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Compactness of the Difference between the Thermoviscoelastic Semigroup and its Decoupled Semigroup

Wei-Jiu Liu
Department of AMES
University of California at San Diego
La Jolla, CA 92093-0411
Fax: 858-822-3107
Phone: 858-822-2406
weiliu@cherokee.ucsd.edu

Abstract

By using the theory of semigroups, we prove the compactness of the difference between the semigroups generated by the system of thermoviscoelasticity type and its decoupled system, respectively.

Key Words: Compactness; thermoviscoelasticity; semigroups.

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1 Introduction

Consider the following thermoviscoelastic model

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u & \\ \quad +\mu g * \Delta u + (\lambda+\mu) g * \nabla \mathrm{div} u + \nabla\theta = 0 & \text{in } \Omega\times(0,\infty), \\ \theta_t - \Delta\theta + \mathrm{div}\ u_t = 0 & \text{in } \Omega\times(0,\infty), \\ u = 0,\ \theta = 0 & \text{on } \Gamma\times(0,\infty), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x) & \text{in } \Omega, \\ u(x,0) - u(x,-s) = w^0(x,s) & \text{in } \Omega\times(0,\infty), \end{cases} \tag{1.1}$$

where the sign "$*$" denotes the convolution product in time, which is defined by

$$g * v(t) = \int_{-\infty}^{t} g(t-s) v(x,s) ds. \tag{1.2}$$

System (1.1) is a model for a linear viscoelastic body Ω of the Boltzmann type with thermal damping. The body Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ (say C^2) and is assumed to be linear, homogeneous, and isotropic. $u(x,t) = (u_1(x,t), \cdots, u_n(x,t)),\ \theta(x,t)$ represent displacement and temperature deviations, respectively, from the natural state of the reference configuration at position x and time t. $\lambda,\ \mu > 0$ are Lamé's constants. $g(t)$ denotes the relaxation function, $w^0(x,s)$ is a specified " history", and $u^0(x),\ u^1(x),\ \theta^0(x)$ are initial data. The subscript t denotes the derivative with respect to the time variable. $\Delta,\ \nabla$, div denote the Laplace, gradient, and divergence operators in the space variables, respectively. We refer to [12] for the derivation of model (1.1).

In [10, 11], we studied the problem of stabilization and controllability of system (1.1). In this paper, we address the problem of compactness of the difference between the C_0 (i.e., strongly continuous) semigroup $S(t)$ generated by system (1.1) and the C_0 semigroup $S_d(t)$ generated by its decoupled system

$$\begin{cases} \bar{u}_{tt} - \mu\Delta \bar{u} - (\lambda+\mu)\nabla \mathrm{div} \bar{u} & \\ \quad +\mu g * \Delta \bar{u} + (\lambda+\mu) g * \nabla \mathrm{div} \bar{u} + \nabla\Delta^{-1}\mathrm{div}\ \bar{u}_t = 0 & \text{in } \Omega\times(0,\infty), \\ \bar{\theta}_t - \Delta\bar{\theta} + \mathrm{div}\bar{u}_t = 0 & \text{in } \Omega\times(0,\infty), \\ \bar{u} = 0,\ \bar{\theta} = 0 & \text{on } \Gamma\times(0,\infty), \\ \bar{u}(x,0) = u^0(x),\ \bar{u}_t(x,0) = u^1(x),\ \bar{\theta}(x,0) = \theta^0(x) & \text{in } \Omega, \\ \bar{u}(x,0) - \bar{u}(x,-s) = w^0(x,s) & \text{in } \Omega\times(0,\infty). \end{cases} \tag{1.3}$$

This problem is motivated when we study the essential spectrum $\sigma_e(S(t))$ of $S(t)$ (for the definition of essential spectrum, we refer to [5, p.39]). Indeed, if we can prove that the difference $S(t) - S_d(t)$ is compact, then it follows from Theorem 4.1 of [5, p.40] that $\sigma_e(S(t)) = \sigma_e(S_d(t))$. Moreover, $\sigma_e(S_d(t))$ is easier to be calculated as system (1.3) is decoupled, much simpler than system (1.1). The reason why we use the term $\nabla\Delta^{-1}\mathrm{div}\ \bar{u}_t$ to decouple system (1.1) is because it is a dissipative term. Therefore, the main theme of this paper is to prove that the difference $S(t) - S_d(t)$ is compact. This result generalizes the similar result of [7].

