function of n in Fig. 4. Note that ratio(n) remains bounded, thus its lim sup is finite. Our calculation suggests the numerical value

$$\limsup_{n\to\infty} \text{ratio}(n) = a \approx 0.04.$$

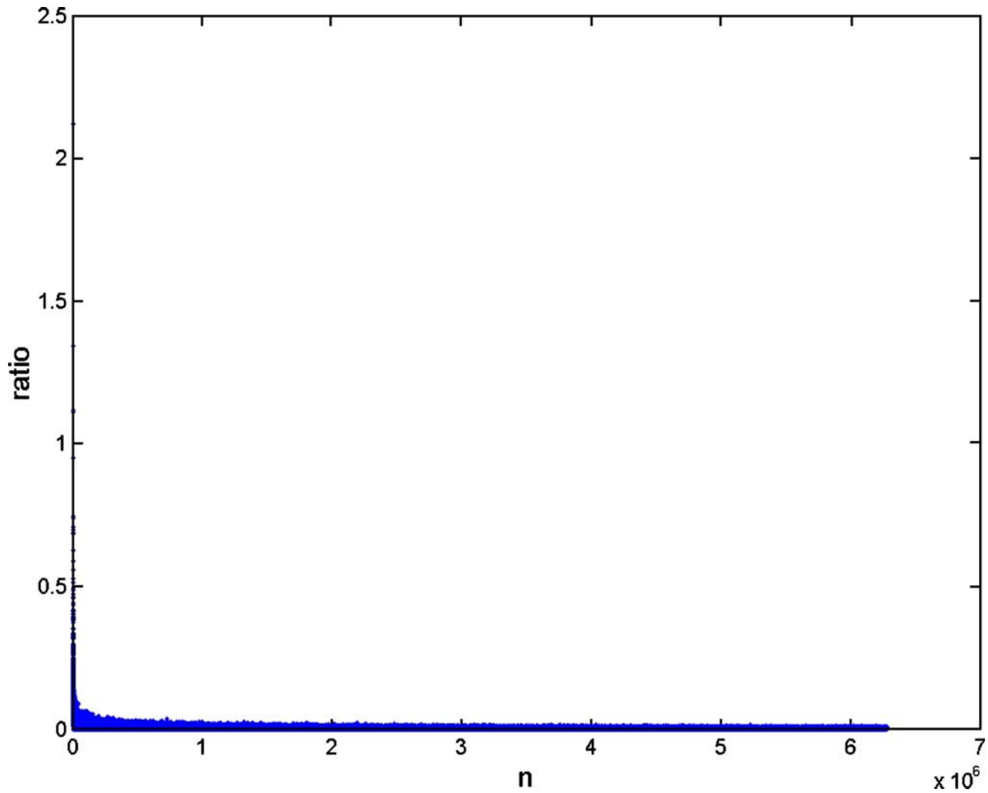

Fig. 4. The graph of ratio(n) for n values up to 6.3×10^6. The maximum of the graph is approximately 2.1.

We then have

$$\limsup_{n\to\infty} \frac{\mu_{n+1} - \mu_n - (1/2)a\sqrt{\mu_n}}{\sqrt{\mu_n}} = \frac{a}{2} > 0,$$

thus (34) implies

$$\limsup_{n\to\infty} (\mu_{n+1} - \mu_n - \tfrac{1}{2}a\sqrt{\mu_n}) = \infty.$$

As a result, for

$$\frac{v_0}{\kappa} < \frac{a}{2} \approx 0.02,$$

we have

$$\limsup_{n\to\infty} \left(\mu_{n+1} - \mu_n - \frac{v_0}{\kappa}\sqrt{\mu_n}\right) \geq \limsup_{n\to\infty} \left(\mu_{n+1} - \mu_n - \frac{1}{2}a\sqrt{\mu_n}\right) = \infty.$$

Hence the gap condition (32) is satisfied for the domain $S = [0, \pi] \times [0, \pi]$ if $\kappa > 50v_0$.

Theorem 4 clarifies some of the views in the literature about qualitatively different stages of tracer evolution. Some authors distinguish between super-exponential decay, exponential decay, and near-constant behavior for the tracer variance, noting that the strange eigenmode description may only be valid for certain types of flows (those with no barriers), or for certain stages of the tracer evolution (asymptotic time scale).

By our results, no such distinction is justified: under condition (23), which holds for the two-dimensional geometries considered in the literature, any concentration $c(\mathbf{x}, t)$ is close to a finite set of strange eigenmodes *regardless* of the mixing properties of the flow, and regardless of the time that has elapsed. Generically, a single mode with the weakest decay will prevail in the end, but the intermediate stages of tracer mixing are also governed by finitely many additional strange eigenmodes that have yet to decay to invisibility.

Example 6 (Variance decay patterns generated by strange eigenmodes). To illustrate the above discussion, we consider the function

$$C(t) = \|c(\mathbf{x}, t)\|^2 = \mathrm{e}^{-2\lambda_0 t} + \mathrm{e}^{-2\lambda_1 t} + \mathrm{e}^{-2\lambda_2 t},$$

to model the tracer variance evolution on a three-dimensional inertial manifold $\mathcal{M}(T)$ of the Poincaré map $c(\mathbf{x}, t) \mapsto c(\mathbf{x}, t + T)$. Here the inertial manifold is spanned by three eigenmodes with Floquet exponents $-\lambda_k$. For simplicity, we have assumed

$$\|\varphi_k(\mathbf{x}, T)\|^2 = 1, \qquad \int_S \varphi_k(\mathbf{x}, T)\varphi_j(\mathbf{x}, T)|_{j\neq k}\,\mathrm{d}V = 0.$$

We show the decay of the model-variance, $C(t)$, for three different choices of the Floquet exponents in Fig. 5.

Fig. 5a shows the type of variance decay that is widely considered to be the indication of a single strange eigenmode (see, e.g. [30]). Fig. 5b shows the type of decay seen in flows with mixing barriers, for which the strange eigenmode description is generally viewed inapplicable (see, e.g. [3,30]). Finally, Fig. 5c shows what is often thought to be a super-exponential transient preceding the strange eigenmode state (see, e.g. [3,11,35]).

Despite the above views, all three phenomena shown in Fig. 5 occur on inertial manifolds spanned by Floquet modes. All three types of variance decay are exponential: it is only the ratio of the participating exponents that is different in each case.

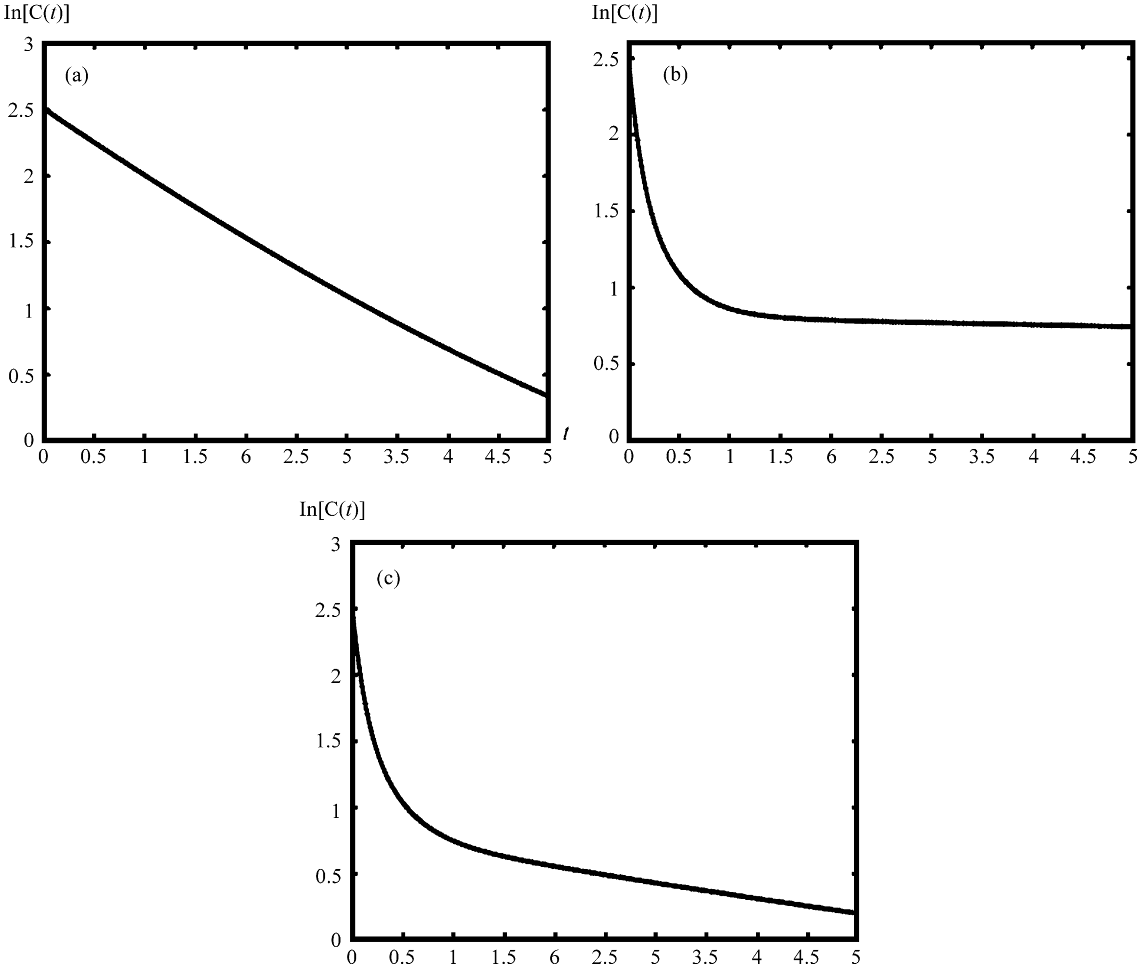

Fig. 5. Decay of the logarithm of the model-variance $C(t)$ for different parameter values: (a) $\lambda_0 = 0.25, \lambda_1 = 0.3, \lambda_2 = 0.35$; (b) $\lambda_0 = 0.01, \lambda_1 = 3, \lambda_2 = 10$; (c) $\lambda_0 = 0.1, \lambda_1 = 3, \lambda_2 = 10$.

4.4. Statistical implication: self-similarity of tracer PDFs

We define the asymptotic tracer PDF as

$$\mathrm{PDF}_t^\infty(z) = \langle p\{c_\infty(\mathbf{x}, t) < z\}\rangle,$$

where $p\{B\}$ measures the probability of event B, the operation $\langle\cdot\rangle$ refers to spatial averaging over the domain S, and $c_\infty(\mathbf{x}, t)$ is the asymptotic tracer concentration on the invariant manifold $\mathcal{M}(t)$ (cf. (29)). Since we have

$$\begin{aligned}\mathrm{PDF}_{t+T}^\infty(z) &= \langle p\{c_\infty(\mathbf{x}, t+T) < z\}\rangle = \langle p\{\mathrm{e}^{-\lambda_0(t+T)}\varphi_0(\mathbf{x}, t+T) < z\}\rangle \\ &= \langle p\{c_\infty(\mathbf{x}, t) < \mathrm{e}^{\lambda_0 T} z\}\rangle = \mathrm{PDF}_t^\infty(\mathrm{e}^{\lambda_0 T} z),\end{aligned}$$

we can write

$$\|\mathrm{PDF}_{t+T}^\infty\|^2 = \int_{-\infty}^{+\infty}[\mathrm{PDF}_{t+T}^\infty(z)]^2\,\mathrm{d}z = \int_{-\infty}^{+\infty}[\mathrm{PDF}_t^\infty(w)]^2\frac{\mathrm{d}w}{\mathrm{e}^{\lambda_0 T}} = \mathrm{e}^{-\lambda_0 T}\|\mathrm{PDF}_t^\infty\|^2.$$

We therefore obtain

$$\frac{\|\mathrm{PDF}_{t+T}^{\infty}\|^2}{\|c_\infty(\mathbf{x},t+T)\|^2}=\frac{\|\mathrm{PDF}_t^{\infty}\|^2}{\|c_\infty(\mathbf{x},t)\|^2}, \tag{35}$$

because $c_\infty(\mathbf{x},t+T)=\mathrm{e}^{-\lambda T}c_\infty(\mathbf{x},t)$.

Under the spectral gap condition (23), all solutions converge to the inertial manifold, and hence formula (35) gives asymptotic self-similarity in the sense of Sukhatme and Pierrehumbert [30] for *any* concentration field.

4.5. Generic form of strange eigenmodes

We now consider the generic case in which the concentration $c(\mathbf{x},t)$ converges to a simple Floquet solution as described in (29). From now on, an overbar will refer to time-averaging over the interval $[0,T]$, i.e., we write

$$\bar{a}=\frac{1}{T}\int_0^{\mathrm{T}}a(t)\,\mathrm{d}t$$

for any function $a(t)$. We have the following result on the relation between Floquet exponents and the corresponding strange eigenmodes.

