Published in *Bull. Australian Math. Soc.* 57 (1998), 305-327.

THE EXPONENTIAL STABILITY OF THE PROBLEM OF TRANSMISSION OF THE WAVE EQUATION

Weijiu Liu and Graham H. Williams

Department of Mathematics
University of Wollongong
Northfields Avenue, Wollongong
NSW 2522, Australia
E-mail: w.liu@uow.edu.au; ghw@uow.edu.au

Abstract

The problem of exponential stability of the problem of transmission of the wave equation with lower-order terms is considered. Making use of the classical energy method and multiplier technique, we prove that this problem of transmission is exponentially stable.

Key Words: Wave equation, Problem of transmission, Exponential stability.

AMS subject classification: 35B35, 35L05.

1. Introduction

Throughout this paper, let Ω be a bounded domain (open, nonempty, and connected) in $\mathbb{R}^n$ ($n \geq 1$) with a boundary $\Gamma = \partial\Omega$ of class C^2 which consists of two parts, S_1 and S_2 (see Figure 1 below). S_1 is assumed to be either empty or to have a nonempty interior and $S_2 \neq \emptyset$ and relatively open in Γ. Assume $\overline{S}_1 \cap \overline{S}_2 = \emptyset$. Let S_0 with $\overline{S}_0 \cap \overline{S}_1 = \overline{S}_0 \cap \overline{S}_2 = \emptyset$ be a regular hypersurface of class C^2, which separates Ω into two domains, Ω_1 and Ω_2, such that $S_1 \subset \Gamma_1 = \partial\Omega_1$ and $S_2 \subset \Gamma_2 = \partial\Omega_2$. For $T > 0$, set $Q = \Omega \times (0,T)$, $Q_1 = \Omega_1 \times (0,T)$, $Q_2 = \Omega_2 \times (0,T)$, $\Sigma_i = S_i \times (0,T)$ $(i = 0,1,2)$. The following figure is a typical domain of this kind.

In this paper we shall be concerned with the problem of rate of exponential decay of energy for the problem of transmission of the wave equation with lower-order terms and with dissipative boundary condition of Robin type:

$$\begin{cases} u_i'' - a_i\Delta u_i + qu_i = 0 & \text{in } \Omega_i \times (0,\infty), \\ u_i(x,0) = u_i^0(x), \quad u_i'(x,0) = u_i^1(x) & \text{in } \Omega_i, \qquad i = 1,2, \\ u_1 = 0 & \text{on } S_1 \times (0,\infty), \\ \dfrac{\partial u_2}{\partial \nu} + \alpha(x)u_2 + \sigma(x)u_2' = 0, & \text{on } S_2 \times (0,\infty), \\ u_1 = u_2, \quad a_1\dfrac{\partial u_1}{\partial \nu} = a_2\dfrac{\partial u_2}{\partial \nu} & \text{on } S_0 \times (0,\infty). \end{cases} \tag{1.1}$$

In (1.1), ν denotes the unit normal on Γ and S_0 directing towards the exterior of Ω and Ω_1, a_1 and a_2 are positive constants, the functions $q : \Omega \to I\!R$, α, σ : $S_2 \to I\!R$ are nonnegative and satisfy

$$q \in L^\infty(\Omega), \quad \alpha,\ \sigma \in C^1(S_2). \tag{1.2}$$

There has been extensive work on energy decay for the wave equation. The poineering work (see [17], [23]) was first performed in the mid-seventies in studies aimed at achieving energy decay rates for the wave equation exterior to a bounded obstacle (the so-called "exterior" problem). In contrast, the "interior" problem is more difficult than the "exterior" problem, since the latter enjoys the advantage that the energy distributes itself over an infinite region as $t \to \infty$. Russell [21] made a conjecture in 1974 concerning uniform energy decay rates for the interior problem. This conjecture was (see [2-6]) verified by Chen (see [2-6]) under some natural geometrical conditions on Ω. Lagnese [11] further relaxed the geometrical conditions on Ω by obtaining a key inequality which is of independent interest. More recently, Bardos, Lebeau, and Rauch [1] considered general second order hyperbolic equations but with smooth coefficients.

We note that in the previous work the coefficients of the equation are required to be sufficiently smooth and it seems that the problem of transmission has not been considered yet. Therefore, by applying the classical energy method and multiplier technique, we here discuss this problem and generalize some known results to the case of transmission.

Set

$$u = \begin{cases} u_1, & x \in \Omega_1, \\ u_2, & x \in \Omega_2, \end{cases} \qquad u^0 = \begin{cases} u_1^0, & x \in \Omega_1, \\ u_2^0, & x \in \Omega_2, \end{cases} \qquad u^1 = \begin{cases} u_1^1, & x \in \Omega_1, \\ u_2^1, & x \in \Omega_2. \end{cases}$$

We define the energy of system (1.1) by

$$\begin{aligned} E(u,t) = &\frac{1}{2}\int_\Omega \Big[| \, u'(x,t) \, |^2 + a(x) \mid \nabla u(x,t) \mid^2 + q(x)|u(x,t)|^2 \Big] dx \\ &+ \frac{1}{2}\int_{S_2} a(x)\alpha(x)|u|^2 d\Gamma. \end{aligned}$$

Let $H^s(\Omega)$ always denote the usual Sobolev space and $\|\cdot\|_{s,\Omega}$ its norm for any $s \in I\!R$. Let

$$L^2(\Omega, S_1) = \begin{cases} \{u \in L^2(\Omega) \ : \ \int_\Omega u(x)\ dx = 0\}, & \text{if } S_1 = \emptyset,\ q \equiv 0, \text{ and } \alpha \equiv 0, \\ L^2(\Omega), & \text{otherwise;} \end{cases} \tag{1.3}$$

$$H^1(\Omega, S_1) = \begin{cases} \{u \in H^1(\Omega) \ : \ \int_\Omega u(x)\ dx = 0\}, & \text{if } S_1 = \emptyset,\ q \equiv 0, \text{ and } \alpha \equiv 0, \\ \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } S_1\}, & \text{otherwise;} \end{cases} \tag{1.4}$$

$$H^1_{S_1}(\Omega) = \begin{cases} H^1(\Omega), & \text{if } S_1 = \emptyset, \\ \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } S_1\}, & \text{otherwise.} \end{cases} \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.1. *Let ν denote the unit normal on Γ and S_0 directing towards the exterior of Ω and Ω_1. Assume there is a vector field $l(x) = (l_1(x), \cdots,\ l_n(x))$ of class $C^2(\overline{\Omega})$ such that*

(i) $l \cdot \nu \le 0$ a.e. on S_1 with respect to the (n-1)-dimensional surface measure;

(ii) $l \cdot \nu \ge \eta > 0$ a.e. on S_2 with respect to the (n-1)-dimensional surface measure;

(iii) $(a_1 - a_2) l \cdot \nu \ge 0$ a.e. on S_0 with respect to the (n-1)-dimensional surface measure;

(iv) the matrix $(\dfrac{\partial l_i}{\partial x_j} + \dfrac{\partial l_j}{\partial x_i})$ is uniformly positive definite on Ω;

(v) there exists a constant $\sigma_0 > 0$ such that

$$\sigma \ge \sigma_0 \qquad \text{on } S_2.$$

Then there are positive constants M, τ such that

$$E(u,t) \le M e^{-\tau t} E(u,0), \qquad \text{for all } t \ge 0 \tag{1.6}$$

for all solutious u of (1.1) with $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$.

