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EXACT NEUMANN BOUNDARY CONTROLLABILITY FOR SECOND ORDER HYPERBOLIC EQUATIONS

BY

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ABSTRACT

Using HUM, we study the problem of exact controllability with Neumann boundary conditions for second order hyperbolic equations. We prove that these systems are exactly controllable for all initial states in $L^2(\Omega)\times(H^1(\Omega))'$ and we derive estimates for the control time T.

Key Words: Exact controllability; Second order hyperbolic equations; Neumann boundary condition; HUM

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0. Introduction and Main Result. Let Ω be a bounded domain (open, connected, and nonempty) in $I\!R^n$ ($n \geq 1$) with suitably smooth boundary $\Gamma = \partial\Omega$. For $T > 0$, set $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$.

The aim of this paper is to discuss the problem of exact controllability for second order hyperbolic equations with Neumann boundary control

$$\begin{cases} y'' - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial y}{\partial x_j}\Big) = 0 & \text{in Q,} \\ y(0) = y^0, \qquad y'(0) = y^1 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu_A} = \phi & \text{on } \Sigma. \end{cases} \tag{0.1}$$

In (0.1), $a_{ij}(x,t)$ are suitably smooth real valued functions and $a_{ij}(x,t) = a_{ji}(x,t)$, $i,j = 1,2,\cdots,n$,

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\Big), \tag{0.2}$$

the co-normal derivative $\dfrac{\partial y}{\partial \nu_A}$ with respect to A is equal to $\displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)\nu_i\frac{\partial y}{\partial x_j}$, and $\nu = (\nu_1,\nu_2,\cdots,\nu_n)$ is the unit normal on Γ pointing towards the exterior of Ω, $y' = \dfrac{\partial y}{\partial t}$, $y(0) = y(x,0)$, $y'(0) = y'(x,0)$, and ϕ is a boundary control function.

More precisely, the problem of exact controllability can be stated as follows.

Given $T > 0$, for any initial state (y^0, y^1) and any terminal state (z^0, z^1) in a suitable Hilbert space $\mathcal{H}$, find a boundary control ϕ such that the solution $y = y(x,t;\phi)$ of (0.1) satisfies

$$y(x,T;\phi) = z^0, \qquad y'(x,T;\phi) = z^1 \qquad \textit{in } \Omega. \tag{0.3}$$

Since system (0.1) is linear, it is sufficient to look for controls driving the system (0.1) to rest, i.e.,

$$y(x,T;\phi) = 0, \qquad y'(x,T;\phi) = 0 \qquad \text{in } \Omega. \tag{0.4}$$

Throughout this paper, we will adopt the following notation. Let $x^0(t) \in C^1([0,\infty); I\!R^n)$, and set

$$m(x,t) = x - x^0(t) = (x_1 - x_1^0(t), \cdots, x_n - x_n^0(t)) = (m_1(x,t), \cdots, m_n(x,t)), \tag{0.5}$$

$$\Sigma(x^0) = \{(x,t) \in \Sigma \ : \ m(x,t)\cdot\nu(x) = \sum_{k=1}^{n} m_k(x,t)\nu_k(x) > 0\}, \tag{0.6}$$

$$\Sigma_*(x^0) = \Sigma - \Sigma(x^0), \tag{0.7}$$

$$\Gamma(x^0(0)) = \{x \in \Gamma \ : \ m(x,0)\cdot\nu(x) > 0\}, \tag{0.8}$$

$$\Sigma(x^0(0)) = \Gamma(x^0(0)) \times (0,T), \tag{0.9}$$

$$R(t) = \max_{x\in\overline{\Omega}} \mid m(x,t) \mid = \max_{x\in\overline{\Omega}} \mid \sum_{k=1}^{n} (x_k - x_k^0(t))^2 \mid^{\frac{1}{2}}, \tag{0.10}$$

$$R_1(t) = \max_{x\in\overline{\Omega}} \mid m'(x,t) \mid = \max_{x\in\overline{\Omega}} \mid \sum_{k=1}^{n} ((x_k^0)'(t))^2 \mid^{\frac{1}{2}}, \tag{0.11}$$

$$R_0 = \max_{0\le t\le\infty} R(t). \tag{0.12}$$

Before stating the main results of this paper, we impose certain conditions on a_{ij}. We suppose

$$\begin{cases} a_{ij}(x,t), \ a'_{ij}(x,t), \ a''_{ij}(x,t) \in C([0,\infty); L^\infty(\Omega)), \\ \dfrac{\partial a_{ij}(x,t)}{\partial x_k} \in L^\infty(\Omega\times(0,\infty)), \qquad i,j,k = 1,2,\cdots,n. \end{cases} \tag{0.13}$$

and there exists a constant $\alpha > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \ge \alpha|\xi|^2, \qquad \forall\xi \in I\!R^n, \ \forall(x,t) \in Q. \tag{0.14}$$

Here and in the sequel, we use the summation convention for repeated indices, for example,

$$a_{ij}(x,t)\xi_i\xi_j = \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j.$$

Set

$$a(t) = \frac{n}{\alpha} \max_{1\le i,j\le n} \max_{x\in\Omega} |a'_{ij}(x,t)|, \tag{0.15}$$

$$b(t) = \frac{n}{\alpha} \max_{1\le i,j,k\le n} \max_{x\in\Omega} \Big|\frac{\partial a_{ij}(x,t)}{\partial x_k}\Big|. \tag{0.16}$$

If

$$a(t),\ b(t),\ R_1(t) \in L^1(0,+\infty), \tag{0.17}$$

we set

$$T_0 = \Big(R_0 \parallel b \parallel_{0,1} + \frac{R_0}{\sqrt{\alpha}}(1+e^{-\|a\|_{0,1}}) + \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}}\Big)e^{2\|a\|_{0,1}}, \tag{0.18}$$

where $\|\cdot\|_{0,1}$ denotes the norm of $L^1(0,+\infty)$. Furthermore, if

$$a'_{ij}(x,t)\xi_i\xi_j \le 0, \qquad \forall (x,t)\in\Omega\times[0,\infty), \quad \xi\in I\!R^n, \tag{0.19}$$

or

$$a'_{ij}(x,t)\xi_i\xi_j \ge 0, \qquad \forall (x,t)\in\Omega\times[0,\infty), \quad \xi\in I\!R^n, \tag{0.20}$$

then T_0 can be refined slightly to

$$T_0 = \Big(R_0 \parallel b \parallel_{0,1} + \frac{2R_0}{\sqrt{\alpha}} + \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}}\Big)e^{\|a\|_{0,1}}, \tag{0.21}$$

or

$$T_0 = \Big(R_0 \parallel b \parallel_{0,1} + \frac{R_0}{\sqrt{\alpha}}(1+e^{-\|a\|_{0,1}}) + \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}}\Big)e^{\|a\|_{0,1}}. \tag{0.22}$$

If

$$a(t),\ b(t),\ R_1(t) \in L^\infty(0,+\infty), \tag{0.23}$$

we suppose

$$3R_0 \parallel a \parallel_{0,\infty} + R_0\sqrt{\alpha} \parallel b \parallel_{0,\infty} + \parallel R_1 \parallel_{0,\infty} < \sqrt{\alpha}, \tag{0.24}$$

where $\|\cdot\|_{0,\infty}$ denotes the norm of $L^\infty(0,+\infty)$.

In the sequel, $W^{s,p}(\Omega)$ denotes the usual Sobolev space and $\|\cdot\|_{s,p}$ its norm for any $s\in I\!R$ and $1\le p\le\infty$. We write $H^s(\Omega)$ for $W^{s,2}(\Omega)$ and $\|\cdot\|_s$ for $\|\cdot\|_{s,2}$.

We now state the main result as follows.