Theorem 5. *For a generic, two-dimensional, time-periodic, incompressible velocity field defined on the spatial domain S, the concentration $c(\mathbf{x},t)-\langle c_0\rangle$ converges to a Floquet solution of the form*

$$c_\infty(\mathbf{x},t)=\exp\left[-\kappa\left(\frac{\|\nabla\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2}+\mathrm{i}\frac{\|\boldsymbol{\nabla}\bar{\varphi}_0\|^2\|\mathrm{Re}\,\bar{\varphi}_0\|^2-\|\bar{\varphi}_0\|^2\|\mathrm{Re}\,\boldsymbol{\nabla}\bar{\varphi}_0\|^2}{\varkappa\|\bar{\varphi}_0\|^2\overline{\langle\mathrm{Re}\,\varphi_0\,\mathrm{Im}\,\varphi_0\rangle}}\right)t\right]\varphi_0(\mathbf{x},t), \tag{36}$$

where $\varphi_0(\mathbf{x},t)$ and $\boldsymbol{\nabla}\varphi_0(\mathbf{x},t)$ are square-integrable complex functions for all $t>0$, and the constant $\varkappa$ is defined in (3).

We prove the above theorem in Appendix E. Note that (36) implies

$$\|\bar{c}_\infty\|^2=\exp\left[-2\kappa\frac{\|\nabla\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2}t\right]\|\overline{\varphi_0}\|^2,\qquad \|\boldsymbol{\nabla}\bar{c}_\infty\|^2=\exp\left[-2\kappa\frac{\|\boldsymbol{\nabla}\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2}t\right]\|\boldsymbol{\nabla}\bar{\varphi}_0\|^2. \tag{37}$$

4.6. Quasiperiodic and subharmonic eigenmodes

Formula (36) shows that c_∞ is either time-periodic (as observed originally by Pierrehumbert [26]) or *time-quasiperiodic*. The latter case occurs if $\mathrm{Im}(\lambda_0)$, the imaginary part of the exponent in (36), is nonzero and rationally independent of $2\pi/T$. If $\mathrm{Im}(\lambda_0)$ is rationally related to $2\pi/T$, then the resulting pattern is again time-periodic, but with a period equal to the maximum of $2\pi/\mathrm{Im}(\lambda_0)$ and T. Thus, if $2\pi/\mathrm{Im}(\lambda_0)>T$, then we have a *subharmonic* eigenmode.

Example 7 (Subharmonic eigenmode). The velocity field

$$v_1(x,y,t)=\sin(\pi x)[\cos(\pi y)\cos(\pi x)+\cos(2\pi t)\sin(\pi y)]+\sin(\pi x)\sin(\pi y)\cos(2\pi t)\cos(\pi y),$$
$$v_2(x,y,t)=-\cos(\pi x)[\sin(\pi y)\cos(\pi x)+\cos(2\pi t)\sin(\pi y)]+\sin^2(\pi x)\sin(\pi y),$$

whose time-period is $T=1$, gives rise to a strange eigenmode whose period is $\hat{T}=7$. We show this period-seven eigenmode in Fig. 6 after a long spin-up time of $\Delta t=43$. For reasons of symmetry, $\mathrm{Im}(\lambda_0)=2\pi/\hat{T}$ is rationally related to $2\pi/T$, resulting in a pattern whose period is longer than the period of the velocity field.

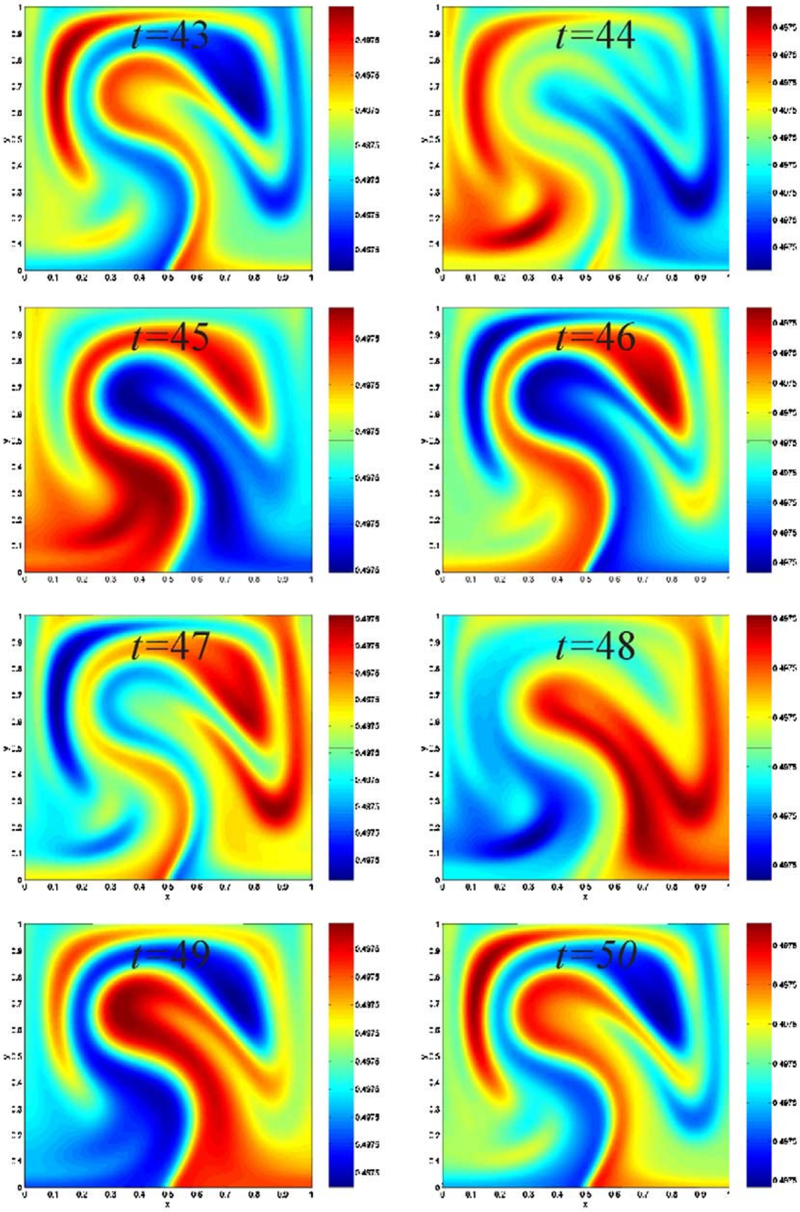

Fig. 6. Snapshots of a strange eigenmode with time-period $\hat{T} = 7$, taken at multiples of the velocity period $T = 1$. Note that the exact pattern at $t = 43$ only comes back—with weaker intensity—at $t = 50$. (The parameters are $\kappa = 0.001$, $f \equiv 0$, $S = [0, 1] \times [0, 1]$; initial condition as in Fig. 3.)

4.7. Numerical extraction of the dominant Floquet exponent

Here we discuss how the dominant (i.e., slowest-decaying) Floquet exponent can be extracted from simulated or measured concentration data *without* assuming that the general Floquet expansion (30) simplifies to (33). In other words, we do not assume here that the advection–diffusion equation reduced to $\mathcal{M}(t)$ admits a semisimple Floquet decomposition.

Under the conditions of Theorem 3, if $-\lambda_0$ is a Floquet exponent for Eq. (6), then there exists a nonzero initial condition c_0 that leads to a solution

$$c(\mathbf{x},t) = \langle c_0\rangle + \mathrm{e}^{-\lambda_0 t}[\varphi_0^0(\mathbf{x},t) + t\varphi_0^1(\mathbf{x},t) + \cdots + t^{l(0)}\varphi_0^{l(0)}(\mathbf{x},t)].$$

Subtracting $\langle c_0\rangle$, multiplying both sides by their complex conjugates, and integrating over the domain S, we obtain

$$\|c - \langle c_0\rangle\| = \mathrm{e}^{-\mathrm{Re}\,\lambda_0 t}\|\varphi_0^0 + t\varphi_0^1 + \cdots + t^{l(0)}\varphi_0^{l(0)}\|.$$

Thus

$$t\,\mathrm{Re}\,\lambda_0 = -\ln\frac{\|c-\langle c_0\rangle\|}{\|\varphi_0^0 + t\varphi_0^1 + \cdots + t^{l(0)}\varphi_0^{l(0)}\|} = -\ln\frac{\|c-\langle c_0\rangle\|}{c_0-\langle c_0\rangle\|} + \ln\frac{\|\varphi_0^0 + t\varphi_0^1 + \cdots + t^{l(0)}\varphi_0^{l(0)}\|}{\|c_0-\langle c_0\rangle\|}.$$

Therefore, by the boundedness of $\varphi_0^0, \ldots, \varphi_0^{l(0)}$, we obtain

$$\mathrm{Re}\,\lambda_0 = \limsup_{t\to\infty}\frac{1}{t}\ln\frac{\|c_0-\langle c_0\rangle\|}{\|c-\langle c_0\rangle\|}. \tag{38}$$

Calculating the above exponent for a general concentration $c(\mathbf{x},t)$ will render the real part of the weakest-decaying strange eigenmode.

Example 8 (Dominant Floquet exponent in Example 3). We now reconsider the velocity field (14) from Section 2.2 with the same initial condition and diffusivity. Beyond $V = \|c - \langle c_0\rangle\|$ and the upper estimate (9), Fig. 7 shows the exponential decay rate we extracted using formula (38). Note that the final asymptotic decay of the concentration variance is indeed dominated by the exponent $\mathrm{Re}\,\lambda_0$ given in (38).

Formula (37) gives another way to identify the dominant strange eigenmode for time-periodic velocity fields. In particular, (37) gives

$$\frac{\|\nabla\bar{c}_\infty\|^2}{\|\bar{c}_\infty\|^2} = \frac{\|\nabla\bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2},$$

implying the asymptotic relation (cf. (29))

$$c(\mathbf{x},t) \approx \langle c_0\rangle + \exp\left[-\kappa\frac{\|\nabla\bar{c}\|^2}{\|\bar{c}-\langle c_0\rangle\|^2}t\right]\varphi_0(\mathbf{x},t) \qquad \text{as } t\to\infty. \tag{39}$$

We then obtain the following expressions for the weakest-decaying Floquet exponent and the corresponding eigenmode:

$$\mathrm{Re}\,\lambda_0 = \lim_{t\to\infty}\frac{\kappa\|\int_t^{t+T}\nabla c\,\mathrm{d}t\|^2}{\|\int_t^{t+T}(c-\langle c_0\rangle)\,\mathrm{d}t\|^2}, \qquad \varphi_0(\mathbf{x},t) = \lim_{t\to\infty}[c(\mathbf{x},t)-\langle c_0\rangle]\,\mathrm{e}^{\lambda_0 t}.$$

We close by noting that the exponent in formula (39) clarifies why the quantity of $\|\nabla c\|^2/\|c\|^2$, proposed by Pattanayak [25] is indeed a relevant indicator of the strange eigenmode stage in the evolution of c.

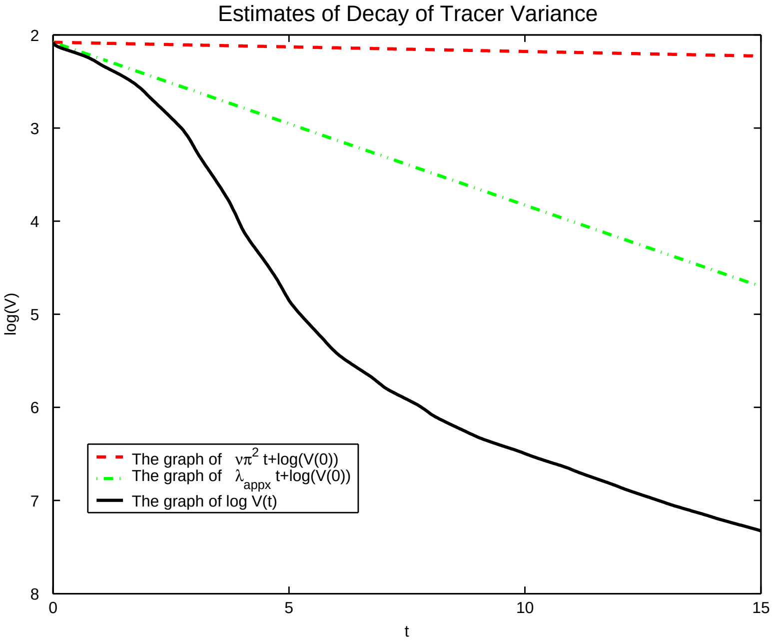

Fig. 7. Decay of tracer variance (solid line), universal upper estimate for the decay (dashed lined), and decay rate extracted using formula (38) (dash dotted line).

4.8. The conservative limit of strange eigenmodes

While an analytic computation of strange eigenmodes appears beyond reach, one may try to approximate them for small $\kappa \geq 0$ in the form

$$\boldsymbol{\varphi}_0(\mathbf{x},t) = \phi_0(\mathbf{x},t) + \kappa\phi_1(\mathbf{x},t) + \mathcal{O}(\kappa^2), \qquad \phi_k(\mathbf{x},t) = \phi_k(\mathbf{x},t+T), \tag{40}$$

where $\phi_0(\mathbf{x},t)$ is a solution of the conservative limiting equation (4).