The rest of this paper is organized as follows. We present our main results in Section 2. Via the semigroup theory, we prove them in Section 3 and 4.

2 Main Results

In what follows, $H^s(\Omega)$ denotes the usual Sobolev space (see, e.g., [1, 9]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω with compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T];X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

Let us introduce a general abstract system which includes system (1.1) as a particular example. For this, let H_1 and H_2 be two Hilbert spaces. Let $A_1 : D(A_1) \subset H_1 \to H_1$ and $A_2 : D(A_2) \subset H_2 \to H_2$ be self adjoint positive operators with compact inverses and $B : D(B) \subset H_2 \to H_1$ a closed operator with adjoint B^* such that $D(A_2^{1/2}) \subset D(B)$ and $D(A_1^{1/2}) \subset D(B^*)$. We consider the following system of thermoviscoelasticity type

$$\begin{cases} u_{tt} + kA_1 u + \int_0^\infty g(s)A_1 w(t,s)ds + B\theta = 0, \\ \theta_t + A_2\theta - B^* u_t = 0, \\ w_t - u_t + w_s = 0, \\ u(0) = u^0,\ u_t(0) = u^1,\ \theta(0) = \theta^0,\ w(0,s) = w^0(s), \end{cases} \tag{2.1}$$

where k is a positive constant and $g(s)$ is a given function.

In order to see that abstract system (2.1) includes system (1.1) as a particular example, we set

$$H_1 = (L^2(\Omega))^n, \quad H_2 = L^2(\Omega), \tag{2.2}$$
$$w(x,t,s) = u(x,t) - u(x,t-s), \tag{2.3}$$

and define the operators A_1, A_2 and B by

$$A_1 = -\mu\Delta - (\lambda+\mu)\nabla\mathrm{div}, \tag{2.4}$$
$$A_2 = -\Delta, \tag{2.5}$$
$$B = \nabla, \tag{2.6}$$

with domains given by

$$D(A_1) = (H^2(\Omega) \cap H_0^1(\Omega))^n, \tag{2.7}$$
$$D(A_2) = H^2(\Omega) \cap H_0^1(\Omega), \tag{2.8}$$
$$D(B) = H_0^1(\Omega). \tag{2.9}$$

It is easy to see that the adjoint B^* of B is given by

$$B^* = -\mathrm{div} \tag{2.10}$$

with the domain

$$D(B^*) = (H_0^1(\Omega))^n. \tag{2.11}$$

It is also clear that

$$D(A_1^{1/2}) = (H_0^1(\Omega))^n, \tag{2.12}$$
$$D(A_2^{1/2}) = H_0^1(\Omega). \tag{2.13}$$

Thus, the operators A_1, A_2 and B satisfy all above conditions. In order to transform the first equation of (1.1) into the first equation of (2.1), we need to impose basic conditions on the function $g(t)$ as follows (see [2, 3]):

(H_1) $g \in C^1[0,\infty) \cap L^1(0,\infty)$;

(H_2) $g(t) \geq 0$ and $g'(t) \leq 0$ for $t > 0$;

(H_3) $k = 1 - \int_0^\infty g(s)ds > 0$.

Under these conditions, we have

$$\begin{aligned}\int_{-\infty}^{t} g(t-s)\Delta u(x,s)ds &= \int_0^\infty g(s)\Delta u(x,t-s)ds \\ &= \int_0^\infty g(s)\Delta(u(x,t-s)-u(x,t))ds + \int_0^\infty g(s)\Delta u(x,t)ds \\ &= -\int_0^\infty g(s)\Delta w(x,t,s)ds + (1-k)\Delta u(x,t), \end{aligned} \tag{2.14}$$

and similar expression for $g * \nabla \mathrm{div} u$. Thus, the first equation of (1.1) can be written in the form of (2.1) and then system (1.1) can be transformed into (2.1).