Theorem 0.1. *Let Ω be a bounded domain in R^n with the boundary Γ of class C^2. Suppose (0.13) and (0.14) hold and $\Sigma(x^0(0))\subset\Sigma(x^0)$. If either (0.17) holds and $T>T_0$ or (0.23) and (0.24) hold and T is large enough so that*

$$3R_0 \parallel a \parallel_{0,\infty} + R_0\sqrt{\alpha} \parallel b \parallel_{0,\infty} + \parallel R_1 \parallel_{0,\infty} < \frac{\sqrt{\alpha}T - 2R_0}{T}, \tag{0.25}$$

then for all initial states

$$(y^0, y^1) \in L^2(\Omega) \times (H^1(\Omega))',$$

there exists a control

$$\phi = \begin{cases} \phi_0 & \text{on } \Sigma(x^0), \\ \phi_1 & \text{on } \Sigma_*(x^0), \end{cases}$$

with $\phi_0 \in (H^1(\Sigma(x^0)))'$ *and* $\phi_1 \in (H^1(\Sigma_*(x^0)))'$ *such that the solution* $y = y(x,t;\phi)$ *of (0.1) satisfies (0.4).*

Corollary 0.2. *Under the conditions of Theorem 0.1, if* $\Sigma_*(x^0) = \emptyset$*, then for all initial states*

$$(y^0, y^1) \in L^2(\Omega) \times (H^1(\Omega))',$$

there exists a control

$$\phi \in (H^1(0,T; L^2(\Gamma)))',$$

such that the solution $y = y(x,t;\phi)$ *of (0.1) satisfies (0.4).*

Remark 0.3. $\Sigma_*(x^0) = \emptyset$ if $x^0(t) \equiv x_0$ and Ω is star-shaped with respect to x^0 (see [13]).

The method of proof of Theorem 0.1 uses multiplier techniques and the Hilbert Uniqueness Method (HUM for short) introduced by Lions [9].

We now compare our result with the existing literature. The problem of exact controllability for second order hyperbolic equations for both Dirichlet and Neumann boundary controls has been extensively studied. The first work for Dirichlet boundary controls was done probably by Komornik [5], who dealt with the wave equation with variable coefficients but not depending on time by using HUM. Later the time-dependent case was considered by Apolaya [1] and Miranda [11]. In addition, making use of the theory of pseudodifferential operators, Bardos, Lebeau and Rauch [2] considered the Neumann boundary controllability with rather smooth coefficients and domains Ω. The control considered in this paper is of Neumann type and the coefficients and domain Ω are required to be less smooth. Generally speaking, Neumann control is more delicate than the Dirichlet one. We also allow for the case that $\Sigma(x^0)$ is not a cylinder of a form $\Sigma(x^0) = \Gamma(x^0) \times (0,T)$, where x^0 is independent of t, and give delicate estimates for the control time T_0 as given in (0.18) and (0.25). Further, the condition (0.24) generalizes condition (3) of [5].

The rest of this paper is divided into four parts. Section 1 is devoted to a discussion of the regularity of solutions of Neumann boundary value problems. We then establish an identity for the solution in section 2. Using the identity, we obtain an observability inequality in section 3. We prove Theorem 0.1 in section 4.

1. Regularity of Solutions. We first give some preliminary results on solutions of the following Neumann boundary value problem

$$\begin{cases} u'' - Au = f & & \text{in } Q, \\ u(0) = u^0, & u'(0) = u^1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_A} = 0 & & \text{on } \Sigma. \end{cases} \tag{1.1}$$

Throughout this paper, it is assumed that there is $\alpha > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha \mid \xi \mid^2, \qquad \forall \xi \in R^n, \quad (x,t) \in \Omega \times [0,T]. \tag{1.2}$$

Let X be a Banach space. We denote by $C^k([0,T],X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T],X)$ for $C^0([0,T],X)$.

By example 3 of chapter XVIII of [3], we have

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary Γ. Suppose that*

$$a_{ij}(x,t),\ a'_{ij}(x,t) \in C([0,T],L^\infty(\Omega)), \quad i,j = 1,2,\cdots,n. \tag{1.3}$$

Then, for $(u^0,u^1,f) \in H^1(\Omega)\times L^2(\Omega)\times L^1(0,T;L^2(\Omega))$, problem (1.1) has a unique solution with

$$u \in C([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega)). \tag{1.4}$$

Moreover, there exists a constant $c = c(T)$ such that

$$\begin{aligned} &\| u \|_{C([0,T];H^1(\Omega))} + \| u' \|_{C([0,T];L^2(\Omega))} \\ &\leq c[\| u^0 \|_1 + \| u^1 \|_0 + \| f \|_{L^1(0,T;L^2(\Omega))}]. \end{aligned} \tag{1.5}$$

A solution to (1.1) which satisfies (1.4) is called a *weak solution.*

Set

$$W^{1,1}(0,T;L^2(\Omega)) = \{f \ : \ f,f' \in L^1(0,T;L^2(\Omega))\} \tag{1.6}$$

with norm

$$\| f \|_{W^{1,1}} = \left(\| f \|^2_{L^1(0,T;L^2(\Omega))} + \| f' \|^2_{L^1(0,T;L^2(\Omega))}\right)^{\frac{1}{2}}, \tag{1.7}$$

and

$$D(A) = \left\{u \in H^2(\Omega) \ : \ \frac{\partial u}{\partial \nu_A} = 0\right\}. \tag{1.8}$$

We will need the following regularity result.

Theorem 1.2. *Let Ω be a bounded domain in R^n with boundary Γ of class C^2. Suppose that*

$$\begin{cases} a_{ij}(x,t),\ a'_{ij}(x,t),\ a''_{ij}(x,t) \in C([0,T];L^\infty(\Omega)), \\ \dfrac{\partial a_{ij}(x,t)}{\partial x_k} \in L^\infty(Q), \quad i,j,k = 1,2,\cdots,n. \end{cases} \tag{1.9}$$

Assume that $\{u^0,u^1\} \in D(A) \times H^1(\Omega)$.

(i) If $f \in W^{1,1}(0,T;L^2(\Omega))$, then problem (1.1) has a unique solution with

$$u \in C([0,T];D(A)) \cap C^1([0,T];H^1(\Omega)) \cap C^2([0,T];L^2(\Omega)). \tag{1.10}$$

Moreover, there exists a constant $c = c(T)$ such that

$$\begin{aligned} &\| u''(t) \|_0 + \| u'(t) \|_1 \\ &\le c\Big[\| u^0 \|_2 + \| u^1 \|_1 + \| f' \|_{L^1(0,T;L^2(\Omega))} \Big], \qquad \forall t \in [0,T]. \end{aligned} \tag{1.11}$$

(ii) If $f \in L^1(0,T;H^1(\Omega))$, then problem (1.1) has a unique solution with

$$u \in C([0,T];D(A)) \cap C^1([0,T];H^1(\Omega)). \tag{1.12}$$

Moreover, there exists a constant $c = c(T)$ such that

$$\begin{aligned} &\| u(t) \|_2 + \| u'(t) \|_1 \\ &\le c\Big[\| u^0 \|_2 + \| u^1 \|_1 + \| f \|_{L^1(0,T;H^1(\Omega))} \Big], \qquad \forall t \in [0,T]. \end{aligned} \tag{1.13}$$

A solution satisfying (1.12) is called a *strong solution.*

Proof. We first prove (1.11). To this end, we first suppose that $f \in \mathcal{D}((0,T);L^2(\Omega))$ (the space of all infinitely differentiable functions with supports in $(0,T)$ and values in $L^2(\Omega)$). Set

$$a(t;u(t),v(t)) = \int_\Omega a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_i}\frac{\partial v(x,t)}{\partial x_j}dx, \tag{1.14}$$

$$a'(t;u(t),v(t)) = \int_\Omega a'_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_i}\frac{\partial v(x,t)}{\partial x_j}dx, \tag{1.15}$$

and

$$a''(t;u(t),v(t)) = \int_\Omega a''_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_i}\frac{\partial v(x,t)}{\partial x_j}dx. \tag{1.16}$$

Let $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$. For any $v \in H^1(\Omega)$, multiplying (1.1) by v and integrating over Ω, we obtain

$$(u''(t),v) + a(t;u(t),v) = (f(t),v). \tag{1.17}$$

Differentiating (1.17) with respect to t, we obtain

$$(u'''(t),v) + a'(t;u(t),v) + a(t;u'(t),v) = (f'(t),v). \tag{1.18}$$

Replacing v by $u''(t)$ in (1.18) gives

$$\begin{aligned} &[(u''(t),u''(t)) + a(t;u'(t),u'(t))]' + 2a'(t;u(t),u''(t)) - a'(t;u'(t),u'(t)) \\ &= 2(f'(t),u''(t)). \end{aligned} \tag{1.19}$$