As we show below, however, an asymptotic expansion (40) with bounded ϕ_1 may only exist for *completely integrable*velocity fields, i.e., for those that generate formally integrable particle motions. Thus, Sukhatme and Pierrehumbert [30] are quite correct when they expect the $\kappa \to 0$ behavior of strange eigenmodes to be delicate by analogy with scalar dynamo problems.

Theorem 6. *Assume that an asymptotic expansion of the form* (40) *exists for a Floquet solution of the advection–diffusion equation* (6). *Let D denote the set in space–time on which* $\nabla\phi_0(\mathbf{x},t) \neq \mathbf{0}$. *Then the domain D is invariant under the velocity field* $\mathbf{v}(\mathbf{x},t)$, *and* $\mathbf{v}(\mathbf{x},t)$ *is completely integrable on D.*

We prove this theorem in Appendix F. The main consequence of this result is that strange eigenmodes in chaotic particle mixing cannot be expanded in terms of the diffusivity parameter κ. In other words, strange eigenmodes are nondifferentiable with respect to the diffusivity at $\kappa = 0$.

Moffatt and Proctor [22] obtained results similar to Theorem 6 for kinematic dynamos. They showed that a topological constraint, the conservation of magnetic helicity, precludes a nonzero growth rate for the magnetic field in the limit of vanishing magnetic diffusivity.

5. Strange eigenmodes in the presence of sources and sinks

Here we briefly consider the long-term behavior of a diffusive tracer $c(\mathbf{x}, t)$ in the full advection–diffusion equation (6), including a nonzero source distribution $f(\mathbf{x}, t)$ on the right-hand side. Let $\bar{c}$ be the solution of

$$\bar{c}_t + \nabla\bar{c}\cdot\mathbf{v} = \kappa\Delta\bar{c} + f - \langle f\rangle, \qquad \bar{c}(\mathbf{x}, t_0) = 0, \qquad \left.\frac{\partial\bar{c}}{\partial n}\right|_{\partial S} = 0$$

and $\hat{c}$ be the solution of

$$\hat{c}_t + \nabla\hat{c}\cdot\mathbf{v} = \kappa\Delta\hat{c}, \qquad \hat{c}(\mathbf{x}, t_0) = c_0(x) - \langle c_0\rangle, \qquad \left.\frac{\partial\hat{c}}{\partial n}\right|_{\partial S} = 0. \tag{41}$$

Then we have

$$c = \langle c_0\rangle + \hat{c} + \bar{c} + \int_0^t \langle f\rangle\,\mathrm{d}\tau,$$

where $c(\mathbf{x}, t)$ is the solution of (6).

The above shows that the full concentration field $c - \langle c_0\rangle$ is the superposition of the particular solution $\bar{c} + \int_0^t \langle f\rangle\,\mathrm{d}\tau$ and the solution $\hat{c}$ of the homogeneous system (41). The homogeneous solution $\hat{c}$ admits strange eigenmodes under the conditions we described earlier, and these strange eigenmodes are linearly superimposed onto the spatiotemporal patterns of the particular solution. Thus, *strange eigenmodes govern the way in which the tracer field converges to the particular solution* $\bar{c} + \int_0^t \langle f\rangle\,\mathrm{d}\tau$.

6. Conclusions

We have shown that if the spectrum of the Laplacian Δ admits large enough gaps, then the advection–diffusion equation (1) possesses a finite-dimensional attracting invariant manifold. This inertial manifold is spanned by generalized strange eigenmodes that simplify to the Floquet-type recurrent eigenmodes of Pierrehumbert [26] in the case of velocity fields with periodic and continuous time-dependence. The slowest-decaying such eigenmode leads to a self-similar tracer PDF in the time-periodic case.

Our results imply that flows with mixing barriers also admit strange eigenmodes, but a single dominant eigenmode may take longer time to emerge. Furthermore, strange eigenmodes compete throughout the whole evolution of the tracer variance, with all but one mode decaying to invisibility over long enough time scales. These results hold for general velocity fields in two- and three-dimensions, although in specific three-dimensional examples our main spectral gap condition may only hold for large enough diffusivity or for small enough velocities.

The zero-diffusivity limit of strange eigenmodes is a natural candidate for perturbation theory, yet may only lead to consistent results for completely integrable velocity fields (cf. Section 4.8). Numerical experiments suggest that strange eigenmodes are supported over invariant sets of the velocity field in the $\kappa \to 0$ limit. This is in agreement with the findings of Voth et al. [34] who observe a perfect relation between unstable manifolds of the velocity field and lines of large gradients in the strange eigenmodes.

The experimental findings of Voth et al. [34] also point to a close connection between large gradients of the finite-time Lyapunov exponent distribution and those of the strange eigenmodes. Thus, while a quantitative prediction of asymptotic tracer decay appears to need more than just Lyapunov exponent statistics (see [11,35]), the spatial distribution of Lyapunov exponents seems intimately linked to strange eigenmodes. The recent work of Pikovsky and Popovych [28] and Gilbert [13] (as well as the references therein) may offer new ways to approximate eigenmodes in the vanishing diffusion limit.

Finding a sharp analytic prediction for the asymptotic decay rate of the tracer variance (i.e., for the exponent of the weakest Floquet mode) remains a challenge, because the $\kappa = 0$ limit of the advection–diffusion equation is singular. A sharp analytic estimate may be possible to derive by exploiting the form of strange eigenmodes in the universal estimates of Section 2.2.

A conceptual question is whether the weakest Floquet exponent indeed becomes a nonzero constant as $\kappa \to 0$. As we noted earlier, Pierrehumbert [26] and Antonsen et al. [3] report asymptotic constancy for two-dimensional flows, but Toussaint et al. [32] observe logarithmic or power-law dependence in κ for some three-dimensional steady flows. Pikovsky and Popovych [28] find that the decay exponent approaches zero in their standard map example. At the same time, Fereday and Haynes [12] find the same limiting exponent to be nonzero for alternating sinusoidal shear-flow maps. On the strict analytic side, even the existence of $\lim_{\kappa\to 0+} \lambda_0(\kappa)$ is questionable, let alone the asymptotic flatness of $\lambda_0(\kappa)$ at $\kappa = 0$.

The asymptotic self-similarity of the tracer PDF also remains to be established for velocity fields with aperiodic time-dependence. We have shown that the tracer concentration converges to solutions of a linear system of ODEs, but this system has general time-dependence, and hence the existence of an exponentially decaying solution—one that would generate PDF self-similarity—is unknown. More work is needed, therefore, to explore the structure of the reduced linear operator $\mathbf{M}(t)$ in (26), which may give clues about the self-similarity observed by Hu and Pierrehumbert [16] and Sukhatme and Pierrehumbert [30].

A further open question is the existence of strange eigenmodes for chemically or biologically active tracers. A promising starting point is the numerical evidence of Muzzio and Liu [23] and Tél et al. [31], both showing recurrent asymptotic patterns for active tracers.

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Appendix A

Here we prove Proposition 1. First, we integrate Eq. (4) over the spatial domain S to obtain

$$0 = \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle + \int_S \boldsymbol{\nabla} c\cdot\mathbf{v}\,\mathrm{d}V = \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle + \int_S [\nabla\cdot(c\mathbf{v}) - c\nabla\cdot\mathbf{v}]\,\mathrm{d}V = \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle + \int_{\partial S} c\mathbf{v}\cdot\mathrm{d}\mathbf{n} = \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle, \quad \text{(A.1)}$$

where we used the boundary conditions assumed for $\mathbf{v}$, as well as the incompressibility of $\mathbf{v}$. But (A.1) implies statement (i) of Proposition 1.

Next we multiply Eq. (4) by c and integrate over S to find

$$\begin{aligned} 0 &= \int_S [cc_t + c\boldsymbol{\nabla} c\cdot\mathbf{v}]\,\mathrm{d}V = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_S c^2\,\mathrm{d}V + \frac{1}{2}\int_S [\nabla\cdot(c^2\mathbf{v}) - c^2\nabla\cdot\mathbf{v}]\,\mathrm{d}V \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c\|^2 + \frac{1}{2}\int_{\partial S} c^2\mathbf{v}\cdot\mathrm{d}\mathbf{n} - \frac{1}{2}\int_S c^2\nabla\cdot\mathbf{v}\,\mathrm{d}V = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c\|^2, \end{aligned}$$

which proves statement (ii) of Proposition 1.

$$\|c\| = \|c_0\|. \quad \text{(A.2)}$$

To prove statement (iii) of the proposition, we recall that the gradient of c satisfies the linear differential equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\nabla c = -\nabla \mathbf{v}^{\mathrm{T}}\nabla c, \tag{A.3}$$

where $\mathrm{D}/\mathrm{D}t$ refers to the material derivative along a fluid trajectory $\mathbf{x}(t;t_0,\mathbf{x}_0)$, and $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{A}$. Using the flow map $\mathbf{F}_{t_0}^{t}(\mathbf{x}_0) = \mathbf{x}(t;t_0,\mathbf{x}_0)$, the solution of Eq. (A.3) can be written as

$$\nabla c(\mathbf{x},t) = [\nabla \mathbf{F}_{t_0}^{t}(\mathbf{x}_0)]^{-\mathrm{T}}\nabla c_0(\mathbf{x}_0) \tag{A.4}$$

with the notation $\mathbf{A}^{-\mathrm{T}} = (\mathbf{A}^{-1})^{\mathrm{T}}$. Multiplying (A.4) by ∇c^{T} gives

$$|\nabla c(\mathbf{x},t)|^2 = \nabla c_0(\mathbf{x}_0)^{\mathrm{T}}[\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)]^{-1}\nabla c_0(\mathbf{x}_0), \tag{A.5}$$

where

$$\mathbf{C}_{t_0}^{t}(\mathbf{x}_0) = [\nabla \mathbf{F}_{t_0}^{t}(\mathbf{x}_0)]^{\mathrm{T}}\nabla \mathbf{F}_{t_0}^{t}(\mathbf{x}_0), \tag{A.6}$$

is the Cauchy–Green strain-tensor. Eq. (A.5) leads to the estimate

$$\Lambda_{\min}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0))|\nabla c_0(\mathbf{x}_0)|^2 \le |\nabla c(\mathbf{x},t)|^2 \le \Lambda_{\max}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0))|\nabla c_0(\mathbf{x}_0)|^2, \tag{A.7}$$

where $\Lambda_{\max}(\mathbf{A})$ and $\Lambda_{\min}(\mathbf{A})$ denote the largest eigenvalue of $\mathbf{A}$. Because $\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)$ is a symmetric, positive definite matrix with determinant one, all its eigenvalues are positive, and

$$0 < \Lambda_{\min}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)) \le 1 \le \Lambda_{\max}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)).$$

By definition, the largest and smallest finite-time Lyapunov exponents associated with the trajectory $\mathbf{x}(t;t_0,\mathbf{x}_0)$ are

$$\lambda_{+}(t;t_0,\mathbf{x}_0) = \frac{1}{2(t-t_0)}\log \Lambda_{\max}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)) > 0, \qquad \lambda_{-}(t;t_0,\mathbf{x}_0) = \frac{1}{2(t-t_0)}\log \Lambda_{\min}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)) < 0.$$

(For two-dimensional flows, we have $\Lambda_{\min}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0)) = 1/\Lambda_{\max}(\mathbf{C}_{t_0}^{t}(\mathbf{x}_0))$, therefore $\lambda_{-} = -\lambda_{+}$.) Using these Lyapunov exponents, we rewrite the inequality (A.7) as

$$\mathrm{e}^{2\lambda_{-}(t;t_0,\mathbf{x}_0)}|\nabla c_0(\mathbf{x}_0)|^2 \le |\nabla c(\mathbf{x},t)|^2 \le \mathrm{e}^{2\lambda_{+}(t;t_0,\mathbf{x}_0)}|\nabla c_0(\mathbf{x}_0)|^2,$$

which, after integration over S, proves statement (iii) of Proposition 1.

Appendix B

Here we prove Proposition 2. Integrating Eq. (6) over S gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle c\rangle = \frac{1}{\varkappa}\int_S (\kappa\Delta c + f)\,\mathrm{d}V = \frac{\kappa}{\varkappa}\int_{\partial S}\nabla c\cdot \mathrm{d}\mathbf{n} + \langle f\rangle = \langle f\rangle,$$

where we used the boundary conditions on $\mathbf{v}$ and c. Integrating the above equation in time yields statement (i) of Proposition 2.