In the proof of Theorem 1.1 below, condition (iii) is crucial. Whether Theorem 1.1 still holds if condition (iii) fails is an open problem. The vector field $l(x)$ was first introduced in [4] and further improved in [11]. We here give an example of $l(x)$ which satisfies conditions (i)-(iv) of Theorem 1.1. Let $\Omega = \{x \in I\!R^2 \ : \ 1 < |x| < 3\}$ and $S_0 = \{x \in I\!R^2 \ : \ |x| = 2\}$. Then $S_1 = \{x \in I\!R^2 \ : \ |x| = 1\}$ and $S_2 = \{x \in I\!R^2 \ : \ |x| = 3\}$. It is easy to see that $l(x) = x$ is the vector field as required.

In comparison with existing results, Theorem 1.1 generalizes the result of Lagnese [11] to the case of transmission with Robin boundary conditions. Also, it generalizes Theorem 1 of [8] and Theorem 8.15 of [12, p.117] in three aspects: firstly, the vector field $m(x) = x - x^0 = (x_1 - x_1^0, \cdots, x_n - x_n^0)$ in the previous theorems is replaced by more general vector field $l(x)$; secondly the condition $\min\limits_{S_2} \alpha(x) > 0$ of

Theorem 8.15 of [12] has been moved off; thirdly, we have considered the problem of transmission. In addition, the most interesting part of this paper may be the strategy for handling the case where α is not necessarily small.

The rest of this paper is divided into two sections. In Section 2, we briefly discuss the well-posedness of problem (1.1) via the theory of semigroups of linear bounded operators. In Section 3, we prove Theorem 1.1.

2. Well-posedness

The well-posedness of problem (1.1) is by now well known in the case where $a_1 = a_2$ (see [2], [12], [13, p.137-139]), and can be similarly treated without any difficulty in the case where $a_1 \neq a_2$. For completeness, we give an outline.

Set

$$u = \begin{cases} u_1, & x \in \Omega_1, \\ u_2, & x \in \Omega_2, \end{cases} \qquad u^0 = \begin{cases} u_1^0, & x \in \Omega_1, \\ u_2^0, & x \in \Omega_2, \end{cases} \qquad u^1 = \begin{cases} u_1^1, & x \in \Omega_1, \\ u_2^1, & x \in \Omega_2, \end{cases} \tag{2.1}$$

$$a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases} \tag{2.2}$$

In the sequel, $u,\ u^0,\ u^1$ always means (2.1); an integral of u on a domain Ω means the sum of two integrals of u_1 and u_2 on the subdomains Ω_1 and Ω_2; that an equation related to u holds on a domain Ω means that the equation holds on the subdomains Ω_1 and Ω_2, respectively.

Problem (1.1) can be formulated as an abstract Cauchy problem:

$$\begin{cases} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & I \\ a(x)\Delta - q & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \end{cases} \tag{2.3}$$

in the Hilbert space

$$\mathcal{H}_1 = H^1(\Omega, S_1) \times L^2(\Omega, S_1)$$

for an initial condition (u^0, u^1) with

$$D(A) = \{(u,v)\ :\ (u,v) \in H^2(\Omega_1, \Omega_2, S_1) \times H^1(\Omega, S_1),\ \frac{\partial u}{\partial \nu} + \alpha u + \sigma v = 0 \text{ on } S_2\}.$$

The spaces used for these definition are given by (1.3)-(1.5). In addition,

$$H^2_{S_1}(\Omega_1, \Omega_2) = \Big\{u \in H^1(\Omega)\ :\ u_i \in H^2(\Omega_i),\ i = 1, 2;\ u = 0 \text{ on } S_1,$$

$$\text{and } a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \text{ on } S_0 \Big\};$$

$$H^2(\Omega_1, \Omega_2, S_1) = \begin{cases} \Big\{ u \in H^2_{S_1}(\Omega_1, \Omega_2) \ : \ \sum_{i=1}^{2} \int_{\Omega_i} u_i dx = 0, \\ \qquad \sum_{i=1}^{2} \int_{\Omega_i} (a_i \Delta u_i - q u_i) \, dx = 0 \Big\}, & \text{if } S_1 = \emptyset, \ q \equiv 0, \text{ and } \alpha \equiv 0, \\ H^2_{S_1}(\Omega_1, \Omega_2), & \text{otherwise.} \end{cases}$$

Note that $H^2(\Omega_1, \Omega_2, S_1) \subset H^1(\Omega, S_1)$ because $u_1 = u_2$ on S_0.

If $S_1 = \emptyset$, $q \equiv 0$, and $\alpha \equiv 0$, $L^2(\Omega, S_1)$ in $\mathcal{H}_1$ cannot be replaced by $L^2(\Omega)$ since $H^1(\Omega, S_1)$ is not dense in $L^2(\Omega)$.

In the sequel, we always use the energy scalar product on $\mathcal{H}_1$:

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle = \int_\Omega [a(x) \nabla u \cdot \nabla \bar{\phi} + q u \bar{\phi} + v \bar{\psi}] dx + \int_{S_2} a(x) \alpha(x) u \bar{\phi} d\Gamma, \qquad (2.5)$$

which is equivalent to the scalar product on $\mathcal{H}_1$ induced by $H^1(\Omega) \times L^2(\Omega)$.

As done in [2] or [13, p.137-139], it is easy to verify that the operator A is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}_1$.

We define the energy of system (1.1) by

$$\begin{aligned} E(u,t) = \frac{1}{2} \int_\Omega \Big[| \, u'(x,t) \, |^2 + a(x) \mid \nabla u(x,t) \mid^2 + q(x) |u(x,t)|^2 \Big] dx \\ + \frac{1}{2} \int_{S_2} a(x) \alpha(x) |u|^2 d\Gamma. \end{aligned} \qquad (2.6)$$

Let X be a Banach space. We denote by $C^k([0,T], X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T], X)$
for $C^0([0,T], X)$.

Now an application of the theory of semigroups [19, Chapter 1] gives

Theorem 2.1. *(i) For any initial condition $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$, problem (1.1) has a unique weak solution with*

$$u \in C([0,\infty); H^1_{S_1}(\Omega)) \cap C^1([0,\infty); L^2(\Omega, S_1)). \qquad (2.7)$$

and

$$\frac{\partial u}{\partial \nu}, \quad u' \in L^2(\Sigma_2). \qquad (2.8)$$

Moreover,

$$E(u,t) \le E(u,0), \qquad \forall t \ge 0, \tag{2.9}$$

and there exists a constant $c = c(T) > 0$ such that

$$\| \frac{\partial u}{\partial \nu} \|^2_{L^2(\Sigma_2)} + \| u' \|^2_{L^2(\Sigma_2)} \le cE(u,0). \tag{2.10}$$

(ii) For any initial condition $(u^0, u^1) \in D(A)$, problem (1.1) has a unique strong solution with

$$u \in C([0,\infty); H^2(\Omega_1, \Omega_2, S_1)) \cap C^1([0,\infty); H^1(\Omega, S_1)). \tag{2.11}$$

Moreover, there exists a constant $c = c(T) > 0$ such that for all $t \in [0,T]$

$$\| u'(t) \|_{1,\Omega} + \| u(t) \|_{1,\Omega} + \sum_{i=1}^{2} \| \Delta u_i(t) \|_{0,\Omega_i} \tag{2.12}$$

$$\le c[\| u^1 \|_{1,\Omega} + \| u^0 \|_{1,\Omega} + \sum_{i=1}^{2} \| \Delta u_i^0 \|_{0,\Omega_i}].$$

Proof. If $S_1 \neq \emptyset$, or $q \not\equiv 0$, or $\alpha \not\equiv 0$, then $H^1_{S_1}(\Omega) = H^1(\Omega, S_1)$. It therefore follows from the semigroup theory that problem (1.1) has a unique weak solution u with (2.7) for $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$, and with (2.11) for $(u^0, u^1) \in D(A)$. On the other hand, multiplying the first equation of (1.1) by u_i' and integrating over $\Omega \times (0,T)$, we obtain

$$0 \le \int_{\Sigma_2} a(x)\sigma(x)|u'|^2 d\Sigma = E(u,0) - E(u,T) \le E(u,0). \tag{2.13}$$