Motivated by the decoupled system (1.3), we consider the decoupled system of (2.1)

$$\begin{cases} \bar{u}_{tt} + kA_1\bar{u} + \int_0^\infty g(s)A_1\bar{w}(t,s)ds + BA_2^{-1}B^*\bar{u}_t = 0, \\ \bar{\theta}_t + A_2\bar{\theta} - B^*\bar{u}_t = 0, \\ \bar{w}_t - \bar{u}_t + \bar{w}_s = 0, \\ \bar{u}(0) = u^0,\ \bar{u}_t(0) = u^1,\ \bar{\theta}(0) = \theta^0,\ \bar{w}(0,s) = w^0(s). \end{cases} \tag{2.15}$$

We are going to prove that systems (2.1) and (2.15) generate C_0 semigroups and the difference between them are compact. In doing so, we formulate systems (2.1) and (2.15) as first order Cauchy problems. For this, we introduce the "history space" $L^2(g,(0,\infty),D(A_1^{1/2}))$. Let $\|\cdot\|$ denote the norm of H_1 or H_2. The "history space" $L^2(g,(0,\infty),D(A_1^{1/2}))$ consist of $D(A_1^{1/2})$-valued functions w on $(0,\infty)$ for which

$$\|w\|^2_{L^2(g,(0,\infty),D(A_1^{1/2}))} = \int_0^\infty g(s)\|A_1^{1/2}w(s)\|^2 ds < \infty. \tag{2.16}$$

Set

$$\mathcal{H} = D(A_1^{1/2}) \times H_1 \times H_2 \times L^2(g,(0,\infty),D(A_1^{1/2})) \tag{2.17}$$

with the norm

$$\|(u,v,\theta,w)\|_{\mathcal{H}} = [k\|A_1^{1/2}u\|^2 + \|v\|^2 + \|\theta\|^2 + \|w\|^2_{L^2(g,(0,\infty),D(A_1^{1/2}))}]^{1/2}. \tag{2.18}$$

We define two linear unbounded operators $\mathcal{A}$ and $\mathcal{A}_d$ on $\mathcal{H}$ by

$$\begin{aligned}\mathcal{A}(u,v,\theta,w) &= (v,\ -kA_1u - \int_0^\infty g(s)A_1w(s)ds - B\theta, \\ &\qquad -A_2\theta + B^*v, v - w_s), \end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathcal{A}_d(u,v,\theta,w) &= (v,\ -kA_1u - \int_0^\infty g(s)A_1w(s)ds - BA_2^{-1}B^*v, \\ &\qquad -A_2\theta + B^*v, v - w_s), \end{aligned} \tag{2.20}$$

with the domain

$$
\begin{aligned}
D(\mathcal{A}) &= D(\mathcal{A}_d) \\
&= \{(u, v, \theta, w) \in \mathcal{H} \ : \ \theta \in D(A_2), \ v \in D(A_1^{1/2}), \\
&\qquad ku + \int_0^\infty g(s)w(s)ds \in D(A_1), \\
&\qquad w(s) \in H^1(g, (0, \infty), D(A_1^{1/2}), \ w(0) = 0\} \qquad (2.21)
\end{aligned}
$$

where

$$
\begin{aligned}
&H^1(g, (0, \infty), D(A_1^{1/2})) \\
= \ &\{w \ : \ w, \ w_s \in L^2(g, (0, \infty), D(A_1^{1/2}))\}. \qquad (2.22)
\end{aligned}
$$

Setting

$$v = u_t, \qquad (2.23)$$

we then transform (2.1) and (2.15) into

$$
\begin{cases}
\dfrac{d}{dt}(u, v, \theta, w) = \mathcal{A}(u, v, \theta, w), \\
(u(0), v(0), \theta(0), w(0)) = (u^0, u^1, \theta^0, w^0),
\end{cases} \qquad (2.24)
$$

and

$$
\begin{cases}
\dfrac{d}{dt}(\bar{u}, \bar{v}, \bar{\theta}, \bar{w}) = \mathcal{A}_d(\bar{u}, \bar{v}, \bar{\theta}, \bar{w}), \\
(\bar{u}(0), \bar{v}(0), \bar{\theta}(0), \bar{w}(0)) = (u^0, u^1, \theta^0, w^0),
\end{cases} \qquad (2.25)
$$

respectively.