To prove statement (ii), we first introduce the new variable $\tilde{c} = c - \langle c\rangle$, which has zero spatial mean, and satisfies the advection–diffusion equation

$$\tilde{c}_t + \nabla\tilde{c}\cdot\mathbf{v} = \kappa\Delta\tilde{c} + \tilde{f}, \qquad \tilde{c}(\mathbf{x},t_0) = \tilde{c}_0(\mathbf{x}), \qquad \left.\frac{\partial\tilde{c}}{\partial n}\right|_{\partial S} = 0, \tag{B.1}$$

where $\tilde{f} = f - \langle f \rangle$, and $\tilde{c}_0 = c_0 - \langle c_0 \rangle$. Multiplying (B.1) by $\tilde{c}$ and using the boundary conditions leads to the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}\|^2 = -\kappa\|\nabla\tilde{c}\|^2 + \varkappa\langle\tilde{c}\tilde{f}\rangle. \tag{B.2}$$

Because $\tilde{c}$ has zero mean, the Poincaré inequality (see, e.g. [1]) applies and yields

$$\mu_1\|\tilde{c}\|^2 \leq \|\nabla\tilde{c}\|^2, \tag{B.3}$$

where μ_1 is the smallest eigenvalue of the Laplacian $-\Delta$ over the domain S. We can thus rewrite (B.2) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}\|^2 \leq -\kappa\mu_1\|\tilde{c}\|^2 + \varkappa\langle\tilde{c}\tilde{f}\rangle \leq -\kappa\mu_1\|\tilde{c}\|^2 + \varkappa\langle\tilde{c}\tilde{f}\rangle. \tag{B.4}$$

By Cauchy's inequality (see, e.g. [1]), for any constant $\epsilon > 0$, we have

$$\varkappa\langle\tilde{c}\tilde{f}\rangle = \int_S \tilde{c}\tilde{f}\,\mathrm{d}V \leq \epsilon\|\tilde{c}\|^2 + \frac{1}{4\epsilon}\|\tilde{f}\|^2,$$

thus the inequality (B.4) can further be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}\|^2 \leq -(\kappa\mu_1 - \epsilon)\|\tilde{c}\|^2 + \frac{1}{4\epsilon}\|\tilde{f}\|^2.$$

Because $-(\kappa\mu_1 - \epsilon)$ may be negative, the classic Gronwall inequality does not apply here. Instead, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\tilde{c}\|^2\,\mathrm{e}^{2(\kappa\mu_1-\epsilon)(t-t_0)}] = \left[\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}\|^2 + 2(\kappa\mu_1-\epsilon)\|\tilde{c}\|^2\right]^2 \mathrm{e}^{2(\kappa\mu_1-\epsilon)(t-t_0)} \leq \frac{1}{2\epsilon}\|\tilde{f}\|^2\,\mathrm{e}^{2(\kappa\mu_1-\epsilon)(t-t_0)},$$

which, after integration over $[t_0, t]$, gives

$$\|\tilde{c}\|^2 \leq \|\tilde{c}_0\|^2\,\mathrm{e}^{-2(\kappa\mu_1-\epsilon)(t-t_0)} + \frac{1}{2\epsilon}\int_{t_0}^{t}\mathrm{e}^{-2(\kappa\mu_1-\epsilon)(t-\tau)}\|\tilde{f}\|^2\,\mathrm{d}\tau, \tag{B.5}$$

proving formula (8). The inequality (9) then follows if we take the limit $\epsilon \to 0$ in (B.5) for the case $f \equiv 0$.

To prove statement (iii) of Proposition 2, we multiply (6) by $-\Delta c$ and integrate over S to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_S|\nabla c|^2\,\mathrm{d}V &= -\kappa\int_S|\Delta c|^2\,\mathrm{d}V + \int_S\nabla c\cdot\nabla(\mathbf{v}\cdot\nabla c)\,\mathrm{d}V + \int_S\nabla c\cdot\nabla f\,\mathrm{d}V \\
&= -\kappa\int_S|\Delta c|^2\,\mathrm{d}V - \int_S\nabla c\cdot(\nabla\mathbf{v}^{\mathrm{T}}\nabla c)\,\mathrm{d}V + \int_S\nabla c\cdot\nabla f\,\mathrm{d}V \\
&\quad - \int_S\nabla c\cdot\left(u\frac{\partial\nabla c}{\partial x} + v\frac{\partial\nabla c}{\partial y} + w\frac{\partial\nabla c}{\partial z}\right)\mathrm{d}V \\
&= -\kappa\int_S|\Delta c(t)|^2\,\mathrm{d}A - \int_S\nabla c^{\mathrm{T}}\nabla\mathbf{v}^{\mathrm{T}}\nabla c\,\mathrm{d}V + \int_S\nabla c\cdot\nabla f\,\mathrm{d}V \\
&\leq -(\sigma(t) - \kappa\mu_1)\int_S|\nabla c(t)|^2\,\mathrm{d}V + \int_S\nabla c\cdot\nabla f\,\mathrm{d}V.
\end{aligned} \tag{B.6}$$

By Cauchy's inequality, for any $\epsilon > 0$, we have

$$\left|\int_S\nabla c\cdot\nabla f\,\mathrm{d}V\right| \leq \int_S|\nabla c||\nabla f|\,\mathrm{d}V \leq \epsilon\|\nabla c\|^2 + \frac{1}{4\epsilon}\|\nabla f\|^2,$$

thus (B.6) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S |\nabla c|^2\,\mathrm{d}V \le -2(\sigma(t)-\kappa\mu_1+\epsilon)\|\nabla c\|^2+\frac{1}{2\epsilon}\|\nabla f\|^2,$$

implying formula (10) in statement (iii) of the proposition. Again, the estimate (11) follows if we take the $\kappa \to 0$ limit in (10) for the case of $f(\mathbf{x}, t) \equiv 0$.

Appendix C

Here we prove the existence of a finite-dimensional invariant manifold for a general class of parabolic equations. Setting $\alpha = 1/2$ in our main result (Theorem C.1 below) proves Theorem 1. Our argument follows the ideas of Chow et al. [9] for one-dimensional parabolic equations.

C.1. Preliminary definitions

Let $\tilde{S}$ be the open domain defined in (19) for $\mathbb{R}^2$ or $\mathbb{R}^3$, and let A be the linear operator defined in (17) on the domain (18). Recall that A is a self-adjoint positive operator with the discrete eigenvalues $\mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$ listed in (20), and with the corresponding real eigenfunctions $e_1(\mathbf{x}), \dots, e_n(\mathbf{x}), \dots$ For any constant $\alpha \in [0, 1)$, these eigenfunctions form an orthogonal basis for the function space

$$H^{2\alpha} = \left\{ u = \sum_{i=1}^{\infty} a_i e_i \,\middle|\, \sum_{i=1}^{\infty} |a_i|^2 \lambda_i^{2\alpha} < \infty \right\},$$

which we equip with the norm

$$\|u\|_{H^{2\alpha}} = \left(\sum_{i=1}^{\infty} |a_i|^2 \lambda_i^{2\alpha} \right)^{1/2}.$$

Following Henry [15], we define fractional powers of A by the formula

$$A^\alpha u = \sum_{i=1}^{\infty} a_i \lambda_i^\alpha e_i, \quad \text{where } u = \sum_{i=1}^{\infty} a_i e_i \in H^{2\alpha}.$$

Note that

$$\|u\|_{H^{2\alpha}} = \|A^\alpha u\|_{L^2}. \tag{C.1}$$

For two positive integers $m \le n$, we define the following subspaces of $H^{2\alpha}$ for later reference:

$$H_n^+ = \mathrm{span}\,\{e_i\}_{i=1}^n, \qquad H_{m,n} = \mathrm{span}\,\{e_i\}_{i=n}^m, \qquad H_n^- = \mathrm{span}\{e_i\}_{i=n}^\infty$$

with norms inherited from H^0. Furthermore, we let

$$H_n^{2\alpha+} = H_n^+ \cap H^{2\alpha}, \qquad H_{m,n}^{2\alpha+} = H_{m,n}^+ \cap H^{2\alpha}, \qquad H_n^{2\alpha-} = H_n^- \cap H^{2\alpha}$$

with the norm inherited from $H^{2\alpha}$. Finally, let P_n^+, $P_{m,n}$, P_n^- denote the corresponding orthogonal projections from H^0 to H_n^+, $H_{m,n}$ and H_n^-, respectively, and let $A_n^+ = A|_{H_n^+}$, $A_{m,n} = A|_{H_{m,n}}$, and $A_n^- = A|_{H_n^-}$ denote the appropriate restrictions of A to H_n^+, $H_{m,n}$ and H_n^-.

C.2. A general class of parabolic equations

Let us consider the abstract equation

$$c_t = -\kappa Ac + V(t)A^{\alpha}c, \qquad c(0) = c_0, \tag{C.2}$$

where $0 \leq \alpha < 1$, $V(t)$ is a bounded linear operator on H^0 for every $t \in \mathbb{R}$, and $V(t)$ is a globally bounded function of $t \in \mathbb{R}$. Without loss of generality, we assume that $\langle c_0 \rangle = 0$ in (C.2), which implies $\langle c \rangle = 0$ for all t by an argument similar to the one used in the proof of Proposition 2. (If $\langle c_0 \rangle \neq 0$, we redefine c by letting $c \to c - \langle c_0 \rangle$.)

The advection–diffusion equation (6) is a special case of the abstract Eq. (C.2), which can be seen as follows. Motivated by the identity

$$\|\nabla u\|^2 = -\int_S u\Delta u \, \mathrm{d}A = \sum a_n^2 \mu_n = \|(-\Delta)^{1/2} u\|^2, \tag{C.3}$$

we seek to view the gradient operator ∇ as an equivalent of the operator $A^{1/2} = (-\Delta)^{1/2}$. To this end, we consider the ranges of the two operators, $\nabla(H^1) \subset L^2 \times L^2$ and $(-\Delta)^{1/2}(H^1) \subset L^2$, and define a linear "identification" map $I : \nabla(H^1) \to (-\Delta)^{1/2}(H^1)$ between the two ranges by letting

$$I(\nabla u) = (-\Delta)^{1/2} u.$$

We then have $(-\Delta)^{1/2} = I \circ \nabla$, and from (C.3) we obtain $\|I\| = 1$, i.e., I is a bounded linear map. Furthermore, using the velocity field $\mathbf{v}(\mathbf{x}, t)$, we define the bounded linear operator $V = \mathbf{v} \circ I^{-1}$, which enables us to write

$$(\mathbf{v} \circ \nabla)u = (\mathbf{v} \circ I^{-1} \circ I \circ \nabla)u = (V \circ (-\Delta)^{1/2})u.$$

This last equation shows that the advection–diffusion equation (6) is a special case of the abstract equation (C.2) with $\alpha = 1/2$.

C.3. Existence of two invariant manifolds

Here we prove the existence of a finite-dimensional and an infinite-dimensional invariant manifold for Eq. (C.2), assuming that a gap condition holds for two adjacent eigenvalues of the operator A (cf. (C.6) below). Under a stronger gap condition, the finite-dimensional invariant manifold becomes the inertial manifold described in Theorem 1.

To state the main result, we define the quantity

$$M(\alpha) = (2^{3-\alpha} + 2^{4-3\alpha}) \frac{\Gamma(\alpha)\, \mathrm{e}^{\alpha-1}}{(1-\alpha)^{\alpha}}, \tag{C.4}$$

where $\alpha \in [0, 1)$ is a parameter, and Γ denotes the classical Gamma function. Since $\alpha = 1/2$ is relevant for the advection–diffusion equation (6), we note that

$$M\left(\frac{1}{2}\right) = (2^{5/2} + 2^{5/2}) \frac{\sqrt{\pi}\, \mathrm{e}^{-1/2}}{\sqrt{1/2}} = 16\sqrt{\frac{\pi}{e}}. \tag{C.5}$$

We shall use the condition

$$\frac{M(\alpha)v_0}{[\kappa(\mu_{N+1} - \mu_N)]^{1-\alpha}} < 1, \tag{C.6}$$

where v_0 is defined in (21). Note that (C.6) simplifies to (22) in the case of equation (6).

Theorem C.1. *Suppose that for a positive integer N, the gap condition (C.6) holds. Then the following are satisfied*:

(i) Eq. (C.2) *has two invariant manifolds of the form*

$$\mathcal{M} = \{(t, p + L^N(t)p) | (t, p) \in \mathbb{R} \times H_N^{2\alpha+}\}, \tag{C.7}$$

$$\mathcal{M}^\infty = \{(t, q + L^\infty(t)q) | (t, q) \in \mathbb{R} \times H_{N+1}^{2\alpha-}\}, \tag{C.8}$$

where $L^N(t) : H_N^{2\alpha+} \to H_{N+1}^{2\alpha-}$ *and* $L^\infty(t) : H_{N+1}^{2\alpha-} \to H_N^{2\alpha+}$ *are bounded linear operators that depend continuously on t.*

(ii) *The operators* $L^N(t)$ *and* $L^\infty(t)$ *satisfy the estimates*

$$\|L^\infty(t)\|_{B(H_{N+1}^{2\alpha-}, H_N^{2\alpha+})}, \qquad \|L^N(t)\|_{B(H_N^{2\alpha+}, H_{N+1}^{2\alpha-})} \le \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1} - \mu_N))^{1-\alpha} - M(\alpha)v_0}. \tag{C.9}$$

(iii) *If* $V(t)$ *is T-periodic in t, then* $L^N(t)$ *and* $L^\infty(t)$ *are also T-periodic in t.*

Proof.