This gives (2.9). In addition, by (1.1) we have

$$\int_{\Sigma_2} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Sigma = \int_{\Sigma_2} (\alpha u + \sigma u')^2 d\Sigma, \tag{2.14}$$

and by the trace theorem [15, p.39], we have

$$\int_{\Sigma_2} |u|^2 d\Sigma \le cE(u,0). \tag{2.15}$$

Thus, (2.8) and (2.10) follows from (2.13)-(2.15). To prove (2.12), let $T(t)$ be the semigroup generated by A. Then the solution u of (1.1) can be expressed as

$$\begin{pmatrix} u \\ u' \end{pmatrix} = T(t) \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}.$$

If $(u^0, u^1) \in D(A)$, by the property of semigroup [19, p.4] we have

$$A\begin{pmatrix} u \\ u' \end{pmatrix} = T(t)A\begin{pmatrix} u^0 \\ u^1 \end{pmatrix},$$

that is,

$$\begin{pmatrix} u' \\ a(x)\Delta u - qu \end{pmatrix} = T(t)\begin{pmatrix} u^1 \\ a(x)\Delta u^0 - qu^0 \end{pmatrix}.$$

This yields

$$\| u'(t) \|_{1,\Omega} + \sum_{i=1}^{2} \| \Delta u_i(t) \|_{0,\Omega_i} \tag{2.16}$$

$$\leq c[\| u^1 \|_{1,\Omega} + \| u^0 \|_{0,\Omega} + \sum_{i=1}^{2} \| \Delta u_i^0 \|_{0,\Omega_i} + \| u(t) \|_{0,\Omega}].$$

Hence, (2.12) follows from (2.9) and (2.16).

Suppose $S_1 = \emptyset$, $q \equiv 0$, and $\alpha \equiv 0$. Let $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1) = H^1(\Omega) \times L^2_{S_1}(\Omega)$, and set

$$w^0 = u^0 - \frac{1}{m(\Omega)} \int_\Omega u^0(x)\, dx, \quad w^1 = u^1, \tag{2.17}$$

then $(w^0, w^1) \in \mathcal{H}_1$, where $m(\Omega)$ denotes the Lebesgue measure of Ω. Thus, problem (1.1) has a weak solution w for the initial condition (w^0, w^1). Moreover it satisfies

$$E(w,t) \leq E(w,0), \qquad \forall t \geq 0,$$

and

$$\| \frac{\partial w}{\partial \nu} \|^2_{L^2(\Sigma_2)} + \| w' \|^2_{L^2(\Sigma_2)} \leq cE(w,0).$$

Set

$$u = w + \frac{1}{m(\Omega)} \int_\Omega u^0(x)\, dx. \tag{2.18}$$

It is easy to verify that u is a solution of (1.1) with the initial condition (u^0, u^1) (note that $q \equiv 0$ and $\alpha \equiv 0$). Moreover,

$$E(u,t) = E(w,t) \leq E(w,0) = E(u,0), \qquad \forall t \geq 0,$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu}, \quad u' = w' \in L^2(\Sigma_2),$$

$$\| \frac{\partial u}{\partial \nu} \|^2_{L^2(\Sigma_2)} + \| u' \|^2_{L^2(\Sigma_2)} \leq cE(u,0).$$

⊙

Remark 2.2. If $u^1 \in L^2(\Omega)$ rather than $L^2_{S_1}(\Omega)$, we don't know if problem (1.1) has a solution. Although

$$u = w + \frac{1}{m(\Omega)} \int_\Omega u^0(x)\ dx + \frac{t}{m(\Omega)} \int_\Omega u^1(x)\ dx \tag{2.19}$$

satisfies equation (1.1) and the initial condition (u^0, u^1), it doesn't satisfy the boundary condition on $S_2 \times (0, \infty)$. In (2.19), w is the solution of (1.1) with the initial condition (w^0, w^1) given by

$$w^0 = u^0 - \frac{1}{m(\Omega)} \int_\Omega u^0(x)\ dx, \quad w^1 = u^1 - \frac{1}{m(\Omega)} \int_\Omega u^1(x)\ dx.$$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first generalize an inequality of Lagnese [11, Theorem 2] to the case of transmission. This inequality is the key to proving Theorem 1.1.

In the sequel, all functions are assumed to be real-valued.

Theorem 3.1. *For every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that for every $\delta > 0$,*

$$\int_0^\infty \int_\Omega e^{-2\delta t}(u - I(u^0))^2 dxdt \le c(\varepsilon)E(u,0) + \varepsilon \int_0^\infty \int_\Omega e^{-2\delta t}|u'|^2 dxdt, \tag{3.1}$$

for every solution of (1.1) with $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$, where

$$I(u_0) = \begin{cases} \dfrac{1}{m(\Omega)} \displaystyle\int_\Omega u^0 dx, & \text{if } S_1 = \emptyset,\ q \equiv 0, \text{ and } \alpha \equiv 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Theorem 3.1 will be proven below. We show how it can be used to prove Theorem 1.1.

For convenience, we adopt the following notation. For a vector field $l(x) = (l_1(x), \cdots, l_n(x))$ of class $C^2(\overline{\Omega})$, the additional subscripts in l_{ij} and l_{iij} denote derivatives of the vector field l, e.g., $l_{ij} = \dfrac{\partial l_i}{\partial x_j}$.

Proof of Theorem 1.1. Case I: $\alpha_0 = \max\limits_{x \in S_2} \alpha(x)$ is small enough.

We begin with the case where $(u^0, u^1) \in \mathcal{H}_1$. We may as well assume that $(u^0, u^1) \in D(A)$ since the general case $(u^0, u^1) \in \mathcal{H}_1$ can be handled by a simple limiting process. Then u is a classical solution of (1.1). After a straightforward and tedious calculation, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big[t(\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2) + 2u'(l\cdot\nabla u) + (l_{ii}-1)uu'\Big] \\
&= \operatorname{div}\Big[2ta(x)u'\nabla u + 2a(x)(l\cdot\nabla u)\nabla u + \mid u' \mid^2 l \\
&+ a(x)(l_{ii}-1)u\nabla u - a(x) \mid \nabla u \mid^2 l\Big] \\
&+ 2a(x)(\delta_{ij}-l_{ij})\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - a(x)l_{iij}u\frac{\partial u}{\partial x_j} + (2-l_{ii})q|u|^2 - 2q(l\cdot\nabla u)u,
\end{aligned}
\tag{3.3}
$$

where δ_{ij} denote the Kronecker symbol, i.e.,

$$
\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}
$$

and summation convention is assumed. Set

$$
\begin{aligned}
P(t) = &\int_\Omega \Big[\frac{t}{2}(\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2) + 2u'(l\cdot\nabla u) + (l_{ii}-1)uu'\Big]dx \\
&+ \int_{S_2} \frac{t}{2}a(x)\alpha(x) \mid u \mid^2 d\Gamma.
\end{aligned}
\tag{3.4}
$$