But

$$a'(t;u(t),u''(t)) = [a'(t;u(t),u'(t))]' - a''(t;u(t),u'(t)) - a'(t;u'(t),u'(t)). \quad (1.20)$$

Integrating (1.19) from 0 to t and using (1.20), we have

$$\begin{aligned}
&\| u''(t) \|_0^2 + a(t,u'(t),u'(t)) \\
&= \| u''(0) \|_0^2 + a(0,u'(0),u'(0)) + 2a'(0,u(0),u'(0)) \\
&\quad - 2a'(t,u(t),u'(t)) + 3\int_0^t a'(s,u'(s),u'(s))ds \\
&\quad + 2\int_0^t a''(s,u(s),u'(s))ds + 2\int_0^t (f'(s),u''(s))ds.
\end{aligned} \quad (1.21)$$

It therefore follows from (1.2) and (1.9) and (1.21) that (the following c's denoting various constants depending on $a,\ \alpha,\ T$)

$$\begin{aligned}
&\| u''(t) \|_0^2 + \alpha \| \nabla u'(t) \|_0^2 \\
&\le \frac{\alpha}{2} \| \nabla u'(t) \|_0^2 + c\Big[\| u''(0) \|_0^2 + \| u^1 \|_1^2 + \| u^0 \|_1^2 \\
&\quad + \| u(t) \|_1^2 + \int_0^t (\| u(s) \|_1^2 + \| u'(s) \|_1^2)ds \\
&\quad + \max_{0\le s\le t} \| u''(s) \|_0 \int_0^t \| f'(s) \|_0 \, ds\Big],
\end{aligned}$$

which, by adding $\| u'(t) \|_0^2$ to both sides of the above inequality, implies

$$\begin{aligned}
&\| u''(t) \|_0^2 + \| u'(t) \|_1^2 \\
&\le c\Big[\| u''(0) \|_0^2 + \| u^1 \|_1^2 + \| u^0 \|_1^2 + \| u(t) \|_1^2 + \| u'(t) \|_0^2 \\
&\quad + \int_0^t (\| u(s) \|_1^2 + \| u'(s) \|_1^2)ds + \max_{0\le s\le t} \| u''(s) \|_0 \int_0^t \| f'(s) \|_0 \, ds\Big].
\end{aligned} \quad (1.22)$$

But $u(t) = u^0 + \int_0^t u'(s)ds$ and $u'(t) = u^1 + \int_0^t u''(s)ds$ yield respectively

$$\| u(t) \|_1 \le \| u^0 \|_1 + \int_0^t \| u'(s) \|_1 \, ds, \quad (1.23)$$

and

$$\| u'(t) \|_0 \le \| u^1 \|_0 + \int_0^t \| u''(s) \|_0 \, ds. \quad (1.24)$$

In addition, by (1.1) we have

$$u''(0) = Au^0 + f(0) = Au^0. \quad (1.25)$$

So we deduce from (1.22)-(1.25) that

$$\begin{aligned}
&\| u''(t) \|_0^2 + \| u'(t) \|_1^2 \\
&\leq c\Big[\| u^0 \|_2^2 + \| u^1 \|_1^2 + \| f' \|_{L^1(0,T;L^2(\Omega))} ^2\\
&\quad + \int_0^t (\| u'(s) \|_1^2 + \| u''(s) \|_0^2) ds\Big] + \frac{1}{2} \max_{0\leq s\leq t} \| u''(s) \|_0^2,
\end{aligned} \tag{1.26}$$

from which, setting

$$w(t) = \max_{0\leq s\leq t} \| u''(s) \|_0^2 + \| u'(s) \|_1^2, \tag{1.27}$$

we deduce

$$w(t) \leq c\Big[\| u^0 \|_2^2 + \| u^1 \|_1^2 + \| f' \|_{L^1(0,T;L^2(\Omega))}^2 + \int_0^t w(s) ds\Big]. \tag{1.28}$$

Gronwall's inequality (see [4, p.36]) shows

$$w(t) \leq c\Big[\| u^0 \|_2^2 + \| u^1 \|_1^2 + \| f' \|_{L^1(0,T;L^2(\Omega))}^2\Big]. \tag{1.29}$$

This implies (1.11). By a density argument, we can show (1.11) still holds for $f \in W^{1,1}(0,T;L^2(\Omega))$.

Now we prove (1.10). Using the proof of Theorem 8.2 of [10, Vol.I, p.275] and (1.7) of [10, Vol.II, p.97], we can prove

$$u \in C^1([0,T];H^1(\Omega)) \cap C^2([0,T];L^2(\Omega)). \tag{1.30}$$

On the other hand, by inequality (6.7) of [7, p.66] and Remark 6.2 of [7, p.77], we have

$$\begin{aligned}
\| u(t) \|_2^2 &\leq c_1 \| Au(t) \|_0^2 + c_2 \| u(t) \|_0^2 \\
&\leq c[\| u''(t) \|_0^2 + \| u(t) \|_0^2 + \| f(t) \|_0^2].
\end{aligned} \tag{1.31}$$

It follows from (1.1) and (1.31) that

$$\begin{aligned}
&\| u(t_1) - u(t_2) \|_2^2 \\
&\leq c[\| u''(t_1) - u''(t_2) \|_0^2 + \| u(t_1) - u(t_2) \|_0^2 + \| f(t_1) - f(t_2) \|_0^2].
\end{aligned} \tag{1.32}$$

Thus, the continuity of u'' and f implies

$$u \in C([0,T];D(A)). \tag{1.33}$$

It remains to prove (1.13). Multiplying (1.1) by $(Au)'$ and integrating over Ω, we obtain

$$\begin{aligned}
&(Au(t), (Au(t))') + a(t;u'(t),u''(t)) + a'(t;u(t),u''(t)) \\
&= a(t;u'(t),f(t)) + a'(t;u(t),f(t)).
\end{aligned} \tag{1.34}$$

Combining this and (1.20) gives

$$\begin{aligned}&\frac{1}{2}[(Au(t),Au(t))+a(t;u'(t),u'(t))]'\\&=\frac{3}{2}a'(t;u'(t),u'(t))+a''(t;u(t),u'(t))-[a'(t;u(t),u'(t))]'\\&\quad+a(t;u'(t),f(t))+a'(t;u(t),f(t)).\end{aligned}\tag{1.35}$$

Integrating (1.35) from 0 to t, we have

$$\begin{aligned}&\| Au(t)\|_0^2+a(t,u'(t),u'(t))\\&=\| Au^0\|_0^2+a(0,u'(0),u'(0))+2a'(0,u(0),u'(0))-2a'(t,u(t),u'(t))\\&\quad+3\int_0^t a'(s,u'(s),u'(s))ds+2\int_0^t a''(s,u(s),u'(s))ds\\&\quad+2\int_0^t[a(s;u'(s),f(s))+a'(s;u(s),f(s))]ds.\end{aligned}\tag{1.36}$$

It therefore follows from (1.2), (1.5), and (1.36) that there exists a constant $c=c(T)>0$ such that

$$\begin{aligned}&\| Au(t)\|_0^2+\|\nabla u'(t)\|_0^2\\&\le c\Big[\| u^0\|_2^2+\| u^1\|_1^2+\| f\|^2_{L^1(0,T;H^1(\Omega))}+\int_0^t\|\nabla u'(s)\|_0^2\,ds\Big]\\&\quad+\frac{1}{2}\max_{0\le s\le t}\|\nabla u'(s)\|_0^2\,.\end{aligned}\tag{1.37}$$

from which, as in the proof of (1.29), we deduce

$$\begin{aligned}&\| Au(t)\|_0^2+\|\nabla u'(t)\|_0^2\\&\le c\Big[\| u^0\|_2^2+\| u^1\|_1^2+\| f\|^2_{L^1(0,T;H^1(\Omega))}\Big].\end{aligned}\tag{1.38}$$

Thus (1.13) follows from (1.5), (1.31), and (1.38). Finally, (1.12) is a consequence of (1.13) through a density argument. □

2. An Identity. We are now in a position to establish an identity, which is indispensable for obtaining an observability inequality in the following section.