(A) *Construction of* $\mathcal{M}$. We introduce the constants

$$\gamma = \tfrac{1}{2}(\mu_{N+1} + \mu_N), \qquad \eta = \tfrac{1}{4}(\mu_{N+1} - \mu_N)$$

and define the function space

$$X_{\alpha,\eta}^- = \{f : (-\infty, 0] \to H^{2\alpha} | f \in C^0, \qquad \sup_{t\le 0} \mathrm{e}^{\kappa\eta t}\|f\|_{H^{2\alpha}} < \infty\}. \tag{C.10}$$

This complete metric space contains functions that grow slower in backward time than $\mathrm{e}^{-\kappa\eta t}$ does. Note that if nonempty, $X_{\alpha,\eta}^-$ is an invariant set for (C.2) by definition. We equip $X_{\alpha,\eta}^-$ with the norm

$$\|f\|_{X_{\alpha,\eta}^-} = \sup_{t\le 0} \mathrm{e}^{\kappa\eta t}\|f\|_{H^{2\alpha}}. \tag{C.11}$$

We want to construct an N-dimensional invariant manifold $\mathcal{M}$ for Eq. (C.2) with solutions that do not grow faster than $\mathrm{e}^{-\kappa\eta t}$ in backward time. In other words, we want to solve (C.2) on the space $X_{\alpha,\eta}^-$ to obtain a finite-dimensional "pseudo-stable manifold" of solutions that decay faster to the zero solution than other solutions do.

We rewrite (C.2) in terms of the scaled variable $v = \mathrm{e}^{\kappa\gamma t}c$ to obtain

$$v_t = \kappa(\gamma - A)v + V(t+\theta)A^\alpha v. \tag{C.12}$$

Here we have introduced the phase parameter $\theta \in \mathbb{R}$ to account for solutions launched at an arbitrary initial time $t_0 = \theta$. (Recall that in the definition of $X_{\alpha,\eta}^-$, the time variable t is restricted to nonpositive values.) The manifold $\mathcal{M}$ will be constructed as the set of points through which the solutions of (C.12) do not grow faster than $\mathrm{e}^{-\kappa\eta t}$ does as $t \to -\infty$.

First note that any $v \in X_{\alpha,\eta}^-$ solution of (C.12) satisfies the integral equations

$$\begin{aligned} P_N^+ v(t) &= \mathrm{e}^{\kappa(\gamma - A_N^+)t}p + \int_0^t \mathrm{e}^{\kappa(\gamma - A_N^+)(t-s)}P_N^+ V(\theta+s)A^\alpha v(s)\,\mathrm{d}s, \\ P_{N+1}^- v(t) &= \mathrm{e}^{\kappa(\gamma - A_{N+1}^-)(t-\tau)}P_{N+1}^- v(\tau) + \int_\tau^t \mathrm{e}^{\kappa(\gamma - A_{N+1}^-)(t-s)}P_{N+1}^- V(\theta+s)A^\alpha v(s)\,\mathrm{d}s, \end{aligned} \tag{C.13}$$

by the variation of constants formula. Since, for any $v \in X^-_{\alpha,\eta}$, we have

$$\begin{aligned}\|e^{\kappa(\gamma-A^-_{N+1})(t-\tau)}P^-_{N+1}v(\tau)\|_{H^{2\alpha}} &\le e^{\kappa(\gamma-\mu_{N+1})(t-\tau)}\|v(\tau)\|_{H^{2\alpha}} \le e^{\kappa(\gamma-\mu_{N+1})t}\, e^{\kappa(\mu_{N+1}-\gamma-\eta)\tau}\|v\|_{X^-_{\alpha,\eta}}\\ &= e^{\kappa(\gamma-\mu_{N+1})t}\, e^{\kappa(\mu_{N+1}-\mu_N)\tau/4}\|v\|_{X^-_{\alpha,\eta}},\end{aligned}$$

we obtain

$$\lim_{\tau\to-\infty}\|e^{\kappa(\gamma-A^-_{N+1})(t-\tau)}P^-_{N+1}v(\tau)\|_{H^{2\alpha}} = 0$$

and hence, by taking the $\tau \to -\infty$ limit in (C.13), we can deduce that

$$P^-_{N+1}v(t) = \int_{-\infty}^{t} e^{\kappa(\gamma-A^-_{N+1})(t-s)}P^-_{N+1}V(\theta+s)A^\alpha v(s)\,\mathrm{d}s.$$

Therefore, any solution $v \in X^-_{\alpha,\eta}$ of (C.12) satisfies the integral equation

$$\begin{aligned}v(t) = e^{\kappa(\gamma-A^+_N)t}p &+ \int_0^t e^{\kappa(\gamma-A^+_N)(t-s)}P^+_N V(\theta+s)A^\alpha v(s)\,\mathrm{d}s\\ &+ \int_{-\infty}^{t} e^{\kappa(\gamma-A^-_{N+1})(t-s)}P^-_{N+1}V(\theta+s)A^\alpha v(s)\,\mathrm{d}s\end{aligned} \tag{C.14}$$

with $p = P^+_N v(0)$. Conversely, direct substitution into (C.12) shows that solutions of the integral equation (C.14) are also solutions of (C.12).

We now show that Eq. (C.14) has a unique solution by applying a contraction mapping argument. To this end, we rewrite the integral equation (C.14) as a fixed point problem

$$v = F(v, p, \theta),$$

where

$$\begin{aligned}F(v, p, \theta) = e^{\kappa(\gamma-A^+_N)t}p &+ \int_0^t e^{\kappa(\gamma-A^+_N)(t-s)}P^+_N V(\theta+s)A^\alpha v(s)\,\mathrm{d}s\\ &+ \int_{-\infty}^{t} e^{\kappa(\gamma-A^-_{N+1})(t-s)}P^-_{N+1}V(\theta+s)A^\alpha v(s)\,\mathrm{d}s.\end{aligned}$$

We start by showing that $F(v, p, \theta)$ maps $X^-_{\alpha,\eta} \times H^+_N \times \mathbb{R}$ into $X^-_{\alpha,\eta}$. To see this, we estimate the first term in $F(v, p, \theta)$ as

$$\|e^{\kappa(\gamma-A^+_N)t}p\|_{H^{2\alpha}} \le e^{\kappa(\gamma-\mu_N)t}\|p\|_{H^{2\alpha}} = e^{2\kappa\eta t}\|p\|_{H^{2\alpha}}. \tag{C.15}$$

To estimate the remaining two terms in $F(v, p, \theta)$, we shall use three ingredients. First, we observe that

$$\max_{t\ge 0} t^\delta e^{-bt} = \left(\frac{\delta}{b}\right)^\delta e^{-\delta} \tag{C.16}$$

for any $\delta, b > 0$. Second, we recall from Henry [15] that if A is the generator of an analytic semigroup e^{-At}, and the real part of the spectrum of A satisfies $\mathrm{Re}\,\sigma(A) > \delta > 0$, then we have

$$\|A^\alpha e^{-At}\|_{L^2} \le \Gamma(\alpha)t^{-\alpha} e^{-\delta t} \tag{C.17}$$

with $\Gamma(\alpha)$ denoting the classical gamma function. Third, by (C.1), we have

$$\left\| \int_0^t \mathrm{e}^{\kappa(\gamma - A_N^+)(t-s)} P_N^+ V(\theta+s) A^\alpha v(s)\,\mathrm{d}s \right\|_{H^{2\alpha}} = \left\| A^\alpha \int_0^t \mathrm{e}^{\kappa(\gamma - A_N^+)(t-s)} P_N^+ V(\theta+s) A^\alpha v(s)\,\mathrm{d}s \right\|_{L^2}. \tag{C.18}$$

Using (C.16)–(C.18), we estimate the second term in the definition of $F(v, p, \theta)$ as follows:

$$\begin{aligned}
&\left\| \int_0^t \mathrm{e}^{\kappa(\gamma - A_N^+)(t-s)} P_N^+ V(\theta+s) A^\alpha v(s)\,\mathrm{d}s \right\|_{H^{2\alpha}} \\
&\quad\le \int_t^0 \Gamma(\alpha)(s-t)^{-\alpha}\,\mathrm{e}^{\kappa(\mu_N-\gamma)(s-t)} \|V(\theta+s)A^\alpha v(s)\|_{L^2}\,\mathrm{d}s \\
&\quad\le v_0 \int_t^0 \Gamma(\alpha)(s-t)^{-\alpha}\,\mathrm{e}^{\kappa(\mu_N-\gamma)(s-t)} \|A^\alpha v(s)\|_{L^2}\,\mathrm{d}s \\
&\quad= v_0 \int_t^0 \Gamma(\alpha)(s-t)^{-\alpha}\,\mathrm{e}^{\kappa(\mu_N-\gamma)(s-t)} \|v(s)\|_{H^{2\alpha}}\,\mathrm{d}s \\
&\quad\le v_0 \|v\|_{X^-_{\alpha,\eta}} \int_t^0 \Gamma(\alpha)(s-t)^{-\alpha}\,\mathrm{e}^{\kappa(\mu_N-\gamma)(s-t)-\kappa\eta s}\,\mathrm{d}s \\
&\quad= \Gamma(\alpha) v_0 \|v\|_{X^-_{\alpha,\eta}} \left(\frac{1}{1-\alpha}(-t)^{1-\alpha}\,\mathrm{e}^{\kappa(\gamma-\mu_N)t} - \frac{\kappa(\mu_N-\gamma-\eta)}{1-\alpha} \int_t^0 (s-t)^{1-\alpha}\,\mathrm{e}^{\kappa(\mu_N-\gamma)(s-t)-\kappa\eta s}\,\mathrm{d}s \right) \\
&\quad\le \Gamma(\alpha) \left(\frac{1-\alpha}{\kappa(\gamma-\mu_N)} \right)^{1-\alpha} \mathrm{e}^{\alpha-1} v_0 \|v\|_{X^-_{\alpha,\eta}} \left(\frac{1}{1-\alpha} + \frac{\kappa(\gamma+\eta-\mu_N)}{1-\alpha} \int_t^0 \mathrm{e}^{-\kappa\eta s}\,\mathrm{d}s \right) \\
&\quad\le \Gamma(\alpha) \left(\frac{1-\alpha}{\kappa(\gamma-\mu_N)} \right)^{1-\alpha} \mathrm{e}^{\alpha-1} v_0 \|v\|_{X^-_{\alpha,\eta}} \left(\frac{1}{1-\alpha} + \frac{(\gamma+\eta-\mu_N)}{(1-\alpha)\eta}\,\mathrm{e}^{-\kappa\eta t} \right) \\
&\quad= \Gamma(\alpha) \left(\frac{2(1-\alpha)}{\kappa(\mu_{N+1}-\mu_N)} \right)^{1-\alpha} \mathrm{e}^{\alpha-1} v_0 \|v\|_{X^-_{\alpha,\eta}} \left(\frac{1}{1-\alpha} + \frac{3}{1-\alpha}\,\mathrm{e}^{-\kappa\eta t} \right).
\end{aligned} \tag{C.19}$$

Similarly, the third term in the definition of $F(v, p, \theta)$ can be estimated as follows:

$$\begin{aligned}
&\left\| \int_{-\infty}^t \mathrm{e}^{\kappa(\gamma - A_{N+1}^-)(t-s)} P_{N+1}^- V(\theta+s) A^\alpha v(s)\,\mathrm{d}s \right\|_{H^{2\alpha}} \\
&\quad\le \int_{-\infty}^t \Gamma(\alpha)(t-s)^{-\alpha}\,\mathrm{e}^{\kappa(\gamma-\mu_{N+1})(t-s)} \|V(\theta+s)A^\alpha v(s)\|_{L^2}\,\mathrm{d}s \\
&\quad\le v_0 \int_{-\infty}^t \Gamma(\alpha)(t-s)^{-\alpha}\,\mathrm{e}^{\kappa(\gamma-\mu_{N+1})(t-s)} \|v(s)\|_{H^{2\alpha}}\,\mathrm{d}s \\
&\quad\le v_0 \|v\|_{X^-_{\alpha,\eta}} \int_{-\infty}^t \Gamma(\alpha)(t-s)^{-\alpha}\,\mathrm{e}^{\kappa(\mu_{N+1}-\gamma)(s-t)-\kappa\eta s}\,\mathrm{d}s \\
&\quad= \Gamma(\alpha) v_0 \|v\|_{X^-_{\alpha,\eta}} \frac{\kappa(\mu_{N+1}-\gamma-\eta)}{1-\alpha} \int_{-\infty}^t (t-s)^{1-\alpha}\,\mathrm{e}^{\kappa(\mu_{N+1}-\gamma)(s-t)-\kappa\eta s}\,\mathrm{d}s \\
&\quad= \Gamma(\alpha) v_0 \|v\|_{X^-_{\alpha,\eta}} \frac{\kappa\eta}{1-\alpha} \int_{-\infty}^t (t-s)^{1-\alpha}\,\mathrm{e}^{-\kappa(\mu_{N+1}-\gamma)t+(1/4)\kappa(\mu_{N+1}-\gamma)s}\,\mathrm{e}^{(3/4)\kappa(\mu_{N+1}-\gamma)s-\kappa\eta s}\,\mathrm{d}s
\end{aligned}$$

$$\times\,(\text{apply (C.16) to } (t-s)^{1-\alpha}\,\mathrm{e}^{(1/4)\nu(\lambda_{N+1}-k)s})$$
$$\le \Gamma(\alpha)\left(\frac{4(1-\alpha)}{\kappa(\mu_{N+1}-\gamma)}\right)^{1-\alpha}\mathrm{e}^{\alpha-1}v_0\|v\|_{X^-_{\alpha,\eta}}\frac{\kappa\eta}{1-\alpha}\mathrm{e}^{-3\kappa\eta t/2}\int_{-\infty}^{t}\mathrm{e}^{\kappa\eta s/2}\,\mathrm{d}s$$
$$\le \frac{2\Gamma(\alpha)}{1-\alpha}\left(\frac{8(1-\alpha)}{\kappa(\mu_{N+1}-\mu_N)}\right)^{1-\alpha}\mathrm{e}^{\alpha-1}v_0\|v\|_{X^-_{\alpha,\eta}}\,\mathrm{e}^{-\kappa\eta t}. \tag{C.20}$$

The estimates (C.15), (C.19) and (C.20) imply that $F(v, p, \theta)$ is bounded in the norm (C.11), and hence $F(v, p, \theta)$ indeed maps into $X^-_{\alpha,\eta}$.