Since

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega t(\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx \\
&= \int_\Omega (\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx \\
&+ 2t\int_\Omega (u'u'' + a(x)\nabla u\cdot\nabla u' + quu')dx \\
&= \int_\Omega (\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx + 2t\int_{S_2} a(x)u'\frac{\partial u}{\partial \nu}d\Gamma,
\end{aligned}
$$

and

$$
\frac{\partial}{\partial t}\int_{S_2}\frac{t}{2}a(x)\alpha(x) \mid u \mid^2 d\Gamma = \int_{S_2}\frac{1}{2}a(x)\alpha(x) \mid u \mid^2 d\Gamma + \int_{S_2} ta(x)\alpha(x)uu'd\Gamma,
$$

it follows from (3.3) and the divergence theorem that

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= -\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega t(\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx \\
&+ \int_\Omega 2a(x)(\delta_{ij} - l_{ij})\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx \qquad (= I_2) \\
&+ \int_\Omega \Big[-a(x)l_{iij}u\frac{\partial u}{\partial x_j} + (2 - l_{ii})q|u|^2 - 2q(l\cdot\nabla u)u\Big]dx \qquad (= I_3) \\
&+ \int_{S_1}\Big[2a_1(l\cdot\nabla u_1)\frac{\partial u_1}{\partial \nu} - a_1 \mid \nabla u_1 \mid^2 l\cdot\nu\Big]d\Gamma \qquad (= I_4) \\
&+ \int_{S_2}\Big[a_2tu_2'(2\frac{\partial u_2}{\partial \nu} + \alpha(x)u_2) + 2a_2(l\cdot\nabla u_2)\frac{\partial u_2}{\partial \nu} + \mid u_2' \mid^2 l\cdot\nu \\
&+ a_2(l_{ii} - 1)u_2\frac{\partial u_2}{\partial \nu} - a_2 \mid \nabla u_2 \mid^2 l\cdot\nu + \frac{1}{2}a_2\alpha(x)|u_2|^2\Big]d\Gamma \\
&+ \int_{S_0}\Big[2a_1(l\cdot\nabla u_1)\frac{\partial u_1}{\partial \nu} - 2a_2(l\cdot\nabla u_2)\frac{\partial u_2}{\partial \nu} \\
&+ (a_2 \mid \nabla u_2 \mid^2 - a_1 \mid \nabla u_1 \mid^2)l\cdot\nu\Big]d\Gamma \qquad (= I_6) \\
&= -\frac{1}{2}\int_\Omega(\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx - \frac{1}{2}\int_{S_2} a\alpha \mid u \mid^2 d\Gamma \qquad (= I_1) \\
&+ I_2 + I_3 + I_4 \\
&+ \int_{S_2}\Big[a_2tu_2'(\frac{\partial u_2}{\partial \nu} + \alpha(x)u_2) + 2a_2(l\cdot\nabla u_2)\frac{\partial u_2}{\partial \nu} + \mid u_2' \mid^2 l\cdot\nu \\
&+ a_2(l_{ii} - 1)u_2\frac{\partial u_2}{\partial \nu} - a_2 \mid \nabla u_2 \mid^2 l\cdot\nu + a_2\alpha(x)|u_2|^2\Big]d\Gamma \qquad (= I_5) \\
&+ I_6 \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

Since, for any positive constant c, cl still satisfies the conditions $(i) - (iv)$ of Theorem 1.1, we may assume that $(2\delta_{ij} - l_{ij} - l_{ji})$ is negative definite in Ω by multiplying l by an enough large positive constant. Thus we have $I_2 \le 0$. From condition (i) of Theorem 1.1 and the fact $\nabla u = \frac{\partial u}{\partial \nu}\nu$ on S_1 it follows that

$$
I_4 = \int_{S_1} a_1 \mid \frac{\partial u_1}{\partial \nu} \mid^2 l\cdot\nu d\Gamma \le 0. \tag{3.5}
$$

Concerning I_5, it follows from the fourth equation of (1.1) and the condition (ii) of

Theorem 1.1 that

$$
\begin{aligned}
I_5 = & \int_{S_2} \Big[-a_2\sigma t|u_2'|^2 - 2a_2(l\cdot\nabla u_2)(\alpha u_2 + \sigma u_2') + \mid u_2' \mid^2 l\cdot\nu \\
& - a_2(l_{ii}-1)u_2(\alpha u_2 + \sigma u_2') - a_2 \mid \nabla u_2 \mid^2 l\cdot\nu + a_2\alpha(x)|u_2|^2 \Big] d\Gamma \\
\le & \int_{S_2} \Big[-a_2\sigma t|u_2'|^2 + \frac{a_2\eta}{4}|\nabla u_2|^2 + c(a,l,\eta)\alpha^2|u_2|^2 \\
& + \frac{a_2\eta}{4}|\nabla u_2|^2 + c(a,l,\eta,\sigma)|u_2'|^2 + \mid u_2' \mid^2 l\cdot\nu \\
& - a_2(l_{ii}-1)\alpha|u_2|^2 + \varepsilon|u_2|^2 + c(\varepsilon,a,l,\sigma)|u_2'|^2 - a_2\eta \mid \nabla u_2 \mid^2 + a_2\alpha(x)|u_2|^2 \Big] d\Gamma \\
\le & \int_{S_2} \Big[(-a_2 t\sigma_0 + l\cdot\nu + c(\varepsilon,a,l,\sigma,\eta))|u_2'|^2 - \frac{a_2\eta}{2}|\nabla u_2|^2 \Big] d\Gamma \\
& + \int_{S_2} (\varepsilon + c(a,l,\eta)(\alpha_0+\alpha_0^2))|u_2|^2 d\Gamma \\
= & I_{51} + I_{52}.
\end{aligned}
$$

By the trace theorem, we have

$$
I_{52} \le c_1(\varepsilon + c(a,l,\eta)(\alpha_0+\alpha_0^2))\|u\|^2_{H^1(\Omega,S_1)}. \tag{3.6}
$$

Concerning I_3, we have

$$
I_3 \le \varepsilon \int_\Omega |\nabla u|^2 dx + c(\varepsilon,a,l,q)\int_\Omega |u|^2 dx. \tag{3.7}
$$

If ε and α_0 are small enough, then by (3.6)-(3.7) we have

$$
\begin{aligned}
I_3 + I_{52} \le & \frac{1}{4}\int_\Omega (\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx + \frac{1}{4}\int_{S_2} a\alpha \mid u \mid^2 d\Gamma \\
& + c(\varepsilon,a,l,q)\int_\Omega |u|^2 dx.
\end{aligned}
$$

It therefore follows that

$$
\begin{aligned}
I_1 + I_3 + I_{52} \le & -\frac{1}{4}\int_\Omega (\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx \\
& - \frac{1}{4}\int_{S_2} a\alpha \mid u \mid^2 d\Gamma + c(\varepsilon,a,l,q)\int_\Omega |u|^2 dx,
\end{aligned}
$$

if ε and α_0 are small enough. Fix ε and α_0, then $I_{51} < 0$ if t is large enough. We also prove that $I_6 \le 0$. Since $u_1 = u_2$ on S_0, we have

$$
\nabla(u_2 - u_1) = \frac{\partial(u_2 - u_1)}{\partial \nu}\nu, \qquad \text{on } S_0,
$$

then,

$$
\begin{aligned}
|\nabla u_2|^2 &= |\nabla u_1|^2 + 2\Big(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu}\Big)\frac{\partial u_1}{\partial \nu} + \Big(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu}\Big)^2 \\
&= |\nabla u_1|^2 + \Big(\frac{\partial u_2}{\partial \nu}\Big)^2 - \Big(\frac{\partial u_1}{\partial \nu}\Big)^2.
\end{aligned}
$$