We define the energy of the solution u of (1.1) with $f=0$ by

$$E(t)=\frac{1}{2}\int_\Omega|u'(x,t)|^2\,dx+\frac{1}{2}a(t;u(t),u(t)),\tag{2.1}$$

then,

$$E'(t)=\frac{1}{2}a'(t;u(t),u(t)),\tag{2.2}$$

and

$$E(t) = E(0) + \frac{1}{2}\int_0^t a'(s; u(s), u(s))ds, \tag{2.3}$$

where $a(t; u(t), u(t))$ and $a'(t; u(t), u(t))$ are given by (1.14) and (1.15), respectively.

For the coming calculation, we introduce the notion of tangential differential operators with respect to A which are similar to those introduced in [9, p.137].

Let Ω be a bounded domain with a Lipschitz boundary Γ. Since by (1.2) we have $a_{ij}(x,t)\nu_i\nu_j \geq \alpha$, the vector $\nu_A = \Big\{\sum_{i=1}^{n} a_{ij}(x,t)\nu_i\Big\}_{j=1}^{n}$ is not tangential to Γ for almost all $x \in \Gamma$. Thus, we can define a tangential vector field $\{\tau_A^k(x)\}_{k=1}^{n-1}$ such that $\{\nu_A(x), \tau_A^1(x), \cdots, \tau_A^{n-1}(x)\}$ forms a basis in $I\!R^n$ for almost all $x \in \Gamma$.

For a smooth function u, there exist β_A^j, $\gamma_A^{k,j}$ $(j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, n-1)$ depending on $\{\nu_A(x), \tau_A^1(x), \cdots, \tau_A^{n-1}(x)\}$ such that

$$\frac{\partial u}{\partial x_j} = \beta_A^j \frac{\partial u}{\partial \nu_A} + \sum_{k=1}^{n-1} \gamma_A^{k,j} \frac{\partial u}{\partial \tau_A^k}, \qquad \text{on } \Gamma, \qquad j = 1, 2, \cdots, n. \tag{2.4}$$

Set

$$\sigma_j^A u = \sum_{k=1}^{n-1} \gamma_A^{k,j} \frac{\partial u}{\partial \tau_A^k}, \qquad j = 1, 2, \cdots, n, \tag{2.5}$$

then,

$$\frac{\partial u}{\partial x_j} = \beta_A^j \frac{\partial u}{\partial \nu_A} + \sigma_j^A u. \tag{2.6}$$

Evidently, σ_j^A $(j = 1, 2, \cdots, n)$ are independent of the choice of the tangential vector field $\{\tau_A^k(x)\}_{k=1}^{n-1}$. Therefore we obtain a family of first order tangential differential operators σ_j^A $(j = 1, 2, \cdots, n)$ on Γ with respect to A. We can define the tangential gradient of u on Γ by

$$\nabla_{\sigma^A} u = \{\sigma_j^A u\}_{j=1}^n. \tag{2.7}$$

For any subset Σ_1 of Σ, σ_j^A $(j = 1, 2, \cdots, n)$ are linear and continuous from $H^1(\Sigma_1) \to L^2(\Sigma_1)$. Set

$$-\Delta_{\Sigma_1}^A = \sum_{j=1}^{n} (\sigma_j^A)^* \sigma_j^A, \tag{2.8}$$

where $(\sigma_j^A)^*$ denotes the adjoint of σ_j^A. Then the operator $-\Delta_{\Sigma_1}^A$ is linear and continuous from $H^1(\Sigma_1) \to (H^1(\Sigma_1))'$ and satisfies

$$\langle -\Delta_{\Sigma_1}^A u, v\rangle = \int_{\Sigma_1} \nabla_{\sigma^A} u \nabla_{\sigma^A} v d\Sigma, \qquad \forall u, v \in H^1(\Sigma_1). \tag{2.9}$$

Lemma 2.1. *Let Ω be a bounded domain in $I\!R^n$ with boundary Γ of class C^2. Let $q = (q_k)$ be a vector field in $[C^1(\overline{\Omega} \times [0, \infty))]^n$. Suppose u is the weak solution of (1.1). Then the following identity holds:*

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma q_k\nu_k\Big(\mid u' \mid^2 - a_{ij}(x,t)\sigma_i^A u \sigma_j^A u\Big) d\Sigma \\
=&\Big(u'(t), q_k\frac{\partial u}{\partial x_k}\Big)\Big|_0^T + \int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k} dxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\Big(\mid u' \mid^2 - a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big) dxdt \\
&- \frac{1}{2}\int_Q q_k \frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dxdt - \int_Q q_k\frac{\partial u}{\partial x_k} f dxdt - \int_Q u' q_k' \frac{\partial u}{\partial x_k} dxdt,
\end{aligned}
\tag{2.10}
$$

where

$$
\Big(u'(t), q_k\frac{\partial u}{\partial x_k}\Big) = \int_\Omega u'(t) q_k \frac{\partial u}{\partial x_k} dx.
$$

Remark 2.2. If $n = 1$, then (2.10) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma q\nu \mid u' \mid^2 d\Sigma \\
=&\Big(u'(t), q\frac{\partial u}{\partial x}\Big)\Big|_0^T + \int_Q a(x,t)\Big|\frac{\partial u}{\partial x}\Big|^2 \frac{\partial q}{\partial x} dxdt \\
&+ \frac{1}{2}\int_Q \frac{\partial q}{\partial x}\Big(\mid u' \mid^2 - a(x,t)\Big|\frac{\partial u}{\partial x}\Big|^2\Big) dxdt \\
&- \frac{1}{2}\int_Q q\frac{\partial a(x,t)}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt - \int_Q q\frac{\partial u}{\partial x} f dxdt - \int_Q u' q' \frac{\partial u}{\partial x} dxdt.
\end{aligned}
\tag{2.10$'$}
$$

Proof. We first prove (2.10) in the case of strong solutions, that is, we assume initial conditions $(u^0, u^1) \in D(A) \times H^1(\Omega)$ and $f \in L^1(0, T; H^1(\Omega))$. Multiplying (1.1) by $q_k \dfrac{\partial u}{\partial x_k}$ and integrating on Q, we have

$$
\int_Q q_k\frac{\partial u}{\partial x_k} u'' dxdt - \int_Q q_k \frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) dxdt = \int_Q q_k\frac{\partial u}{\partial x_k} f dxdt. \tag{2.11}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
\int_Q q_k\frac{\partial u}{\partial x_k} u'' dxdt =& \Big(u'(t), q_k\frac{\partial u}{\partial x_k}\Big)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k} \mid u' \mid^2 dxdt \\
&- \int_Q u' q_k' \frac{\partial u}{\partial x_k} dxdt - \frac{1}{2}\int_\Sigma q_k \nu_k \mid u' \mid^2 d\Sigma,
\end{aligned}
\tag{2.12}
$$

and

$$\begin{aligned}
&\int_Q q_k \frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big)dxdt\\
&= -\int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\Big(q_k\frac{\partial u}{\partial x_k}\Big)dxdt \qquad (2.13)\\
&= -\int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big(\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k} + q_k\frac{\partial^2 u}{\partial x_k \partial x_i}\Big)dxdt.
\end{aligned}$$

But,

$$\begin{aligned}
&\int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j} q_k \frac{\partial^2 u}{\partial x_k\partial x_i}dxdt\\
&= \frac{1}{2}\int_Q q_k\Big(\frac{\partial}{\partial x_k}(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}) - \frac{\partial a_{ij}(x,t)}{x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt\\
&= \frac{1}{2}\int_\Sigma q_k\nu_k a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}d\Sigma - \frac{1}{2}\int_Q q_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt \qquad (2.14)\\
&\quad - \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt.
\end{aligned}$$

By (2.13) and (2.14), and noting that $\dfrac{\partial u}{\partial x_i} = \sigma_i^A u$ on Σ, we have

$$\begin{aligned}
&\int_Q q_k\frac{\partial u}{\partial x_k}\frac{\partial}{x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big)dxdt\\
&= -\frac{1}{2}\int_\Sigma q_k\nu_k a_{ij}(x,t)\sigma_i^A u\sigma_j^A u d\Sigma - \int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt \qquad (2.15)\\
&\quad + \frac{1}{2}\int_Q q_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt + \frac{1}{2}\int_Q\frac{\partial q_k}{\partial x_k}a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt.
\end{aligned}$$

It follows from (2.11) , (2.12), and (2.15) that

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma q_k\nu_k\Big(\Big|u'\Big|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma\\
&=\Big(u'(t), q_k\frac{\partial u}{\partial x_k}\Big)\Big|_0^T + \int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt\\
&\quad + \frac{1}{2}\int_Q\frac{\partial q_k}{\partial x_k}\Big(\mid u'\mid^2 - a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt\\
&\quad - \frac{1}{2}\int_Q q_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt - \int_Q q_k\frac{\partial u}{\partial x_k}f dxdt - \int_Q u' q_k'\frac{\partial u}{\partial x_k}dxdt.
\end{aligned}$$

This is (2.10).