Next we want to show that F defines a contraction mapping on $X^-_{\alpha,\eta}$. From (C.19) and (C.20) we see that for any $v_1, v_2 \in X^-_{\alpha,\eta}$

$$\|F(v_1, p, \theta) - F(v_2, p, \theta\|_{X^-_{\alpha,\eta}} \le (2^{3-\alpha}+2^{4-3\alpha})\frac{\Gamma(\alpha)\,\mathrm{e}^{\alpha-1}}{1-\alpha}\left(\frac{1-\alpha}{\kappa(\mu_{N+1}-\mu_N)}\right)^{1-\alpha}v_0\|v_1-v_2\|_{X^-_{\alpha,\eta}}$$
$$= \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}\|v_1-v_2\|_{X^-_{\alpha,\eta}}, \tag{C.21}$$

where $M(\alpha)$ is defined in (C.4). But the estimate (C.21) and condition (C.6) together establish that F is a contraction mapping on the complete metric space $X^-_{\alpha,\eta}$.

As a contraction mapping, F has a unique fixed point $v(t; p, \theta)$ in $X^-_{\alpha,\eta}$ for any $\theta \in \mathbb{R}$ and for any $p \in H^+_N$, implying a unique solution for (C.14) in $X^-_{\alpha,\eta}$. For such a fixed point $v(t; p, \theta)$, the estimates (C.15) and (C.21) imply that

$$\|v\|_{X^-_{\alpha,\eta}} = \|F(v, p, \theta)\|_{X^-_{\alpha,\eta}} \le \|F(v, p, \theta) - F(0, p, \theta)\|_{X^-_{\alpha,\eta}} + \|F(0, p, \theta)\|_{X^-_{\alpha,\eta}}$$
$$\le \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}\|v\|_{X^-_{\alpha,\eta}} + \|p\|_{H^{2\alpha}},$$

which in turn gives

$$\|v\|_{X^-_{\alpha,\eta}} \le \frac{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha} - M(\alpha)v_0}\|p\|_{H^{2\alpha}}. \tag{C.22}$$

Based on (C.20) and (C.22), the linear operator

$$L^N(\theta)p = P^-_{N+1}v(0; p, \theta) = \int_{-\infty}^{0}\mathrm{e}^{\kappa(\gamma-A^-_{N+1})(t-s)}P^-_{N+1}V(\theta+s)A^\alpha v(s; p, \theta)\,\mathrm{d}s, \tag{C.23}$$

satisfies the estimate

$$\|L^N(\theta)p\|_{H^{2\alpha}} \le \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}\|v\|_{X^-_{\alpha,\eta}}$$
$$\le \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}\frac{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}-M(\alpha)v_0}\|p\|_{H^{2\alpha}}$$
$$= \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}-M(\alpha)v_0}\|p\|_{H^{2\alpha}},$$

which implies

$$\|L^N(\theta)\|_{B(H^{2\alpha+}_N, H^{2\alpha-}_{N+1})} \le \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}-M(\alpha)v_0}.$$

In addition, if $V(t)$ is T-periodic, then by (C.14) so is $v(s; p, \theta)$in θ, and hence, by (C.23), $L(\theta)$ is T-periodic in θ. A similar argument establishes quasiperiodicity for $\mathcal{M}(t)$ if $V(t)$ is quasiperiodic, i.e., we have

$V(t) = \tilde{V}(\omega_1 t, \dots, \omega_l t)$, where $\tilde{V}$ is T-periodic in each phase variable $\omega_j t$. Finally, the set $\mathcal{M}$ defined by (C.7) is T-periodic in t. Because $\mathcal{M}$ contains full solutions of (C.2), $\mathcal{M}$ is an invariant manifold for Eq. (C.2).

(B) *Construction of* $\mathcal{M}^{\infty}$. The construction of the invariant set $\mathcal{M}^{\infty}$ follows a procedure similar to the one described above for $\mathcal{M}^{\infty}$. This time we build $\mathcal{M}^{\infty}$ from the solutions of Eq. (C.12) on the function space

$$X^{+}_{\alpha,\eta} = \{f : [0,\infty) \to H^{2\alpha}(S) | f \in C^0, \qquad \sup_{t\geq 0} \mathrm{e}^{-\kappa\eta t}\|f\|_{H^{2\alpha}} < \infty\},$$

equipped with the weighted norm

$$\|f\|_{X^{+}_{\alpha,\eta}} = \sup_{t\geq 0} \mathrm{e}^{-\kappa\eta t}\|f\|_{H^{2\alpha}}.$$

Thus, $\mathcal{M}^{\infty}$ consists of all initial conditions for which the solutions of (C.12) do not grow faster than $\mathrm{e}^{\kappa\eta t}$ as $t \to \infty$. Again, we obtain $\mathcal{M}^{\infty}$ from a fixed point argument as above. □

C.4. Continuity of $\mathcal{M}$ in time

The continuity of $\mathcal{M}(t)$ will follow form the uniform continuity of the map $\theta \mapsto v(\cdot;p,\theta)$, with p varying on bounded sets. Specifically, for any fixed θ_0, we need to show that

$$v(\cdot;p,\theta) \to v(\cdot;p,\theta_0), \tag{C.24}$$

in the space $X^{-}_{\alpha,\eta}$ as $\theta \to \theta_0-$, which establishes the left-continuity for the map $\theta \mapsto v(\cdot;p,\theta)$ at $\theta = \theta_0$. (The proof of right-continuity is similar.) We shall sketch this procedure up to a point from which a similar argument by Chow et al. [9] for one-dimensional parabolic equations applies and completes the proof. In the interest of brevity, we omit this lengthy final part of the argument and refer the reader to [9] for details.

For every $\theta \leq \theta_0$, Eq. (C.14) implies

$$v(t;p,\theta) - v(t;p,\theta_0) = F(v(\cdot;p,\theta),p,\theta) - F(v(\cdot;p,\theta_0),p,\theta) + I_1 + I_2, \tag{C.25}$$

where

$$I_1 = \int_0^t \mathrm{e}^{\kappa(\gamma - A_N^+)(t-s)} P_N^+ (V(\theta+s) - V(\theta_0+s)) A^{\alpha} v(s;p,\theta_0)\,\mathrm{d}s,$$
$$I_2 = \int_{-\infty}^t \mathrm{e}^{\kappa(\gamma - A_{N+1}^-)(t-s)} P_{N+1}^- (V(\theta+s) - V(\theta_0+s)) A^{\alpha} v(s;p,\theta_0)\,\mathrm{d}s.$$

Using estimate (C.21) and the definition of I_1 and I_2, we deduce from (C.25) the estimate

$$\|v(\cdot;p,\theta) - v(\cdot;p,\theta_0)\|_{X^{-}_{\alpha,\eta}} \leq \frac{M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}\|v(\cdot;p,\theta) - v(\cdot;p,\theta_0)\|_{X^{-}_{\alpha,\eta}} + \|I_1\|_{X^{-}_{\alpha,\eta}} + \|I_2\|_{X^{-}_{\alpha,\eta}},$$

which implies

$$\|v(\cdot;p,\theta) - v(\cdot;p,\theta_0)\|_{X^{-}_{\alpha,\eta}} \leq \frac{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha}}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha} - M(\alpha)v_0}(\|I_1\|_{X^{-}_{\alpha,\eta}} + \|I_2\|_{X^{-}_{\alpha,\eta}}).$$

Thus, to prove (C.24), it suffices to prove that

$$I_1, I_2 \to 0 \quad \text{in } X^{-}_{\alpha,\eta} \text{ as } \theta \to \theta_0^-,$$

which is discussed by Chow et al. [9] in detail.

C.5. Decoupling the evolution equation

Next we want to find conditions under which the invariant manifold $\mathcal{M}(t)$ attracts all solutions of the advection–diffusion equation, i.e., under which $\mathcal{M}(t)$ is an inertial manifold. To establish the decay of all solutions to $\mathcal{M}(t)$, we need to decouple the abstract equation (C.2) into coordinates aligned with the two invariant manifolds $\mathcal{M}$ and $\mathcal{M}^{\infty}$ (cf. Theorem C.1). As a second step, we need to estimate the decay of the coordinates aligned with $\mathcal{M}^{\infty}$. The steps of this construction follow closely the steps used by Chow et al. [9] for parabolic equations with a one-dimensional spatial variable. For this reason, we omit the proofs of some technical lemmas and refer the interested reader to [9] for further details.

In order to decouple equation (C.2) into a finite-dimensional fast-decaying and an infinite-dimensional slow-decaying component, we define the linear operator $\Lambda_N(t) : H^{2\alpha} \to H^{2\alpha}$ by letting

$$\Lambda_N(t)c = L^N(t)p + L^{\infty}(t)q, \tag{C.26}$$

where $p = P_N^+ c \in H_N^{2\alpha+}$ and $q = P_{N+1}^- c \in H_{N+1}^{2\alpha-}$. If the gap condition (C.6) is satisfied, Theorem C.1 guarantees that $\Lambda_N(t)$ is a bounded linear operator with

$$\|\Lambda_N(t)\|_{B(H^{2\alpha},H^{2\alpha})} \le \frac{2M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha} - M(\alpha)v_0} \tag{C.27}$$

and that $\Lambda_N(t)$ is continuous and T-periodic in t whenever $V(t)$ is T-periodic. By (C.27), the operator

$$\Phi_N(t) = (I + \Lambda_N(t))^{-1},$$

is well-defined if $K(\alpha, \kappa, N, V)$ defined by (C.6) is less than 1/2. We now state without proof a few properties of $\Phi_N(t)$ that are simple to establish (cf. [9]).

Lemma C.1. *If $K(\alpha, \kappa, N, V)$ defined by (C.6) is less than 1/2, then*

1. *$\Phi_N(t)$ is a bounded linear map from $H^{2\alpha}$ to $H^{2\alpha}$.*
2. *$\Phi_N(t)$ is continuous and T-periodic in t if $V(t)$ is T-periodic.*
3. *$\lim_{N\to\infty}\Phi_N(t) = \lim_{N\to\infty}\Phi_N^{-1}(t) = I$ uniformly with respect to $t \in \mathbb{R}$.*

To decouple the equation (C.2), we shall project an arbitrary concentration $c \in H^{2\alpha}$ onto the invariant subspaces

$$\mathcal{M}(t) = \{c|(t, c) \in \mathcal{M}\}, \qquad \mathcal{M}^{\infty}(t) = \{c|(t, c) \in \mathcal{M}^{\infty}\},$$

where $\mathcal{M}$ and $\mathcal{M}^{\infty}$ are the invariant manifolds given by (C.7) and (C.8), respectively. Adapting the proof of Lemma 5.1 of Chow et al. [9], we obtain the following result on the projection of c onto the above subspaces.

Lemma C.2. *Suppose that there exists $N > 0$ such that*

$$0 < \frac{2M(\alpha)v_0}{(\kappa(\mu_{N+1}-\mu_N))^{1-\alpha} - M(\alpha)v_0} < 1. \tag{C.28}$$

Then for each $t \in \mathbb{R}$ we have the direct sum

$$H^{2\alpha} = \mathcal{M}(t) \oplus \mathcal{M}^{\infty}(t).$$

Moreover, the associated projections $\Pi^N(t) : H^{2\alpha} \to \mathcal{M}(t)$ and $\Pi^\infty(t) : H^{2\alpha} \to \mathcal{M}^\infty(t)$ *are given by*

$$\Pi^N(t)u = P_N^+(I + \Lambda_N(t))^{-1}u + L^N(t)P_N^+(I + \Lambda_N(t))^{-1}u,$$
$$\Pi^\infty(t)u = P_{N+1}^-(I + \Lambda_N(t))^{-1}u + L^\infty(t)P_{N+1}^-(I + \Lambda_N(t))^{-1}u.$$

With the help of this last lemma, we can now decouple equation (C.2). For any $c \in H^{2\alpha}$, we write

$$c = p + q,$$

where $p = P_N^+ c \in H_N^{2\alpha+}$ and $q = P_{N+1}^- c \in H_{N+1}^{2\alpha-}$. Then Eq. (C.2) can be written as

$$p_t = -\kappa A_N^+ p + P_N^+ V(t)A^\alpha c, \tag{C.29}$$

$$q_t = -\kappa A_{N+1}^- q + P_{N+1}^- V(t)A^\alpha c. \tag{C.30}$$

Repeating the proof of Lemma 5.2 of Chow et al. [9], we obtain the following final decoupled form.