So,

$$
\begin{aligned}
& 2a_1(l\cdot\nabla u_1)\frac{\partial u_1}{\partial \nu} - 2a_2(l\cdot\nabla u_2)\frac{\partial u_2}{\partial \nu} + (a_2 \mid \nabla u_2 \mid^2 - a_1 \mid \nabla u_1 \mid^2) l\cdot\nu \\
&= 2a_1(l\cdot\nabla u_1)\frac{\partial u_1}{\partial \nu} - 2a_2\Big[l\cdot\nabla u_1 + \Big(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu}\Big)l\cdot\nu\Big]\frac{\partial u_2}{\partial \nu} \\
&+ \Big[a_2\Big(\mid \nabla u_1 \mid^2 + \Big(\frac{\partial u_2}{\partial \nu}\Big)^2 - \Big(\frac{\partial u_1}{\partial \nu}\Big)^2\Big) - a_1 \mid \nabla u_1 \mid^2\Big] l\cdot\nu \\
&= 2a_1(l\cdot\nabla u_1)\frac{\partial u_1}{\partial \nu} - 2a_1\Big[l\cdot\nabla u_1 + \Big(\frac{a_1}{a_2}\frac{\partial u_1}{\partial \nu} - \frac{\partial u_1}{\partial \nu}\Big)l\cdot\nu\Big]\frac{\partial u_1}{\partial \nu} \\
&+ \Big[a_2\Big(\mid \nabla u_1 \mid^2 + \frac{a_1^2}{a_2^2}\Big(\frac{\partial u_1}{\partial \nu}\Big)^2 - \Big(\frac{\partial u_1}{\partial \nu}\Big)^2\Big) - a_1 \mid \nabla u_1 \mid^2\Big] l\cdot\nu \\
&= (a_2 - a_1) \mid \nabla u_1 \mid^2 l\cdot\nu - \frac{(a_2-a_1)^2}{a_2}\Big(\frac{\partial u_1}{\partial \nu}\Big)^2 l\cdot\nu.
\end{aligned}
$$

This show that $I_6 \le 0$ because of (iii) of Theorem 1.1. It therefore follows that

$$
\begin{aligned}
\frac{dP}{dt} \le & -\frac{1}{4}\int_\Omega (\mid u' \mid^2 + a(x) \mid \nabla u \mid^2 + q|u|^2)dx \\
& - \frac{1}{4}\int_{S_2} a\alpha \mid u \mid^2 d\Gamma + c(\varepsilon)\int_\Omega |u|^2 dx, \qquad t \ge T_1,
\end{aligned} \tag{3.8}
$$

if T_1 is large enough.

On the other hand, there exist T_2 sufficiently large such that

$$
0 \le P(t) \le c(t+1)E(u,t), \qquad t \ge T_2, \tag{3.9}
$$

where c is a constant independent of t, u.

Let $\delta > 0$ be fixed. Set $T = \max\{T_1, T_2\}$. Multiplying (3.8) by $e^{-2\delta t}$ and integrating from T to $+\infty$ we get

$$
2\delta\int_T^\infty e^{-2\delta t}P(t)dt + \frac{1}{8}\int_T^\infty e^{-2\delta t}E(u,t)dt \le c_1E(u,0) + c_2\int_T^\infty\int_\Omega e^{-2\delta t}u^2 dxdt, \tag{3.10}
$$

where c_1, c_2 are independent of δ. It therefore follows from (3.9) and (3.10) that

$$
\int_0^\infty e^{-2\delta t}E(u,t)dt \le c_1E(u,0) + c_2\int_0^\infty\int_\Omega e^{-2\delta t}u^2 dxdt.
$$

Applying Theorem 3.1, we conclude that

$$\int_0^\infty e^{-2\delta t} E(u,t)dt \le cE(u,0).$$

Letting $\delta \to 0$, we obtain

$$\int_0^\infty E(u,t)dt \le cE(u,0).$$

By Theorem 4.1 of [19, p.116], there are positive constants $M,\ \tau$ such that

$$E(u,t) \le Me^{-\tau t}E(u,0), \qquad t \ge 0. \tag{3.11}$$

If $S_1 = \emptyset,\ q \equiv 0,\ \alpha \equiv 0$, and $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2_{S_1}(\Omega)$ rather than $\mathcal{H}_1$, then we take (w^0, w^1) as in (2.17). Let w be the solution of (1.1) with the initial condition (w^0, w^1), then

$$u = w + \frac{1}{m(\Omega)} \int_\Omega u^0(x)\ dx$$

is the solution of (1.1) with the initial condition (u^0, u^1) and (3.11) holds for w. Therefore,

$$E(u,t) = E(w,t) \le Me^{-\tau t}E(w,0) = Me^{-\tau t}E(u,0).$$

We will use the control-theoretic method given in [2] and [16] to prove Theorem 1.1 in the case that α_0 is arbitrary. Therefore we now employ Russell's " controllability via stabilizability " principle (see [21]) to solve the following exact controllability problem:

For (y_i^0, y_i^1) in a suitable Hilbert space and T large enough, find a control function $\phi(x,t)$ such that the solution of

$$\begin{cases} y_i'' - a_i \Delta y_i + q y_i = 0 & \textit{in } Q_i, \\ y_i(0) = y_i^0, \quad y_i'(0) = y_i^1 & \textit{in } \Omega_i,\ i = 1,2, \\ y_1 = 0 & \textit{on } \Sigma_1, \\ \dfrac{\partial y_2}{\partial \nu} + \alpha y_2 = \phi & \textit{on } \Sigma_2, \\ y_1 = y_2, \qquad a_1 \dfrac{\partial y_1}{\partial \nu} = a_2 \dfrac{\partial y_2}{\partial \nu} & \textit{on } \Sigma_0, \end{cases} \tag{3.12}$$

satisfies

$$y_i(x,T;\phi) = y_i'(x,T;\phi) = 0 \qquad \textit{in}\ \ \Omega_i,\ i = 1,2. \tag{3.13}$$

Because the problem is linear, this is equivalent to steering any initial state to any terminal state. This controllability problem was discussed in [1] in the general case of second order hyperbolic equations, but the coefficients of the equations are required to be smooth enough. Thus, the problem (3.12) (3.13) here is not covered by [1]. With the help of the Hilbert Uniqueness Method, the problem (3.12) (3.13) was also considered in [18] but with $\alpha = 0$.

Theorem 3.2. *Suppose all assumptions of Theorem 1.1 are satisfied. Suppose* α_0 *is small enough and* T *is given large enough. Then for any* $(y^0, y^1) \in \mathcal{H}_1$, *there exists a boundary control function*

$$\phi(x,t) \in L^2(\Sigma_2)$$

such that the solution of (3.12) satisfies (3.13). Moreover, there exist positive constants $c_1(T)$, $c_2(T)$ *such that*

$$c_1 E(y,0) \leq \| \phi \|^2_{L^2(\Sigma_2)} \leq c_2 E(y,0). \tag{3.14}$$

Proof. We first consider the problem:

$$\begin{cases} u_i'' - a_i \Delta u_i + q u_i = 0 & \text{in } \Omega_i \times (0,\infty), \\ u_i(0) = u_i^0, \quad u_i'(0) = u_i^1 & \text{in } \Omega_i, \ i = 1,2, \\ u_1 = 0 & \text{on } S_1 \times (0,\infty), \\ \dfrac{\partial u_2}{\partial \nu} + \alpha u_2 + \sigma u_2' = 0 & \text{on } S_2 \times (0,\infty), \\ u_1 = u_2, \quad a_1 \dfrac{\partial u_1}{\partial \nu} = a_2 \dfrac{\partial u_2}{\partial \nu} & \text{on } S_0 \times (0,\infty), \end{cases} \tag{3.15}$$

which has a unique weak solution with

$$u(t) \in C([0,\infty); H^1(\Omega, S_1)) \cap C^1([0,\infty); L^2(\Omega, S_1))$$

and

$$u' \in L^2(\Sigma_2)$$

for any $(u^0, u^1) \in \mathcal{H}_1$ thanks to Theorem 2.1.