We now consider the general case of weak solutions with $(u^0, u^1) \in H^1(\Omega) \times L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$. We take $(u_n^0, u_n^1) \in D(A) \times H^1(\Omega)$ and $f_n \in L^1(0,T;H^1(\Omega))$ such that

$(u_n^0, u_n^1) \to (u^0, u^1) \qquad$ in $H^1(\Omega) \times L^2(\Omega)$,

$f_n \to f \qquad$ in $L^1(0,T; L^2(\Omega))$.

Now for strong solutions u_n with initial conditions (u_n^0, u_n^1), and right hand side f_n, the identity (2.10) holds. Due to Theorem 1.1, we have

$u_n \to u$ in $C([0,T], H^1(\Omega))$,

$u_n' \to u'$ in $C([0,T], L^2(\Omega))$.

Thus, as in the proof of Lemma 1.3 of chapter 3 of [9, p.139], taking the limit in (2.10) we deduce that (2.10) still holds in this case. □

3. Observability Inequality. To establish an observability inequality, we need the following lemma.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary Γ of class C^2. Then for all weak solutions u of (1.1) with $f=0$ the following hold:*

(i) If $n>1$, then

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
&=\Big(u'(t), m_k\frac{\partial u}{\partial x_k} + \frac{n-1}{2}u(t)\Big)\Big|_0^T + \int_0^T E(t)dt \\
&\quad - \frac{1}{2}\int_Q m_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt - \int_Q u'm_k'\frac{\partial u}{\partial x_k}dxdt.
\end{aligned} \tag{3.1}$$

If $n=1$, then

$$\begin{aligned}
\frac{1}{2}\int_\Sigma m\nu \mid u' \mid^2 d\Sigma =& \Big(u'(t), m\frac{\partial u}{\partial x}\Big)\Big|_0^T + \int_0^T E(t)dt \\
&- \frac{1}{2}\int_Q m\frac{\partial a(x,t)}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt - \int_Q u'm'\frac{\partial u}{\partial x}dxdt.
\end{aligned} \tag{3.2}$$

(ii) If $n>1$, then

$$\begin{aligned}
&\Big|\Big(u'(t), m_k\frac{\partial u}{\partial x_k} + \frac{n-1}{2}u(t)\Big)\Big| \\
&\le \frac{R_0}{\sqrt{\alpha}}E(t) + \frac{\sqrt{\alpha}(1-n^2)}{8R_0}\int_\Omega \mid u(t)\mid^2 dx \\
&\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_\Gamma m_k\nu_k \mid u(t)\mid^2 d\Gamma, \qquad \forall t\in[0,T].
\end{aligned} \tag{3.3}$$

If $n = 1$, then for $\gamma \in (0, 1)$

$$\begin{aligned}
&\Big|\Big(u'(t), m\frac{\partial u}{\partial x} + \frac{1-\gamma}{2}u(t)\Big)\Big| \\
&\le \frac{R_0}{\sqrt{\alpha}}E(t) + \frac{\sqrt{\alpha}(\gamma^2-1)}{8R_0}\int_\Omega |\ u(t)\ |^2\, dx \\
&\quad + \frac{\sqrt{\alpha}(1-\gamma)}{4R_0}\int_\Gamma m\nu\ |\ u(t)\ |^2\, d\Gamma, \qquad \forall t \in [0, T].
\end{aligned} \tag{3.4}$$

Proof. We prove the lemma only in the case of $n > 1$. It is similar in the case of $n = 1$.

(i) Taking $q_k = m_k$ in (2.10), we have

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
=&\Big(u'(t), m_k\frac{\partial u}{\partial x_k}\Big)\Big|_0^T + \int_Q a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}dxdt \\
&+ \frac{n}{2}\int_Q\Big(\ |\ u'\ |^2 - a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt \\
&- \frac{1}{2}\int_Q m_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt - \int_Q u'm_k'\frac{\partial u}{\partial x_k}dxdt \\
=&\Big(u'(t), m_k\frac{\partial u}{\partial x_k}\Big)\Big|_0^T + \frac{n-1}{2}\int_Q\Big(\ |\ u'\ |^2 - a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt \\
&+ \frac{1}{2}\int_Q\Big(\ |\ u'\ |^2 + a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt \\
&- \frac{1}{2}\int_Q m_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt - \int_Q u'm_k'\frac{\partial u}{\partial x_k}dxdt.
\end{aligned} \tag{3.5}$$

But

$$\int_Q\Big(\ |\ u'\ |^2 - a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxdt = (u, u')\Big|_0^T. \tag{3.6}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
=&\Big(u'(t), m_k\frac{\partial u}{\partial x_k} + \frac{n-1}{2}u(t)\Big)\Big|_0^T + \int_0^T E(t)dt \\
&- \frac{1}{2}\int_Q m_k\frac{\partial a_{ij}(x,t)}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxdt - \int_Q u'm_k'\frac{\partial u}{\partial x_k}dxdt.
\end{aligned} \tag{3.7}$$

This shows (3.1)

(ii) From the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\Big|\Big(u'(t), m_k\frac{\partial u}{\partial x_k}+\frac{n-1}{2}u(t)\Big)\Big| \\
&\leq \frac{R_0}{2\sqrt{\alpha}}\int_\Omega \mid u'(t)\mid^2 dx + \frac{\alpha}{2R_0\sqrt{\alpha}}\int_\Omega \mid m_k\frac{\partial u}{\partial x_k}+\frac{n-1}{2}u(t)\mid^2 dx.
\end{aligned}
\tag{3.8}
$$

As shown in [6] by Komornik, we have

$$
\begin{aligned}
&\int_\Omega \mid m_k\frac{\partial u}{\partial x_k}+\frac{n-1}{2}u(t)\mid^2 dx \\
&= \int_\Omega \mid m_k\frac{\partial u}{\partial x_k}\mid^2 dx + \frac{(n-1)^2}{4}\int_\Omega \mid u(t)\mid^2 dx + (n-1)\Big(m_k\frac{\partial u}{\partial x_k}, u(t)\Big).
\end{aligned}
\tag{3.9}
$$

However,

$$
\begin{aligned}
\Big(m_k\frac{\partial u}{\partial x_k}, u(t)\Big) &= \int_\Omega m_k\frac{\partial u}{\partial x_k}u(t)dx \\
&= \frac{1}{2}\int_\Omega m_k\frac{\partial}{\partial x_k}(\mid u(t)\mid^2)dx \\
&= -\frac{n}{2}\int_\Omega \mid u(t)\mid^2 dx + \frac{1}{2}\int_\Gamma m_k\nu_k \mid u(t)\mid^2 d\Gamma.
\end{aligned}
\tag{3.10}
$$

Combining (3.9) and (3.10), we obtain

$$
\begin{aligned}
&\int_\Omega \mid m_k\frac{\partial u}{\partial x_k}+\frac{n-1}{2}u(t)\mid^2 dx \\
&= \int_\Omega \Big|m_k\frac{\partial u}{\partial x_k}\Big|^2 dx + \frac{1-n^2}{4}\int_\Omega \mid u(t)\mid^2 dx + \frac{n-1}{2}\int_\Gamma m_k\nu_k \mid u(t)\mid^2 d\Gamma \\
&\leq R_0^2\int_\Omega \mid \nabla u(t)\mid^2 dx + \frac{1-n^2}{4}\int_\Omega \mid u(t)\mid^2 dx \\
&\quad + \frac{n-1}{2}\int_\Gamma m_k\nu_k \mid u(t)\mid^2 d\Gamma.
\end{aligned}
\tag{3.11}
$$