Theorem 2.1. *Suppose that the function g satisfies (H_1) and (H_2). Then $\mathcal{A}$ and $\mathcal{A}_d$ are infinitesimal generators of C_0 semigroups $e^{\mathcal{A}t}$ and $e^{\mathcal{A}_d t}$ of contractions on $\mathcal{H}$, respectively.*

We now consider the difference $e^{\mathcal{A}t} - e^{\mathcal{A}_d t}$. Let

$$(u, v, \theta, w) = e^{\mathcal{A}t}(u^0, u^1, \theta^0, w^0), \ \ (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}) = e^{\mathcal{A}_d t}(u^0, u^1, \theta^0, w^0). \qquad (2.26)$$

Then by (2.1) and (2.15) we have

$$
\begin{pmatrix} u(t) - \bar{u}(t) \\ v(t) - \bar{v}(t) \\ \theta(t) - \bar{\theta}(t) \\ w(t) - \bar{w}(t) \end{pmatrix} = \int_0^t e^{\mathcal{A}(t-s)} \begin{pmatrix} 0 \\ BA_2^{-1}B^*\bar{v}(s) - B\bar{\theta}(s) \\ 0 \\ 0 \end{pmatrix} ds. \qquad (2.27)
$$

Due to the smoothness effect of temperature component $\bar{\theta}$, we can expect certain compact properties of the difference $e^{\mathcal{A}t} - e^{\mathcal{A}_d t}$. Indeed, we have

Theorem 2.2. *Suppose that the function g satisfies (H_1) and (H_2). Suppose that $A_2^{-1}B^*A_1^{1/2} : H_1 \to H_2$ and $BA_2^{-1/2} : H_2 \to H_1$ are bounded and $BA_2^{-\gamma} : H_2 \to H_1$ is compact for some $\gamma < 1$. Then, for any $T > 0$, $e^{\mathcal{A}t} - e^{\mathcal{A}_d t} : \mathcal{H} \to C([0, T]; \mathcal{H})$ is a compact operator.*

Remark 2.1. The operators A_1, A_2 and B defined by (2.4), (2.5) and (2.6) satisfy conditions of Theorem 2.2 with $\gamma = 3/4$.

3 Proof of Theorem 2.1

In this section we use the Lumer-Phillips theorem from the theory of semigroups (see [13, p.14]) to prove Theorem 2.1. We first present two technical lemmas.

Lemma 3.1. *[8, p.491] If the function $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous and is in $L^1(0, \infty)$, then*

$$\lim_{t\to\infty} f(t) = 0. \tag{3.1}$$

Lemma 3.2. *Suppose that the function g satisfies (H_1) and (H_2). If $w \in H^1(g, (0, \infty), D(A_1^{1/2}))$, then*

$$g'(s)\|A_1^{1/2}w(s)\|^2 \in L^1(0, \infty), \tag{3.2}$$

and

$$\lim_{s\to\infty} g(s)\|A_1^{1/2}w(s)\|^2 = 0. \tag{3.3}$$

Proof. Since for $w \in H^1(g, (0, \infty), D(A_1^{1/2}))$

$$\begin{aligned} & 2\int_0^t g(s)(w_s(s), w(s))_{D(A_1^{1/2})}ds \\ = & \int_0^t g(s)\frac{\partial}{\partial s}[\|A_1^{1/2}w(s)\|^2]ds \\ = & g(t)\|A_1^{1/2}w(t)\|^2 - g(0)\|A_1^{1/2}w(0)\|^2 - \int_0^t g'(s)\|A_1^{1/2}w(s)\|^2ds, \end{aligned} \tag{3.4}$$

we have for all $t \geq 0$

$$\begin{aligned} & \int_0^t |g'(s)|\|A_1^{1/2}w(s)\|^2ds \\ \leq & 2\Big(\int_0^\infty g(s)\|A_1^{1/2}w_s(s)\|^2ds\Big)^{1/2}\Big(\int_0^\infty g(s)\|A_1^{1/2}w(s)\|^2ds\Big)^{1/2} \\ & +g(0)\|A_1^{1/2}w(0)\|^2. \end{aligned} \tag{3.5}$$