Using the solution u of (3.15), we then consider the backwards problem:

$$\begin{cases} w_i'' - a_i \Delta w_i + q w_i = 0 & \text{in } \Omega_i \times (0,\infty), \\ w_i(T) = u_i(T), \quad w_i'(T) = u_i'(T) & \text{in } \Omega_i, \ i = 1,2, \\ w_1 = 0 & \text{on } S_1 \times (0,\infty), \\ \dfrac{\partial w_2}{\partial \nu} + \alpha w_2 - \sigma w_2' = 0 & \text{on } S_2 \times (0,\infty), \\ w_1 = w_2, \quad a_1 \dfrac{\partial w_1}{\partial \nu} = a_2 \dfrac{\partial w_2}{\partial \nu} & \text{on } S_0 \times (0,\infty). \end{cases}$$

which has a unique weak solution with

$$w \in C([0,T]; H^1(\Omega, S_1)) \cap C^1([0,T]; L^2(\Omega, S_1))$$

and

$$w' \in L^2(\Sigma_2)$$

since $(u(x,T), u'(x,T)) \in \mathcal{H}_1$.

Set

$$y = u - w,$$

and

$$\phi = -\sigma(x)(w' + u') \in L^2(\Sigma_2),$$

then y satisfies

$$\begin{cases} y_i'' - a_i \Delta y_i + q y_i = 0 & \text{in } \Omega_i \times (0,\infty), \\ y_i(0) = u_i^0 - w_i(0), \quad y_i'(0) = u_i^1 - w_i'(0) & \text{in } \Omega_i, \\ y_i(T) = 0, \quad y_i'(T) = 0 & \text{in } \Omega_i, \ i = 1,2, \\ y_1 = 0 & \text{on } S_1 \times (0,\infty), \\ \dfrac{\partial y_2}{\partial \nu} + \alpha y_2 = \phi(x) & \text{on } S_2 \times (0,\infty), \\ y_1 = y_2, \quad a_1 \dfrac{\partial y_1}{\partial \nu} = a_2 \dfrac{\partial y_2}{\partial \nu} & \text{on } S_0 \times (0,\infty). \end{cases}$$

We define an operator Λ by

$$\Lambda(u^0, u^1) = (w(x,0), w'(x,0)).$$

Then it is clear that Λ is a linear operator from $\mathcal{H}_1$ into $\mathcal{H}_1$. Moreover, by Theorem 1.1 (in the case where α_0 is small enough) we have

$$\begin{aligned} \| \Lambda(u^0, u^1) \|_{\mathcal{H}_1}^2 &= E(w,0) \\ &\le M e^{-\tau T} E(w,T) \\ &\le M^2 e^{-2\tau T} \| (u^0, u^1) \|_{\mathcal{H}_1} . \end{aligned} \tag{3.16}$$

Therefore,

$$\| \Lambda \| \le M e^{-\tau T}.$$

Taking T large enough so that $M e^{-\tau T} < 1$, then $I - \Lambda$ is an isomorphism from $\mathcal{H}_1$ onto $\mathcal{H}_1$. Thus, for any $(y^0, y^1) \in \mathcal{H}_1$, there exists a unique $(u^0, u^1) \in \mathcal{H}_1$ such that

$$(y^0, y^1) = (u^0, u^1) - \Lambda(u^0, u^1) = (u^0, u^1) - (w(x,0), w'(x,0)). \tag{3.17}$$

Consequently, we have constructed a control function $\phi = -\sigma(x)(w' + u')$ solving the exact controllability problem (3.12)-(3.13).

On the other hand, multiplying the first equation of (3.15) by u' and integrating over Q, we obtain

$$\int_{\Sigma_2} a(x)\sigma(x)|u'|^2 d\Sigma = E(u,0) - E(u,T).$$

Likewise, we have

$$\int_{\Sigma_2} a(x)\sigma(x)|w'|^2 d\Sigma = E(w,T) - E(w,0).$$

It therefore follows from Theorem 1.1 that there exist positive constants $c_1,\ c_2$ such that

$$c_1(1-Me^{-\tau T})^{\frac{1}{2}} E^{\frac{1}{2}}(u,0) \leq \| u' \|_{L^2(\Sigma_2)} \leq c_2 E^{\frac{1}{2}}(u,0),$$

and

$$c_1(1-Me^{-\tau T})^{\frac{1}{2}} E^{\frac{1}{2}}(w,T) \leq \| w' \|_{L^2(\Sigma_2)} \leq c_2 E^{\frac{1}{2}}(w,T).$$

Noting $E(u,T) = E(w,T)$, we deduce from the triangle inequality and Theorem 1.1 that

$$\begin{aligned} c_1[1-Me^{-\tau T}]^{\frac{1}{2}}[1-(Me^{-\tau T})^{\frac{1}{2}}]E^{\frac{1}{2}}(u,0) &\leq \| u'+w' \|_{L^2(\Sigma_2)} \\ &\leq c_2[1+(Me^{-\tau T})^{\frac{1}{2}}]E^{\frac{1}{2}}(u,0). \end{aligned} \tag{3.18}$$

Since $I - \Lambda$ is an isomorphism, (3.14) follows from (3.17) and (3.18).

Proof of Theorem 1.1. Case II: α_0 is arbitrary.

Let $\varepsilon > 0$ be small enough and T large enough. It then follows from Theorem 3.2 that there exists a control ϕ such that

$$\begin{cases} y_i'' - a_i\Delta y_i + qy_i = 0 & \text{in } Q_i, \\ y_i(0) = 0, \quad y_i'(0) = 0 & \text{in } \Omega_i, \\ y_i(T) = u_i(T), \quad y_i'(T) = u_i'(T) & \text{in } \Omega_i,\ i = 1,2, \\ y_1 = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y_2}{\partial \nu} + \varepsilon y_2 = \phi(x) & \text{on } \Sigma_2, \\ y_1 = y_2, \quad a_1\dfrac{\partial y_1}{\partial \nu} = a_2\dfrac{\partial y_2}{\partial \nu} & \text{on } \Sigma_0. \end{cases} \tag{3.19}$$

According to the proof of Theorem 3.2, y and ϕ can be written as

$$y = v - w, \qquad \phi = \sigma(v' + w'), \tag{3.20}$$

where v and w are respectively the solutions of

$$\begin{cases} v_i'' - a_i\Delta v_i + qv_i = 0 & \text{in } Q_i, \\ v_i(T) = v_i^0, \quad v_i'(T) = v_i^1 & \text{in } \Omega_i,\ i = 1,2, \\ v_1 = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial v_2}{\partial \nu} + \varepsilon v_2 - \sigma v_2' = 0 & \text{on } \Sigma_2, \\ v_1 = v_2, \quad a_1\dfrac{\partial v_1}{\partial \nu} = a_2\dfrac{\partial v_2}{\partial \nu} & \text{on } \Sigma_0, \end{cases} \tag{3.21}$$

$$
\begin{cases}
w_i'' - a_i \Delta w_i + q w_i = 0 & \text{in } Q_i, \\
w_i(0) = v_i(0), \quad w_i'(0) = v_i'(0) & \text{in } \Omega_i, \ i = 1, 2, \\
w_1 = 0 & \text{on } \Sigma_1, \\
\dfrac{\partial w_2}{\partial \nu} + \varepsilon w_2 + \sigma w_2' = 0 & \text{on } \Sigma_2, \\
w_1 = w_2, \quad a_1 \dfrac{\partial w_1}{\partial \nu} = a_2 \dfrac{\partial w_2}{\partial \nu} & \text{on } \Sigma_0.
\end{cases} \tag{3.22}
$$