Thus, by (1.2) and (3.8) we have

$$
\begin{aligned}
&\Big|\Big(u'(t), m_k\frac{\partial u}{\partial x_k}+\frac{n-1}{2}u(t)\Big)\Big| \\
&\leq \frac{R_0}{\sqrt{\alpha}}E(t) + \frac{\sqrt{\alpha}(1-n^2)}{8R_0}\int_\Omega \mid u(t)\mid^2 dx \\
&\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_\Gamma m_k\nu_k \mid u(t)\mid^2 d\Gamma, \quad \forall t\in[0,T].
\end{aligned}
\tag{3.12}
$$

□

Lemma 3.2. *Let Ω be a bounded domain in R^n with boundary Γ of class C^2. Suppose (0.13) and (0.14) hold. If either (0.17) holds and $T > T_0$ or (0.23) and (0.24) hold and T is large enough so that*

$$3R_0 \parallel a \parallel_{0,\infty} + R_0\sqrt{\alpha} \parallel b \parallel_{0,\infty} + \parallel R_1 \parallel_{0,\infty} < \frac{\sqrt{\alpha}T - 2R_0}{T}, \tag{3.13}$$

then for all weak solutions u of (1.1) with $f = 0$ there exists $c = c(T) > 0$ such that

$$\begin{aligned} &\int_\Sigma m_k\nu_k(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u)d\Sigma + \int_\Gamma m_k\nu_k(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \\ &\geq c\Big(\parallel u^0 \parallel_1^2 + \parallel u^1 \parallel_0^2\Big), \qquad \text{for } n > 1, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\int_\Sigma m\nu|u'|^2 d\Sigma + \int_\Gamma m\nu(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \\ &\geq c\Big(\parallel u^0 \parallel_1^2 + \parallel u^1 \parallel_0^2\Big), \qquad \text{for } n = 1. \end{aligned} \tag{3.15}$$

Furthermore, if condition (0.19) or (0.20) is satisfied, then T_0 can be refined slightly to (0.21) or (0.22), respectively.

Proof. (i) Suppose (0.17) holds and $T > T_0$.

Case I: $n > 1$. It follows from (0.14) and (1.15) that

$$-a(t)E(t) \leq E'(t) \leq a(t)E(t), \qquad \forall t \geq 0, \tag{3.16}$$

where $a(t)$ is given by (0.15). Let

$$h(t) = \int_0^t a(s)\, ds, \tag{3.17}$$

then

$$\begin{aligned} \Big(e^h E\Big)' &= e^h E' + h' e^h E \\ &\geq -e^h aE + ae^h E \\ &= 0. \end{aligned} \tag{3.18}$$

Thus,

$$E(t) \geq E(0)e^{-h(t)} \geq E(0)e^{-\|a\|_{0,1}}, \qquad \forall t \geq 0. \tag{3.19}$$

On the other hand, it follows from Gronwall's inequality and (3.16) that

$$E(t) \leq E(0)e^{\|a\|_{0,1}}, \qquad \forall t \geq 0. \tag{3.20}$$

Set

$$Z = \Big(u'(t), m_k\frac{\partial u}{\partial x_k} + \frac{n-1}{2}u(t)\Big)\Big|_0^T. \tag{3.21}$$

It follows from (3.3) and (3.20) that

$$
\begin{aligned}
&| Z | \leq | Z(0) | + | Z(T) | \\
&\quad \leq \frac{R_0}{\sqrt{\alpha}} E(0)(1+e^{\|a\|_{0,1}}) + \frac{\sqrt{\alpha}(1-n^2)}{8R_0} \int_\Omega (| u(0) |^2 + | u(T) |^2) dx \\
&\quad\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0} \int_\Gamma m_k \nu_k (| u(0) |^2 + | u(T) |^2) d\Gamma.
\end{aligned} \tag{3.22}
$$

In addition, by (0.14) and (3.20), we have

$$
\begin{aligned}
\Big| \frac{1}{2} \int_Q m_k \frac{\partial a_{ij}(x,t)}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dxdt \Big| &\leq \frac{R_0}{2} \int_Q b(t) a_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dxdt \\
&\leq R_0 \int_0^T b(t) E(t) dt \\
&\leq R_0 E(0) \parallel b \parallel_{0,1} e^{\|a\|_{0,1}},
\end{aligned} \tag{3.23}
$$

where $b(t)$ is given by (0.16). Also,

$$
\begin{aligned}
\Big| \int_Q u' m_k' \frac{\partial u}{\partial x_k} dxdt \Big| &\leq \int_Q u' R_1(t) \mid \nabla u \mid dxdt \\
&\leq \frac{1}{2\sqrt{\alpha}} \int_Q R_1(t) (| u' |^2 + \alpha | \nabla u |^2) dxdt \\
&\leq \frac{1}{\sqrt{\alpha}} \int_Q R_1(t) E(t) dt \\
&\leq \frac{E(0) \parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}} e^{\|a\|_{0,1}}.
\end{aligned} \tag{3.24}
$$

It therefore follows from (3.1), (3.19), (3.22), (3.23), and (3.24) that

$$
\begin{aligned}
&\frac{1}{2} \int_\Sigma m_k \nu_k \Big(|u'|^2 - a_{ij}(x,t) \sigma_i^A u \sigma_j^A u \Big) d\Sigma \\
&\geq T E(0) e^{-\|a\|_{0,1}} - R_0 E(0) \parallel b \parallel_{0,1} e^{\|a\|_{0,1}} - \frac{R_0}{\sqrt{\alpha}} E(0)(1+e^{\|a\|_{0,1}}) \\
&\quad - \frac{\sqrt{\alpha}(1-n^2)}{8R_0} \int_\Omega (| u(0) |^2 + | u(T) |^2) dx \\
&\quad - \frac{\sqrt{\alpha}(n-1)}{4R_0} \int_\Gamma m_k \nu_k (| u(0) |^2 + | u(T) |^2) d\Gamma - \frac{E(0) \parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}} e^{\|a\|_{0,1}}.
\end{aligned} \tag{3.25}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{2} \int_\Sigma m_k \nu_k \Big(|u'|^2 - a_{ij}(x,t) \sigma_i^A u \sigma_j^A u \Big) d\Sigma \\
&\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0} \int_\Gamma m_k \nu_k (| u(0) |^2 + | u(T) |^2) d\Gamma \\
&\geq \Big(T e^{-\|a\|_1} - R_0 \parallel b \parallel_{0,1} e^{\|a\|_{0,1}} - \frac{R_0}{\sqrt{\alpha}} (1+e^{\|a\|_{0,1}}) \\
&\quad - \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}} e^{\|a\|_{0,1}} \Big) E(0) + \frac{\sqrt{\alpha}(n^2-1)}{8R_0} \int_\Omega | u(0) |^2 \, dx.
\end{aligned} \tag{3.26}
$$

This implies (3.14).

Case II: $n = 1$. By (3.2), we have

$$\begin{aligned}\frac{1}{2}\int_{\Sigma} m\nu \mid u' \mid^2 d\Sigma =&\Big(u'(t), m\frac{\partial u}{\partial x}\Big)\Big|_0^T + \int_0^T E(t)dt \\ &- \frac{1}{2}\int_Q m\frac{\partial a(x,t)}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt - \int_Q u'm'\frac{\partial u}{\partial x}dxdt.\end{aligned} \tag{3.27}$$

Choose $\gamma \in (0,1)$ such that

$$\gamma Te^{-\|a\|_{0,1}} - R_0 \parallel b \parallel_{0,1} e^{\|a\|_{0,1}} - \frac{R_0}{\sqrt{\alpha}}(1 + e^{\|a\|_{0,1}}) - \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}} e^{\|a\|_{0,1}} > 0. \tag{3.28}$$

We write

$$\begin{aligned}\int_0^T E(t)dt = \gamma \int_0^T E(t)dt + \frac{1-\gamma}{2}\int_Q \Big(\mid u' \mid^2 - a(x,t)\Big|\frac{\partial u}{\partial x}\Big|^2\Big) dxdt \\ + \frac{2-2\gamma}{2}\int_Q a(x,t)\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt.\end{aligned} \tag{3.29}$$

Then,

$$\begin{aligned}&\frac{1}{2}\int_{\Sigma} m\nu \mid u' \mid^2 d\Sigma \\ =&\Big(u'(t), m\frac{\partial u}{\partial x} + \frac{1-\gamma}{2}u(t)\Big)\Big|_0^T + \gamma\int_0^T E(t)dt \\ &- \frac{1}{2}\int_Q m\frac{\partial a(x,t)}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt - \int_Q u'm'\frac{\partial u}{\partial x}dxdt + \frac{2-2\gamma}{2}\int_Q a(x,t)\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt \\ \geq&\Big(u'(t), m\frac{\partial u}{\partial x} + \frac{1-\gamma}{2}u(t)\Big)\Big|_0^T + \gamma\int_0^T E(t)dt \\ &- \frac{1}{2}\int_Q m\frac{\partial a(x,t)}{\partial x}\Big|\frac{\partial u}{\partial x}\Big|^2 dxdt - \int_Q u'm'\frac{\partial u}{\partial x}dxdt,\end{aligned} \tag{3.30}$$

from which, as in the case $n > 1$, we can deduce (3.15).