Lemma C.3. *Suppose that the gap condition* (C.28) *is satisfied. Then the transformation* $u = \Phi_N(t)c$ *transforms equations* (C.29) and (C.30) *to the decoupled equations*

$$u_{Nt} = -\kappa A_N^+ u_N + P_N^+ V(t)A^\alpha[I + L^N(t)]u_N, \tag{C.31}$$

$$u_{\infty t} = -\kappa A_{N+1}^- u_\infty + P_{N+1}^- V(t)A^\alpha[I + L^\infty(t)]u_\infty, \tag{C.32}$$

where $u = \Phi_N(t)c$, $u_N = P_N^+ u \in H_N^{2\alpha+}$ and $u_\infty = P_{N+1}^- u \in H_{N+1}^{2\alpha-}$.

C.6. $\mathcal{M}(t)$ is an inertial manifold

We first recall a modified form of the classic Gronwall inequality (see, e.g. [15, Lemma 7.1.1]). Suppose that $a, b \geq 0$, $\delta > 0$, and the function $\phi(t)$ is nonnegative and locally integrable on $[0, +\infty)$, satisfying

$$\phi(t) \leq a + b\int_a^t (t - s)^{\delta-1}\phi(s)\,\mathrm{d}s, \quad 0 \leq t < +\infty.$$

Then

$$\phi(t) \leq aE_\delta(\theta t), \quad 0 \leq t < +\infty, \tag{C.33}$$

where

$$\theta = (b\Gamma(\delta))^{1/\delta}, \qquad E_\delta(z) = \sum_{n=0}^\infty \frac{z^{n\delta}}{\Gamma(n\delta + 1)} \simeq \frac{\mathrm{e}^z}{\delta} \quad \text{as } z \to +\infty.$$

To show the attractivity of $\mathcal{M}(t)$, we first recall formula (C.28) and note that if

$$\limsup_{n\to\infty} \frac{2M(\alpha)v_0}{(\kappa(\mu_{n+1} - \mu_n))^{1-\alpha} - M(\alpha)v_0} < 1, \tag{C.34}$$

is satisfied, then there exist a sequence $N_1 < N_2 < \cdots < N_i < \cdots$ of indices such that

$$\frac{2M(\alpha)v_0}{(\kappa(\mu_{N_i+1} - \mu_{N_i}))^{1-\alpha} - M(\alpha)v_0} < 1, \quad i = 1, 2, \ldots$$

holds. The operators

$$\Phi_{N_i}(t) = (I + \Lambda_{N_i}(t))^{-1}, \qquad i = 1, 2, \dots$$

are therefore well-defined by (C.27), and satisfy the properties listed in Lemma C.1. By Lemma C.3, each $\Phi_{N_i}(t)$ transforms Eqs. (C.29) and (C.30) to the decoupled equations (C.31) and (C.32).

Let c be a solution of (C.2) in $H^{2\alpha}$ with the initial condition

$$c_0 = \sum_{i=1}^{\infty} c_{iN_j}^0 \Phi_{N_j}^{-1}(0) e_i.$$

By Lemma C.1, we have

$$\lim_{j\to\infty} \sum_{i=1}^{\infty} c_{iN_j}^0 e_i = \lim_{j\to\infty} \Phi_{N_j}(0) c_0 = c_0 = \sum_{i=1}^{\infty} c_i^0 e_i.$$

We select a small constant $\epsilon > 0$, and pick N_{j_0} large enough so that

$$\left\| \sum_{i=N_{j_0}+1}^{\infty} c_i^0 e_i \right\|_{H^{2\alpha}} \le \frac{\epsilon}{2}, \qquad \left\| \sum_{i=1}^{\infty} c_{iN_{j_0}}^0 e_i - \sum_{i=1}^{\infty} c_i^0 e_i \right\|_{H^{2\alpha}} \le \frac{\epsilon}{2}.$$

Then the “tail” of $\sum_{i=1}^{\infty} c_{iN_{j_0}}^0 e_i$ obeys the estimate

$$\left\| \sum_{i=N_{j_0}+1}^{\infty} c_{iN_{j_0}}^0 e_i \right\|_{H^{2\alpha}} \le \left\| \sum_{i=N_{j_0}+1}^{\infty} c_{iN_{j_0}}^0 e_i - \sum_{i=N_{j_0}+1}^{\infty} c_i^0 e_i \right\|_{H^{2\alpha}} + \left\| \sum_{i=N_{j_0}+1}^{\infty} c_i^0 e_i \right\|_{H^{2\alpha}} \le \epsilon. \tag{C.35}$$

Next, we select u to be the solution of (C.31) and (C.32) in $H^{2\alpha}$ with initial condition

$$u(0) = \Phi_{N_{j_0}}(0) c_0 = \sum_{i=1}^{\infty} c_{iN_{j_0}}^0 e_i.$$

Then the two components of u appearing in the decoupled equations (C.31) and (C.32) are

$$u_{N_{j_0}}(0) = P_{N_{j_0}}^+ u(0) = \sum_{i=1}^{N_{j_0}} c_{iN_{j_0}}^0 e_i \in H_{N_{j_0}}^{2\alpha+}, \qquad u_\infty(0) = P_{N_{j_0}+1}^- u(0) = \sum_{i=N_{j_0}+1}^{\infty} c_{iN_{j_0}}^0 e_i.$$

Because Eq. (C.31) is a finite-dimensional linear ODE, it admits a fundamental matrix solution $U_{N_{j_0}}(t)$ such that

$$u_{N_{j_0}}(t) = (e_1, \dots, e_{N_{j_0}}) U_{N_{j_0}}(t) c_{N_{j_0}}^0,$$

where $c_{N_{j_0}}^0 = (c_{1N_{j_0}}^0, \dots, c_{N_{j_0}N_{j_0}}^0)^{\mathrm{T}}$. We then have

$$c = \Phi_{N_{j_0}}^{-1}(t) u = \Phi_{N_{j_0}}^{-1}(t)(u_{N_{j_0}} + u_\infty) = \sum_{i=1}^{n} (\Phi_{N_{j_0}}^{-1}(t) e_1, \dots, \Phi_{N_{j_0}}^{-1}(t) e_{N_{j_0}}) U_{N_{j_0}}(t) c_{N_{j_0}}^0 + \Phi_{N_{j_0}}^{-1}(t) u_\infty.$$

Furthermore, Eqs. (C.9), (C.32) and (C.17) imply

$$\begin{aligned}\|u_\infty(t)\|_{H^{2\alpha}} &= \left\| e^{-\kappa A^-_{N_{j_0}+1} t} u_\infty(0) + \int_0^t e^{-\kappa A^-_{N_{j_0}+1}(t-s)} P^-_{N_{j_0}+1} V(s) A^\alpha (I + L^\infty(s)) u_\infty(s)\, ds \right\|_{H^{2\alpha}} \\ &\le e^{-\kappa\mu_{N_{j_0}+1} t} \|u_\infty(0)\|_{H^{2\alpha}} + 2\Gamma(\alpha) v_0 \int_0^t (t-s)^{-\alpha} e^{-\kappa\mu_{N_{j_0}+1}(t-s)} \|u_\infty(s)\|_{H^{2\alpha}}\, ds,\end{aligned}$$

or, equivalently

$$\|u_\infty(t)\|_{H^{2\alpha}} e^{\kappa\mu_{N_{j_0}+1} t} \le \|u_\infty(0)\|_{H^{2\alpha}} + 2\Gamma(\alpha) v_0 \int_0^t (t-s)^{-\alpha} e^{\kappa\mu_{N_{j_0}+1} s} \|u_\infty(s)\|_{H^{2\alpha}}\, ds.$$

Then the generalized Gronwall inequality (C.33) gives

$$\begin{aligned}\|u_\infty(t)\|_{H^{2\alpha}} e^{\kappa\mu_{N_{j_0}+1} t} &\le \|u_\infty(0)\|_{H^{2\alpha}} E_{1-\alpha}((2\Gamma(\alpha) v_0 \Gamma(1-\alpha))^{1/(1-\alpha)} t) \\ &\le C(\alpha) \|u_\infty(0)\|_{H^{2\alpha}} \exp((2\Gamma(\alpha) v_0 \Gamma(1-\alpha))^{1/(1-\alpha)} t),\end{aligned}$$

where $C(\alpha)$ denotes a positive constant depending on α. Then estimate (C.35) yields

$$\begin{aligned}&\left\| c - \sum_{i=1}^n (\Phi^{-1}_{N_{j_0}}(t) e_1, \dots, \Phi^{-1}_{N_{j_0}}(t) e_{N_{j_0}}) U_{N_{j_0}}(t) c^0_{N_{j_0}} \right\|_{H^{2\alpha}} \\ &\quad = \|\Phi^{-1}_{N_{j_0}}(t) u_\infty(t)\|_{H^{2\alpha}} \le C(\alpha) \|\Phi^{-1}_{N_{j_0}}\| \|u_\infty(0)\|_{H^{2\alpha}} \exp(((2\Gamma(\alpha) v_0 \Gamma(1-\alpha))^{1/(1-\alpha)} - \kappa\mu_{N_{j_0}+1}) t) \\ &\quad \le \epsilon C(\alpha) \|\Phi^{-1}_{N_{j_0}}\| \exp(-(\kappa\mu_{N_{j_0}+1} - (2\Gamma(\alpha) v_0 \Gamma(1-\alpha))^{1/(1-\alpha)}) t).\end{aligned} \tag{C.36}$$

This last formula shows that all solutions converge to $\mathcal{M}(t)$ exponentially in the $H^{2\alpha}$ norm for large enough N_{j_0}, and hence $\mathcal{M}(t)$ is an inertial manifold.

Appendix D

Here we first review some facts about infinite-dimensional Floquet theory, then prove Theorem 4.

D.1. Infinite-dimensional Floquet theory

A Floquet theory for the abstract equation (C.2) is described by Kuchment [18] for the case of a T-periodic $V(t)$. Specifically, Kuchment shows that the space of all solutions of the form

$$c(x,t) = e^{\lambda t} \sum_{l=0}^n t^l \varphi_l(x,t), \qquad \lambda \in \mathbb{C}, \qquad \varphi_l(x,t) = \varphi_l(x, t+T),$$

is complete in the function space $L^2(\mathbb{R}^+, H^0)$, provided that

$$\liminf_{n\to\infty} \frac{2v_0(\kappa\mu_{n+1})^\alpha}{\kappa(\mu_{n+1}-\mu_n)} < 1, \tag{D.1}$$

where v_0 is the minimal upper bound on $|V|$ defined in (21).

Originally obtained by Miloslavskii [21], condition (D.1) is too restrictive to cover physically relevant velocity fields in the advection–diffusion equation (6). For instance, for the square domain $S = [0, \pi]\times[0, \pi]$, the eigenvalues of A are $\mu_{m,n} = m^2 + n^2$ ($m, n = 0, 1, 2, \dots$). In this case, after setting $\alpha = 1/2$ in (D.1), we obtain the condition

$$\liminf_{n\to\infty} \frac{\mu_{n+1}^{1/2}}{\mu_{n+1} - \mu_n} < \frac{\sqrt{\kappa}}{2v_0}.$$

The limit on the left-hand side of this inequality seems impossible to compute analytically, but appears to be plus infinity in numerical calculations. If the limit is indeed infinity, then v_0 needs to be zero by condition (D.1), which makes the Miloslavskii–Kuchment result inapplicable to nontrivial velocity fields.

The above shortcoming of existing Floquet theory for parabolic PDEs has prompted us to adapt the approach of Chow et al. [9], who developed a Floquet theory for parabolic equations in one spatial dimension. We have extended their approach to our current higher-dimensional context by first proving the existence of an inertial manifold in $H^1(\tilde{S})$, then applying the classical finite-dimensional Floquet decomposition on the inertial manifold.

D.2. Classical Floquet decomposition on the inertial manifold

If $V(t)$ is T-periodic in t, it then $\Phi_{N_{j0}}^{-1}(t)$ is also T-periodic in t. As a result, by finite-dimensional Floquet theory, there exists a T-periodic $N_{j0} \times N_{j0}$ matrix $P_{N_{j0}}(t)$, and a constant $N_{j0} \times N_{j0}$ matrix $B_{N_{j0}}$ such that

$$U_{N_{j0}}(t) = P_{N_{j0}}(t)\,\mathrm{e}^{B_{N_{j0}}t}. \tag{D.2}$$

Setting $N_{j0} \equiv N$ in (C.36) and using (D.2) with $\alpha = 1/2$, we obtain statement (i) of Theorem 4.