In (3.21), (v^0, v^1) are chosen to be such that

$$
(v^0, v^1) - (w(T), w'(T)) = (v^0, v^1) - \Lambda(v^0, v^1) = (u(T), u'(T)). \tag{3.23}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
0 &= \int_Q \Big[y'(u'' - a\Delta u + qu) + u'(y'' - a\Delta y + qy) \Big] dxdt \\
&= \int_Q \frac{\partial}{\partial t}(u'y' + a\nabla u \cdot \nabla y + quy) dxdt - \int_{\Sigma_2} a(y' \frac{\partial u}{\partial \nu} + u' \frac{\partial y}{\partial \nu}) d\Sigma \\
&= \int_\Omega (|u'(T)|^2 + a|\nabla u(T)|^2 + q|u(T)|^2) dx + \int_{\Sigma_2} a[y'(\alpha u + \sigma u') + u'(\varepsilon y - \phi)] d\Sigma \\
&= 2E(u, T) + \int_{\Sigma_2} a u'(\sigma y' - \alpha y + \varepsilon y - \phi) d\Sigma
\end{aligned} \tag{3.24}
$$

The following constants $c = c(T)$ denote various positive constans depending on T. By the trace theorem and (3.20) and (3.23), we have

$$
\begin{aligned}
\|y\|^2_{L^2(\Sigma_2)} &\le c \int_0^T \|y(t)\|^2_{H^1(\Omega, S_1)} dt \\
&\le c \int_0^T (\|v(t)\|^2_{H^1(\Omega, S_1)} + \|w(t)\|^2_{H^1(\Omega, S_1)}) dt \\
&\le c(E(v, T) + E(w, 0)) \\
&\le cE(v, T) \\
&\le cE(u, T).
\end{aligned} \tag{3.25}
$$

Moreover, since

$$
\int_{\Sigma_2} a\sigma |v'|^2 d\Sigma = E(v, T) - E(v, 0), \tag{3.26}
$$

and

$$
\int_{\Sigma_2} a\sigma |w'|^2 d\Sigma = E(w, 0) - E(w, T). \tag{3.27}
$$

It follows from (3.20) and (3.23) that

$$
\|y'\|^2_{L^2(\Sigma_2)} \le cE(u, T). \tag{3.28}
$$

By (3.14) we have

$$\|\phi\|^2_{L^2(\Sigma_2)} \le cE(u,T). \tag{3.29}$$

By Cauchy-Schwartz inequality we deduce from (3.24) that

$$(E(u,T))^2 \le c\int_{\Sigma_2} |u'|^2 d\Sigma \int_{\Sigma_2} |\sigma y' - \alpha y + \varepsilon y - \phi|^2 d\Sigma, \tag{3.30}$$

which, combining (3.25), (3.28), and (3.29), yields

$$\int_{\Sigma_2} |u'|^2 d\Sigma \ge cE(u,T). \tag{3.31}$$

On the other hand, we have

$$\int_{\Sigma_2} a\sigma |u'|^2 d\Sigma = E(u,0) - E(u,T). \tag{3.32}$$

We then conclude

$$E(u,0) - E(u,T) \ge cE(u,T),$$

so that

$$E(u,T) \le \frac{1}{1+c} E(u,0).$$

Repeating the above reasoning, we get

$$\begin{aligned} E(u,(k+1)T) &\le \frac{1}{1+c} E(u,kT) \\ &\le \frac{1}{(1+c)^{k+1}} E(u,0), \qquad k = 0,1,2,\cdots \end{aligned}$$

This implies (1.6) with

$$M = 1+c, \qquad \tau = \frac{1}{T}\ln(1+c).$$

The proof of Theorem 1.1 is complete. $\odot$

Remark 3.3. From Theorem 1.1 and the proof of Theorem 3.2 we can conclude that Theorem 3.2 still holds true for arbitrary α_0.

If $S_1 = \phi$, $q \equiv 0$, and $\alpha \equiv 0$, then $\mathcal{H}_1$ in Theorem 3.2 cannot be replaced by $H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$ because $(E(u,0))^{\frac{1}{2}}$ is no longer a norm on $H^1(\Omega) \times L^2(\Omega, S_1)$. Nevertheless, (3.16) still holds on the quotient space $\Big(H^1(\Omega) \times L^2(\Omega, S_1)\Big)\Big/N$, where $N = \{(c,0) \; : \; c \in I\!R\}$. Therefore, Theorem 3.2 still holds when $\mathcal{H}_1$ is replaced by $\Big(H^1(\Omega) \times L^2(\Omega, S_1)\Big)\Big/N$. Because the zero element in $\Big(H^1(\Omega) \times L^2(\Omega, S_1)\Big)\Big/N$ is N, we can only drive any initial state $(y^0, y^1) \in H^1(\Omega) \times L^2(\Omega, S_1)$

to a constant function $(c,0)$. In fact, We can explain this in the following way. For any $(y^0,y^1)\in H^1(\Omega)\times L^2(\Omega)$, set

$$w^0=y^0-\frac{1}{m(\Omega)}\int_\Omega y^0(x)\;dx,\quad w^1=y^1-\frac{1}{m(\Omega)}\int_\Omega y^1(x)\;dx.$$

Then $(w^0,w^1)\in\mathcal{H}_1$. By Theorem 3.2, there exists a control function $\phi(x,t)$ such that the solution w of (3.12) with the initial state (w^0,w^1) satisfies (3.13). It is easy to check that

$$y=w+\frac{1}{m(\Omega)}\int_\Omega y^0(x)\;dx+\frac{t}{m(\Omega)}\int_\Omega y^1(x)\;dx$$

is the solution of (3.12) with the initial state (y^0,y^1), but

$$y(x,T;\phi)=\frac{1}{m(\Omega)}\int_\Omega y^0(x)\;dx+\frac{T}{m(\Omega)}\int_\Omega y^1(x)\;dx,\qquad \text{(a constant)},$$

$$y'(x,T;\phi)=\frac{1}{m(\Omega)}\int_\Omega y^1(x)\;dx,\qquad \text{(a constant)}.$$

At last, we want to prove Theorem 3.1. For this, we need the following lemma.

Lemma 3.4. *Let $\phi\in H^{\frac{1}{2}}(\Gamma)$. Then there exists $u\in H^2(\Omega_1,\Omega_2)$ such that*

$$u=0,\qquad \frac{\partial u}{\partial\nu}=\phi,\qquad \text{on }\Gamma,\tag{3.33}$$

and

$$\|u_1\|_{2,\Omega_1}+\|u_2\|_{2,\Omega_2}\le c\|\phi\|_{H^{\frac{1}{2}}(\Gamma)},\tag{3.34}$$

where c is a positive constant and

$$H^2(\Omega_1,\Omega_2)=\left\{u\;:\;u_i=u|_{\Omega_i}\in H^2(\Omega_i),\;i=1,2;\;u_1=u_2,\;a_1\frac{\partial u_1}{\partial\nu}=a_2\frac{\partial u_2}{\partial\nu}\text{ on }S_0\right\}.\;\blacksquare$$

Proof. By the trace theorem, it follows that there exists $w\in H^2(\Omega)$ such that

$$w=0,\qquad \frac{\partial w}{\partial\nu}=\phi,\qquad \text{on }\Gamma,$$

and

$$\|w\|_{2,\Omega}\le c\|\phi\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Since $w\in H^2(\Omega_2)$, again by the trace theorem, we have

$$w\in H^{\frac{3}{2}}(S_0),\qquad \frac{\partial w}{\partial\nu}\in H^{\frac{1}{2}}(S_0).$$

Also by the trace theorem, it follows that there exists $v \in H^2(\Omega_1)$ such that

$$v = 0, \qquad \frac{\partial v}{\partial \nu} = \phi \qquad \text{on } \Gamma \cap \Gamma_1,$$

$$v = w, \qquad \frac{\partial v}{\partial \nu} = \frac{a_2}{a_1}\frac{\partial w}{\partial \nu} \qquad \text{on } S_0,$$

and

$$\begin{aligned}
\|v\|_{2,\Omega_1} &\le c[\|\phi\|_{H^{\frac{1}{2}}(\Gamma\cap\Gamma_1)} + \|w\|_{H^{\frac{3}{2}}(S_0)} + \|\frac{\partial w}{\partial \nu}\|_{H^{\frac{1}{2}}(S_0)}] \\
&\le c[\|\phi\|_{H^{\frac{1}{2}}(\Gamma\cap\Gamma_1)} + \|w\|_{2,\Omega_2}] \\
&\le c[\|\phi\|_{H^{\frac{1}{2}}(\Gamma\cap\Gamma_1)} + \|w\|_{2,\Omega}] \\
&\le c\|\phi\|_{H^{\frac{1}{2}}(\Gamma)}.
\end{aligned}$$

Then, u defined by

$$u = \begin{cases} v, & x \in \Omega_1, \\ w, & x \in \Omega_2, \end{cases}$$

belongs to $H^2(\Omega_1, \Omega_2)$ and satisfies (3.33) and (3.34)

We also need the following unique continuation theorem for elliptic operators given in [7].