Furthermore, if (0.19) is satisfied, then $E'(t) \leq 0$. Consequently,

$$E(t) \leq E(0), \qquad \text{for } t \geq 0. \tag{3.31}$$

Then (3.26) becomes

$$\begin{aligned}&\frac{1}{2}\int_{\Sigma} m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u \sigma_j^A u\Big) d\Sigma \\ &\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_{\Gamma} m_k\nu_k(\mid u(0) \mid^2 + \mid u(T) \mid^2) d\Gamma \\ \geq&\Big(Te^{-\|a\|_{0,1}} - R_0 \parallel b \parallel_{0,1} - \frac{2R_0}{\sqrt{\alpha}} - \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}}\Big)E(0) \\ &\quad + \frac{\sqrt{\alpha}(n^2-1)}{8R_0}\int_{\Omega} \mid u(0) \mid^2 dx.\end{aligned} \tag{3.32}$$

So T_0 can be refined to (0.21).

If (0.20) is satisfied, then $E'(t) \geq 0$. Consequently,

$$E(t) \geq E(0), \qquad \text{for } t \geq 0. \tag{3.33}$$

Then (3.26) becomes

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma} m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
&\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_{\Gamma} m_k\nu_k(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \\
&\geq \Big(T - R_0 \parallel b \parallel_{0,1} e^{\|a\|_{0,1}} - \frac{R_0}{\sqrt{\alpha}}(1+e^{\|a\|_{0,1}}) - \frac{\parallel R_1 \parallel_{0,1}}{\sqrt{\alpha}} e^{\|a\|_{0,1}}\Big)E(0) \\
&\quad + \frac{\sqrt{\alpha}(n^2-1)}{8R_0}\int_{\Omega} \mid u(0)\mid^2 dx.
\end{aligned} \tag{3.34}$$

So T_0 can be refined to (0.22).

(ii) Suppose (0.23) and (0.24) hold and T is large enough so that (3.13) holds.

By (2.3) we deduce

$$E(T) \leq E(0) + \parallel a \parallel_{0,\infty} \int_0^T E(t)dt, \tag{3.35}$$

and

$$\int_0^T E(t)dt \geq TE(0) - T \parallel a \parallel_{0,\infty} \int_0^T E(t)dt. \tag{3.36}$$

It therefore follows from (3.1), (3.19), (3.22), (3.23), and (3.24) that

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma} m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
&\geq \Big(1 - R_0 \parallel b \parallel_{0,\infty} - \frac{R_0 \parallel a \parallel_{0,\infty}}{\sqrt{\alpha}} - \frac{\parallel R_1 \parallel_{0,\infty}}{\sqrt{\alpha}}\Big)\int_0^T E(t)dt - \frac{2R_0}{\sqrt{\alpha}}E(0) \\
&\quad - \frac{\sqrt{\alpha}(1-n^2)}{8R_0}\int_{\Omega}(\mid u(0)\mid^2 + \mid u(T)\mid^2)dx \\
&\quad - \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_{\Gamma} m_k\nu_k(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \\
&\geq \Big(1 - R_0 \parallel b \parallel_{0,\infty} - \frac{R_0 \parallel a \parallel_{0,\infty}}{\sqrt{\alpha}} - \frac{\parallel R_1 \parallel_{0,\infty}}{\sqrt{\alpha}}\Big)\frac{TE(0)}{1+T \parallel a \parallel_{0,\infty}} - \frac{2R_0}{\sqrt{\alpha}}E(0) \\
&\quad - \frac{\sqrt{\alpha}(1-n^2)}{8R_0}\int_{\Omega}(\mid u(0)\mid^2 + \mid u(T)\mid^2)dx \\
&\quad - \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_{\Gamma} m_k\nu_k(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma.
\end{aligned} \tag{3.37}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma m_k\nu_k\Big(|u'|^2 - a_{ij}(x,t)\sigma_i^A u\sigma_j^A u\Big)d\Sigma \\
&\quad + \frac{\sqrt{\alpha}(n-1)}{4R_0}\int_\Gamma m_k\nu_k(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \\
&\geq \frac{\sqrt{\alpha}T - 2R_0 - 3R_0T \parallel a \parallel_{0,\infty} - R_0T\sqrt{\alpha}\parallel b\parallel_{0,\infty} - T\parallel R_1\parallel_{0,\infty}}{\sqrt{\alpha}(1+T\parallel a\parallel_{0,\infty})}E(0) \\
&\quad + \frac{\sqrt{\alpha}(n^2-1)}{8R_0}\int_\Omega \mid u(0)\mid^2 dx.
\end{aligned} \tag{3.38}$$

Taking into account (3.13), this implies (3.14). □

Remark 3.3. If $x^0(t)$ is independent of t, then $R_1(t)\equiv 0$. If a_{ij} are independent of x, then $b(t)\equiv 0$. If a_{ij} are independent of t, then $a(t)\equiv 0$.

Let Γ_0 be any subset of Γ and $\Sigma_0 = \Gamma_0\times(0,T)$. Then

$$\int_{\Gamma_0}(\mid u(0)\mid^2 + \mid u(T)\mid^2)d\Gamma \leq \frac{2(T+1)}{T}\int_{\Sigma_0}(\mid u'\mid^2 + \mid u\mid^2)d\Sigma. \tag{3.39}$$

As a matter of fact, by calculation we have

$$\begin{aligned}
\int_{\Gamma_0} T\mid u(T)\mid^2 d\Gamma &= \int_{\Gamma_0}\int_0^T u^2 dt d\Gamma + \int_{\Gamma_0}\int_0^T t du^2 d\Gamma \\
&\leq (T+1)\int_{\Sigma_0}(\mid u'\mid^2 + \mid u\mid^2)d\Sigma,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Gamma_0} T\mid u(0)\mid^2 d\Gamma &= \int_{\Gamma_0}\int_0^T u^2 dt d\Gamma + \int_{\Gamma_0}\int_0^T (t-T) du^2 d\Gamma \\
&\leq (T+1)\int_{\Sigma_0}(\mid u'\mid^2 + \mid u\mid^2)d\Sigma.
\end{aligned}$$

Therefore (3.39) follows from the above.

By Lemma 3.2, we obtain the following observability inequality.