D.3. Faster convergence outside $\mathcal{M}(t)$ than inside

To prove statement (ii) of Theorem 4, we multiply (C.32) by $u_{N_{j0}}$, integrate over the domain S, and use (C.9) to find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u_{N_{j0}}\|_{L^2}^2 &= -2\kappa\int_S (A_{N_{j0}}^+ u_{N_{j0}})u_{N_{j0}}\,\mathrm{d}V + \int_S [P_{N_{j0}}^+ V(t)A^\alpha(I + L^{N_{j0}}(t))u_{N_{j0}}]u_{N_{j0}}\,\mathrm{d}V\\
&\geq -2\kappa\mu_{N_{j0}}\|u_{N_{j0}}\|_{L^2}^2 - \|P_{N_{j0}}^+ V(t)A^\alpha(I + L^{N_{j0}}(t))u_{N_{j0}}\|_{L^2}\|u_{N_{j0}}\|_{L^2}\\
&\geq -2\kappa\mu_{N_{j0}}\|u_{N_{j0}}\|_{L^2}^2 - v_0\|A^\alpha(I + L^{N_{j0}}(t))u_{N_{j0}}\|_{L^2}\|u_{N_{j0}}\|_{L^2}\\
&= -2\kappa\mu_{N_{j0}}\|u_{N_{j0}}\|_{L^2}^2 - v_0\|(I + L^{N_{j0}}(t))u_{N_{j0}}\|_{H^{2\alpha}}\|u_{N_{j0}}\|_{L^2}\\
&= -2\kappa\mu_{N_{j0}}\|u_{N_{j0}}\|_{L^2}^2 - 2v_0\|u_{N_{j0}}\|_{H^{2\alpha}}\|u_{N_{j0}}\|_{L^2} \geq -2\kappa\mu_{N_{j0}}\|u_{N_{j0}}\|_{L^2}^2 - 2\mu_{N_{j0}}^\alpha v_0\|u_{N_{j0}}\|_{L^2}^2,
\end{aligned}$$

where we used the inequality $\|u_{N_{j0}}\|_{H^{2\alpha}} \leq \mu_{N_{j0}}^\alpha\|u_{N_{j0}}\|$. The above estimate then implies

$$\|u_{N_{j0}}\|_{L^2}^2 \geq \exp[-2(\kappa\mu_{N_{j0}} + \mu_{N_{j0}}^\alpha v_0)t]\|u_{N_{j0}}(0)\|_{L^2}^2.$$

Now the $H^{2\alpha}$ norm and the L^2 norm are equivalent on the finite-dimensional space $H_{N_{j0}}^{2\alpha+}$, thus we have

$$\|u_{N_{j0}}\|_{H^{2\alpha}} \geq \tilde{C}\exp[-(\kappa\mu_{N_{j0}} + \mu_{N_{j0}}^\alpha v_0)t]\|u_{N_{j0}}(0)\|_{H^{2\alpha}} \tag{D.3}$$

for an appropriate $\tilde{C} > 0$. Setting $\alpha = 1/2$ in (C.36) and (D.3), we obtain that for large times $\|u_{N_{j0}}\|_{H^1}$ decays slower than $\|u_\infty\|_{H^1}$ does if

$$\kappa\mu_{N_{j0}} + \mu_{N_{j0}}^\alpha v_0 < \kappa\mu_{N_{j0}+1} - 4\pi^2 v_0^2,$$

or, equivalently

$$\mu_{N_{j_0}+1} - \mu_{N_{j_0}} - \frac{v_0}{\kappa}\sqrt{\mu_{N_{j_0}}} > 4\pi^2 v_0^2/\kappa.$$

But this condition is always satisfied under (32) for large enough N_{j_0}, thus statement (ii) of Theorem 4 follows.

Appendix E

Here we prove Theorem 5. Substituting the limiting Floquet solution $\mathrm{e}^{-\lambda_0 t}\varphi_0(\mathbf{x}, t)$ into the advection–diffusion equation (6) gives

$$\partial_t \varphi_0 - \lambda_0 \varphi_0 + \nabla \varphi_0 \cdot \mathbf{v} = \kappa \Delta \varphi_0. \tag{E.1}$$

Multiplying this equation by φ_0^*, the complex conjugate of φ_0, gives

$$\varphi_0^* \partial_t \varphi_0 - \lambda_0 |\varphi_0|^2 + \varphi_0^* \nabla \varphi_0 \cdot \mathbf{v} = \kappa \varphi_0^* \Delta \varphi_0.$$

Adding this equation to its complex conjugate, and integrating over the domain S leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_0\|^2 &= 2\int_S \mathrm{Re}\,\lambda_0 |\varphi_0|^2\,\mathrm{d}V - 2\int_S \varphi_0^* \nabla \varphi_0 \cdot v\,\mathrm{d}V + 2\kappa \int_S \varphi_0^* \Delta \varphi_0\,\mathrm{d}V \\ &= 2\mathrm{Re}\,\lambda_0 \|\varphi_0\|^2 - 2\int_S \frac{1}{2}[\nabla \cdot (|\varphi_0|^2 \mathbf{v}) - |\varphi_0|^2 \nabla \cdot \mathbf{v}]\,\mathrm{d}V - 2\kappa \int_S [\nabla \cdot (\varphi_0^* \nabla \varphi_0) - |\nabla \varphi_0|^2]\,\mathrm{d}V \\ &= 2\mathrm{Re}\,\lambda_0 \|\varphi_0\|^2 - \int_{\partial S} |\varphi_0|^2 \mathbf{v} \cdot \mathbf{n}\,\mathrm{d}A + 2\int_S |\varphi_0|^2 \nabla \cdot \mathbf{v}\,\mathrm{d}V - 2\kappa \int_{\partial S} \varphi_0^* \nabla \varphi_0 \cdot \mathbf{n}\,\mathrm{d}A + 2\kappa \|\nabla \varphi_0\|^2 \\ &= 2\mathrm{Re}\,\lambda_0 \|\varphi_0\|^2 + 2\kappa \|\nabla \varphi_0\|^2,\end{aligned} \tag{E.2}$$

where we used the incompressibility of $\mathbf{v}$ as well as the boundary conditions on $\mathbf{v}$ and φ_0.

By the T-periodicity of the function φ_0, integration of (E.2) with respect to t over $[0, T]$ gives

$$\mathrm{Re}\,\lambda = \kappa \frac{\|\nabla \bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2}, \tag{E.3}$$

where the bar denotes temporal averaging over $[0, T]$.

Next we split φ_0 and λ_0 into real and imaginary parts by letting

$$\boldsymbol{\varphi}_0(\mathbf{x}, t) = g(\mathbf{x}, t) + \mathrm{i}h(\mathbf{x}, t), \qquad \lambda = \mu + \mathrm{i}\rho,$$

where μ and ρ are real constants, and g and h are real-valued functions that satisfy the boundary conditions, and are T-periodic in t. Substitution into (E.1) then gives the two equations

$$g_t + \nabla g \cdot \mathbf{v} = \mu g - \rho h + \kappa \Delta g, \qquad h_t + \nabla h \cdot \mathbf{v} = \rho g + \mu h + \kappa \Delta h. \tag{E.4}$$

Multiplying the first equation by complex conjugate g^*, integrating over the domain S, then averaging in time as before leads to

$$\rho = \frac{\mu \|\bar{g}\|^2 - \kappa \|\nabla \bar{g}\|^2}{\varkappa \langle \overline{gh} \rangle}. \tag{E.5}$$

Thus, based on (E.3) and (E.5), a simple Floquet solutions of (6) has the general form

$$c(\mathbf{x}, t) = \exp\left[-\kappa \left(\frac{\|\nabla \bar{\varphi}_0\|^2}{\|\bar{\varphi}_0\|^2} + \mathrm{i}\frac{\|\nabla \bar{\varphi}_0\|^2 \|\mathrm{Re}\,\bar{\varphi}_0\|^2 - \|\bar{\varphi}_0\|^2 \|\mathrm{Re}\,\nabla \bar{\varphi}_0\|^2}{\varkappa \|\bar{\varphi}_0\|^2 \langle \overline{\mathrm{Re}\,\varphi_0\ \mathrm{Im}\,\varphi_0} \rangle}\right) t\right] \varphi_0(x, t) \tag{E.6}$$

for some T-periodic, complex-valued H^1 function φ_0.

Appendix F

Here we prove Theorem 6. Substituting (40) into the Floquet solution (36) then, in turn, into Eq. (6) gives

$$\partial_t\phi_0 + \nabla\phi_0 \cdot \mathbf{v} + \kappa[\partial_t\phi_1 + \Lambda_0\phi_0 - \Delta\phi_0] + \mathcal{O}(\kappa^2) = 0,$$

which yields the equation

$$\partial_t\phi_0 + \nabla\phi_0 \cdot \mathbf{v} = 0,$$

in the $\kappa = 0$ limit.

Let $\psi(\mathbf{x}, t)$ denote the streamfunction associated with the time-periodic velocity field $\mathbf{v}(\mathbf{x}, t)$, i.e., let

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{J}\nabla\psi(\mathbf{x}, t), \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, let I be a parameter with values taken from a closed interval $\mathcal{I} \in \mathbb{R}$, and let $\gamma \in S^1$ be a T-periodic coordinate on a circle of perimeter T. We define the Hamiltonian

$$H(\mathbf{x}, \gamma, I) = \psi(\mathbf{x}, \gamma) + I$$

and the associated two-degree-of-freedom Hamiltonian system

$$\dot{\mathbf{x}} = \mathbf{J}\nabla_{\mathbf{x}} H(\mathbf{x}, I, \gamma), \qquad \dot{I} = -\partial_\gamma H(\mathbf{x}, I, \gamma), \qquad \dot{\gamma} = \partial_I H(\mathbf{x}, I, \gamma). \tag{F.1}$$

Because $\phi_0(\mathbf{x}, t)$ is a T-periodic solution of (4), a direct calculation shows that

$$p_1(\mathbf{x}, \gamma) = \phi_0(\mathbf{x}, \gamma),$$

is a first integral for system (F.1). Another first integral for (F.1) is

$$p_2(\mathbf{x}, \gamma, I) = H(\mathbf{x}, \gamma, I),$$

which is in involution with p_1, i.e.

$$\begin{aligned} \{p_1, p_2\} &= \frac{\partial p_1}{\partial x_2}\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_1}\frac{\partial p_2}{\partial x_2} + \frac{\partial p_1}{\partial I}\frac{\partial p_2}{\partial \gamma} - \frac{\partial p_1}{\partial \gamma}\frac{\partial p_2}{\partial I} \equiv \partial_{x_2}\phi_0\partial_{x_1}H - \partial_{x_1}\phi_0\partial_{x_2}H - \partial_t\phi_0 \\ &= -\partial_t\phi_0 + \nabla\phi_0 \cdot \mathbf{v} \equiv 0. \end{aligned} \tag{F.2}$$

In addition, we have

$$\nabla p_1 = (\partial_{x_1}\phi_0, \partial_{x_2}\phi_0, 0, \partial_t\phi_0), \qquad \nabla p_2 = (\partial_{x_1}\psi, \partial_{x_2}\psi, 1, \partial_t\psi),$$

therefore the vectors ∇p_1 and ∇p_2 are linearly independent whenever $\nabla p_1 \neq \mathbf{0}$. Furthermore, any common level set $L(\alpha_1, \alpha_2)$ of p_1 and p_2, defined as

$$L(\alpha_1, \alpha_2) = \{(\mathbf{x}, I, \boldsymbol{\gamma}) \in S \times \mathcal{I} \times S^1 | p_1(\mathbf{x}, I, \boldsymbol{\gamma}) = \alpha_1, \ p_2(\mathbf{x}, I, \boldsymbol{\gamma}) = \alpha_2\},$$

is compact by the compactness of $S \times \mathcal{I} \times S^1$. Then, by the Liouville–Arnold theorem [6], any connected component of the set $L(\alpha_1, \alpha_2)$ is diffeomorphic to a two-dimensional torus, and hence system (F.1) is completely integrable whenever $\nabla p_1|_{L(\alpha_1,\alpha_2)} \neq \mathbf{0}$, or, equivalently

$$\nabla\phi_0|_{L(\alpha_1,\alpha_2)} \neq \mathbf{0}. \tag{F.3}$$

This last inequality follows from $\nabla p_1|_{L(\alpha_1,\alpha_2)} \neq \mathbf{0}$, because $\nabla\phi_0(\mathbf{x}, t) = \mathbf{0}$ implies $\partial_t\phi_0(\mathbf{x}, t) = 0$ by Eq. (4). Because $\nabla\mathbf{F}_{t_0}^t(\mathbf{x}_0)$ is a diffeomorphism, the region D defined in the statement of the theorem contains full trajectories of **v** along which $\nabla c_0(\mathbf{x}, t) \neq \mathbf{0}$ (cf. formula (A.4)). Therefore, condition (F.3) holds for all trajectories within D, making **v** completely integrable on D.

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