Let $A(x) = (a_{ij}(x))$ be a real symmetric matrix-valued function on Ω satisfying the assumptions:

(i) there exists a $\rho \in (0,1)$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\rho|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \rho^{-1}|\xi|^2; \tag{3.35}$$

(ii) there exists a $K > 0$ such that, for every $x, y \in \Omega$,

$$|a_{ij}(x) - a_{ij}(y)| \le K|x-y|, \qquad i,j = 1,2,\cdots,n. \tag{3.36}$$

Let the potential V satisfy the assumption: for every $x_0 \in \Omega$ there exist $r_0 > 0$ and two constants $C_1,\ C_2 > 0$ such that if $V(x) = V^+(x) - V^-(x)$, then

$$0 \le V^+(x) \le \frac{C_1}{|x-x_0|^2}, \tag{3.37}$$

$$0 \le V^-(x) \le \frac{C_2}{|x-x_0|^2}, \tag{3.38}$$

for any $x \in B_{r_0}(x_0) \cap \Omega$, where $B_{r_0}(x_0) = \{x \in \mathbb{R}^n \ :\ |x - x_0| < r_0\}$.

Theorem 3.5. [7, Corollary 1.1] *Assume that $A(x)$ and $V(x)$ satisfy the above assumptions (3.35)-(3.38). Then the operator $L = -\mathrm{div}(A(x)\nabla) + V(x)$ has* *the unique continuation property in Ω, that is, the only $H^1_{loc}(\Omega)$ solution of $Lu = 0$ which can vanish in an open subset of Ω is $u \equiv 0$.*

Proof of Theorem 3.1. Case I: $S_1 \neq \emptyset$, or $q \neq 0$, or $\alpha \neq 0$. The proof is the same as the one of Theorem 2 of [11] except for the following two pionts:

i) In the proof of Theorem 2 of [11], Lagnese used the analyticity of the solution of

$$\Delta W - \omega^2 W = 0 \qquad \text{in } \Omega, \qquad \omega \text{ real}$$

to conclude that $W = 0$ in Ω if $W = 0$ in an open subset of Ω. For the present case of transmission, we use Theorem 3.5 since we now can no longer appeal to the analyticity of solutions.

ii) The usual trace theorem used in the proof of Theorem 2 of [11] is replaced by Lemma 3.4.

Case II: $S_1 = \emptyset$, $q \equiv 0$, $\alpha \equiv 0$. If $(u^0, u^1) \in H^1_{S_1}(\Omega) \times L^2(\Omega, S_1)$ rather than $\mathcal{H}_1$, then we take (w^0, w^1) as in (2.17). Let w be the solution of (1.1) with the initial condition (w^0, w^1), then

$$u = w + \frac{1}{m(\Omega)} \int_\Omega u^0(x)\, dx$$

is the solution of (1.1) with the initial condition (u^0, u^1) and (3.1) holds for w. Therefore,

$$\begin{aligned}
\int_0^\infty \int_\Omega e^{-2\delta t}(u - I(u^0))^2 dxdt &= \int_0^\infty \int_\Omega e^{-2\delta t} w^2 dxdt \\
&\le c(\varepsilon)E(w,0) + \varepsilon \int_0^\infty \int_\Omega e^{-2\delta t}|w'|^2 dxdt \\
&= c(\varepsilon)E(u,0) + \varepsilon \int_0^\infty \int_\Omega e^{-2\delta t}|u'|^2 dxdt.
\end{aligned}$$

$\odot$

Acknowledgments. The first author is grateful for the financial support from the Overseas Postgraduate Research Scholarship from Australian Government, the University of Wollongong Postgraduate Research Award, and the Department of Mathematics Analysis Research Group Scholarship from the Faculty of Informatics at the University of Wollongong.

References

[1] C. Bardos, G. Lebeau, and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim., 30 (1992), pp.1024-1065.

[2] G. Chen, Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain, J. Math. Pures Appl., 58 (1979), pp.249-273.

[3] G. Chen, Control and stabilization for the wave equation in a bounded domain, SIAM J. Control Optim., 17 (1979), pp.66-81.

[4] G. Chen, A note on the boundary stabilization of the wave equation, SIAM J. Control Optim., 19 (1981), pp.106-113.

[5] G. Chen, Control and stabilization for the wave equation in a bounded domain, part II, SIAM J. Control Optim., 19 (1981), pp.114-122.

[6] G. Chen, Control and stabilization for the wave equation, part III: Domain with moving boundary, SIAM J. Control Optim., 19 (1981), pp.123-138.

[7] N. Garofalo and F.H. Lin, Unique continuation for elliptic operators: A geometric-variational approach, Comm. Pure Appl. Math., XL (1987), pp.347-366.

[8] V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation, J. Math. Pures Appl., 69 (1990), pp.33-54.

[9] V. Komornik, Exact controllability and stabilization: The multiplier method, John Wiley & Sons, Masson, Paris, 1994.

[10] J. Lagnese, Control of wave process with distributed control supported on a subregion, SIAM J. Control Optim., 21 (1983), pp.68-85.

[11] J. Lagnese, Decay of solutions of wave equations in a bounded region with boundary dissipation, J. Differential Equations, 50 (1983), pp.163-182.

[12] P. Lax and R.S. Phillips, Scattering theory for dissipative hyperbolic systems, J. Funct. Anal., 14 (1973), pp.172-235.

[13] P. Lax and R.S. Phillips, Scattering theory, Revised Edition, Academic Press, INC., Boston, 1989.

[14] J.L. Lions, Contrôlabilité exacte perturbations et stabilisation de systèmes distribués, Tome 1, Contrôlabilité exacte, Masson, Paris Milan Barcelone Mexico, 1988.

[15] J.L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[16] C.S. Morawetz, Exponential decay of solutions of the wave equation, Comm. Pure Appl. Math., 19 (1966), pp.439-444.

[17] C.S. Morawetz, Decay for solutions of the exterior problem for the wave equation, Comm. Pure Appl. Math., 28 (1975), pp.229-264.

[18] S. Nicaise, Boundary exact controllability of interface problems with singularities: addition of the coefficients singularities, SIAM J. Control Optim., 34 (1996), pp.1512-1532.

[19] A. Pazy, Semigroup of linear operators and applications to partial differential

equations, Springer-Verlag, New York, 1983.

[20] W. Rudin, Functional analysis, McGraw-hill, New York, 1973.

[21] D. L. Russell, Exact boundary value controlability theorems for wave and heat processes in star-complemented regions, In differential games and control theory, Roxin, Liu, and Sternberg, Eds., Marcel Dekker Inc., New York, 1974, pp.291-319.

[22] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Rev., 20 (1978), pp.639-739.

[23] W.A. Strauss, Dispersal of waves vanishing on the boundary of an exterior domain, Comm. Pure Appl. Math., 28 (1975), pp.265-278.