Lemma 3.4. (Observability inequality) *Suppose $\Sigma(x^0(0))\subset\Sigma(x^0)$, and suppose the conditions of Lemma 3.2 are satisfied. Then there exists a constant $c = c(T) > 0$ such that for all strong solutions u of (1.1) with $f = 0$*

$$\int_{\Sigma(x^0)}(|u'|^2 + \mid u\mid^2)d\Sigma + \int_{\Sigma_*(x^0)}\mid\nabla_{\sigma^A}u\mid^2 d\Sigma \geq c[\parallel u^0\parallel_1^2 + \parallel u^1\parallel_0^2]. \tag{3.40}$$

4. Proof of Theorem 0.1. We apply HUM. To do so, we consider the problem:

$$\begin{cases} u'' - Au = 0, & \text{in } Q, \\ u(0) = u^0, u'(0) = u^1, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_A} = 0, & \text{on } \Sigma. \end{cases} \tag{4.1}$$

For any $(u^0, u^1) \in (C^\infty(\overline{\Omega}) \cap D(A)) \times C^\infty(\overline{\Omega})$, problem (4.1) has a unique strong solution due to Theorem 1.2. Define

$$\| (u^0, u^1) \|_{\mathcal{H}} = \Big(\int_{\Sigma(x^0)} (|u'|^2 + | u |^2) d\Sigma + \int_{\Sigma_*(x^0)} | \nabla_{\sigma^A} u |^2 \, d\Sigma \Big)^{\frac{1}{2}}, \tag{4.2}$$

which is a norm on $(C^\infty(\overline{\Omega}) \cap D(A)) \times C^\infty(\overline{\Omega})$ due to Lemma 3.4. Let $\mathcal{H}$ be the completion of $(C^\infty(\overline{\Omega}) \cap D(A)) \times C^\infty(\overline{\Omega})$ with respect to the norm $\| \cdot \|_{\mathcal{H}}$. Then Lemma 3.4 implies that

$$\mathcal{H} \subset H^1(\Omega) \times L^2(\Omega). \tag{4.3}$$

Consequently

$$(H^1(\Omega))' \times L^2(\Omega) \subset \mathcal{H}'. \tag{4.4}$$

According to the definition of $\mathcal{H}$, we have for any $(u^0, u^1) \in \mathcal{H}$,

$$u|_{\Sigma(x^0)}, \quad u'|_{\Sigma(x^0)} \in L^2(\Sigma(x^0)), \qquad \nabla_{\sigma^A} u|_{\Sigma_*(x^0)} \in (L^2(\Sigma_*(x^0)))^n. \tag{4.5}$$

To apply the HUM, we need to consider the backward problem:

$$\begin{cases} v'' - Av = 0, & \text{in } Q, \\ v(T) = 0, v'(T) = 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu_A} = \begin{cases} -u + \frac{\partial}{\partial t} u', & \text{on } \Sigma(x^0), \\ \Delta_{\Sigma_*(x^0)} u, & \text{on } \Sigma_*(x^0). \end{cases} \end{cases} \tag{4.6}$$

The solution of (4.6) can be defined by the transposition method (see [9, 10]) as follows. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathcal{H}$ and $\mathcal{H}'$.

Definition 4.1. *v is said to be an ultraweak solution of (4.6) if there exist $(\rho^1, -\rho^0) \in \mathcal{H}'$ such that v satisfies*

$$\begin{aligned} &\int_Q f v dx dt + \langle (-\rho^1, \rho^0), (\theta^0, \theta^1) \rangle \\ &= - \int_{\Sigma(x^0)} (\theta u + \theta' u') d\Sigma - \int_{\Sigma_*(x^0)} \nabla_{\sigma^A} \theta \nabla_{\sigma^A} u d\Sigma, \end{aligned} \tag{4.7}$$

for any $(\theta^0, \theta^1) \in \mathcal{H}$, $f \in L^1(0, T; H^1(\Omega))$, and where θ is the solution of the following problem

$$\begin{cases} \theta'' - A\theta = f, & \textit{in } Q, \\ \theta(0) = \theta^0, \theta'(0) = \theta^1, & \textit{in } \Omega, \\ \dfrac{\partial \theta}{\partial \nu_A} = 0, & \textit{on } \Sigma. \end{cases} \tag{4.8}$$

We define

$$v(0) = \rho^0, \qquad v'(0) = \rho^1. \tag{4.9}$$

Lemma 4.2. *Problem (4.6) has a unique ultraweak solution in the sense of Definition 4.1 satisfying*

$$v \in L^\infty(0,T;(H^1(\Omega))'), \tag{4.10}$$

$$(v'(0), -v(0)) \in \mathcal{H}'. \tag{4.11}$$

Moreover, there exists $c > 0$ such that

$$\| (v'(0), -v(0)) \|_{\mathcal{H}'} \leq c \| (u^0, u^1) \|_{\mathcal{H}} . \tag{4.12}$$

We assume Lemma 4.2 for the moment. We then define a linear operator Λ by

$$\Lambda(u^0, u^1) = (v'(0), -v(0)). \tag{4.13}$$

Taking $f = 0$ in (4.7), we find

$$\langle \Lambda(u^0,u^1), (u^0,u^1)\rangle = \int_{\Sigma(x^0)} (|u'|^2 + | u |^2) d\Sigma + \int_{\Sigma_*(x^0)} | \nabla_{\sigma^A} u |^2 \, d\Sigma. \tag{4.14}$$

It therefore follows from Lemma 3.4, Lemma 4.2, and the Lax-Milgram Theorem that Λ is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$. This means that for all $(y^1, -y^0) \in \mathcal{H}'$, the equation

$$\Lambda(u^0, u^1) = (y^1, -y^0) \tag{4.15}$$

has a unique solution (u^0, u^1). With this initial condition we solve problem (4.1), and then solve problem (4.6). Then set

$$\phi = \begin{cases} -u + \frac{\partial}{\partial t} u', & \text{on } \Sigma(x^0), \\ \Delta_{\Sigma_*(x^0)} u, & \text{on } \Sigma_*(x^0), \end{cases} \tag{4.16}$$

and

$$y(x,t;\phi) = v(x,t;\phi). \tag{4.17}$$

Then we have constructed a control function ϕ such that the solution $y(x,t;\phi)$ of (0.1) satisfies (0.4). Thus, we have proved Theorem 0.1 provided we can prove Lemma 4.2.

Proof of Lemma 4.2. The solution θ of problem (4.8) can be written as $\theta = \eta + w$, where η and w are solutions of the following problems:

$$\begin{cases} \eta'' - A\eta = 0, & \text{in } Q, \\ \eta(0) = \theta^0, \quad \eta'(0) = \theta^1, & \text{in } \Omega, \\ \dfrac{\partial \eta}{\partial \nu_A} = 0. & \text{on } \Sigma, \end{cases} \tag{4.18}$$

and

$$\begin{cases} w'' - Aw = f, & \text{in } Q, \\ w(0) = w'(0) = 0, & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu_A} = 0. & \text{on } \Sigma. \end{cases} \tag{4.19}$$

Since $\{\theta^0, \theta^1\} \in \mathcal{H}$, we have

$$\| \{\theta^0, \theta^1\} \|_{\mathcal{H}} = \Big(\int_{\Sigma(x^0)} (|\eta'|^2 + | \eta |^2) d\Sigma + \int_{\Sigma_*(x^0)} | \nabla_\sigma \eta |^2 \, d\Sigma \Big)^{\frac{1}{2}}. \tag{4.20}$$

On the other hand, by Theorems 1.1-1.2 and the trace theorem (see [10, Chap. 1]), we have

$$\Big(\int_{\Sigma(x^0)} (|w'|^2 + | w |^2) d\Sigma + \int_{\Sigma_*(x^0)} | \nabla_\sigma w |^2 \, d\Sigma \Big)^{\frac{1}{2}} \leq c \| f \|_{L^1(0,T;H^1(\Omega))} . \tag{4.21}$$

Therefore,

$$\begin{aligned} & \Big| \int_Q f v dx dt + \langle (-\rho^1, \rho^0), (\theta^0, \theta^1) \rangle \Big| \\ & = \Big| \int_{\Sigma(x^0)} (\theta u + \theta' u') d\Sigma + \int_{\Sigma_*(x^0)} \nabla_{\sigma^A} \theta \nabla_{\sigma^A} u d\Sigma \Big| \\ & \leq \Big| \int_{\Sigma(x^0)} (\eta u + \eta' u') d\Sigma + \int_{\Sigma_*(x^0)} \nabla_{\sigma^A} \eta \nabla_{\sigma^A} u d\Sigma \Big| \\ & + \Big| \int_{\Sigma(x^0)} (w u + w' u') d\Sigma + \int_{\Sigma_*(x^0)} \nabla_{\sigma^A} w \nabla_{\sigma^A} u d\Sigma \Big| \\ & \leq c \Big(\| \{\theta^0, \theta^1\} \|_{\mathcal{H}} + \| f \|_{L^1(0,T;H^1(\Omega))} \Big) \| \{u^0, u^1\} \|_{\mathcal{H}} . \end{aligned} \tag{4.22}$$

Thus, there exist $v \in L^\infty(0, T; (H^1(\Omega))')$ and $\{\rho^1, -\rho^0\} \in \mathcal{H}'$ such that (4.7) holds, that is, v is an ultraweak solution of (4.6) and $\{v(0), -v'(0)\} \in \mathcal{H}'$. Taking $f = 0$, (4.22) gives (4.12).

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