Thus

$$g'(s)\|A_1^{1/2}w(s)\|^2 \in L^1(0, \infty). \tag{3.6}$$

On the other hand, for any $0 \leq s_1 < s_2 < \infty$, we have

$$\begin{aligned} & g(s_2)\|A_1^{1/2}w(s_2)\|^2 - g(s_1)\|A_1^{1/2}w(s_1)\|^2 \\ = & \int_{s_1}^{s_2} \frac{d}{ds}[g(s)\|A_1^{1/2}w(s)\|^2]ds \\ = & \int_{s_1}^{s_2} g'(s)\|A_1^{1/2}w(s)\|^2ds + 2\int_{s_1}^{s_2} g(s)(w_s(s), w(s))_{D(A_1^{1/2})}ds, \end{aligned} \tag{3.7}$$

which, combining (3.6), implies that $g(s)\|A_1^{1/2}w(s)\|^2$ is uniformly continuous on $[0, \infty)$. Hence Lemma 3.1 gives

$$\lim_{s\to\infty} g(s)\|A_1^{1/2}w(s)\|^2 = 0. \tag{3.8}$$

□

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. By the Lumer-Phillips theorem from the theory of semigroups (see [13, p.14]), it suffices to prove that $\mathcal{A}$ is dissipative and $I-\mathcal{A}$ is surjective.

In what follows, we denote by $(\cdot,\cdot)$ the inner product of H_1 or H_2.

For any $(u,v,\theta,w)\in D(\mathcal{A})$, we have

$$\begin{aligned}
& (\mathcal{A}(u,v,\theta,w),(u,v,\theta,w))_{\mathcal{H}} \\
= \ & k(A_1^{1/2}v, A_1^{1/2}u) - (kA_1u + \int_0^\infty g(s)A_1w(s)ds + B\theta, v) \\
& + (-A_2\theta + B^*v,\theta) + (v - w_s, w)_{L^2(g,(0,\infty),D(A_1^{1/2}))} \\
= \ & -\int_0^\infty g(s)(A_1^{1/2}w(s), A_1^{1/2}v)ds - \|A_2^{1/2}\theta\|^2 \\
& + \int_0^\infty g(s)(A_1^{1/2}w(s), A_1^{1/2}v)ds - \int_0^\infty g(s)(A_1^{1/2}w_s(s), A_1^{1/2}w)ds \\
= \ & -\|A_2^{1/2}\theta\|^2 - \frac{1}{2}g(s)\|A_1^{1/2}w(s)\|^2\Big|_0^\infty + \frac{1}{2}\int_0^\infty g'(s)\|A_1^{1/2}w(s)\|^2 ds \\
= \ & -\|A_2^{1/2}\theta\|^2 + \frac{1}{2}\int_0^\infty g'(s)\|A_1^{1/2}w(s)\|^2 ds \ \ (\text{use} \quad \text{Lemma 3.2}) \\
\le \ & 0. \qquad (3.9)
\end{aligned}$$

Thus, $\mathcal{A}$ is dissipative.

To prove that $I-\mathcal{A}$ is surjective, we first prove that $\mathcal{A}$ is closed. Let $(u_n,v_n,\theta_n,w_n)\in D(\mathcal{A})$ be such that

$$(u_n,v_n,\theta_n,w_n)\to(u,v,\theta,w) \quad \text{in } \mathcal{H}, \tag{3.10}$$

and

$$\mathcal{A}(u_n,v_n,\theta_n,w_n)\to(\varphi,\psi,\xi,z) \quad \text{in } \mathcal{H}. \tag{3.11}$$

We want to show that

$$\mathcal{A}(u,v,\theta,w)=(\varphi,\psi,\xi,z), \ \ (u,v,\theta,w)\in D(\mathcal{A}). \tag{3.12}$$

By (3.10) and (3.11), we have

$$\begin{aligned}
u_n &\to u \quad \text{in } D(A_1^{1/2}), & (3.13)\\
v_n &\to v \quad \text{in } H_1, & (3.14)\\
\theta_n &\to \theta \quad \text{in } H_2, & (3.15)\\
w_n &\to w \quad \text{in } L^2(g,(0,\infty),D(A_1^{1/2})), & (3.16)
\end{aligned}$$

and

$$\begin{aligned}
v_n &\to \varphi \quad \text{in } D(A_1^{1/2}), & (3.17)\\
-kA_1u_n - \int_0^\infty g(s)A_1w_n(s)ds - B\theta_n &\to \psi \quad \text{in } H_1, & (3.18)\\
-A_2\theta_n - B^*v_n &\to \xi \quad \text{in } H_2, & (3.19)
\end{aligned}$$

$$v_n - w_{ns} \to z \quad \text{in } L^2(g,(0,\infty),D(A_1^{1/2})). \tag{3.20}$$

By (3.14) and (3.17), we deduce

$$v_n \to v \quad \text{in } D(A_1^{1/2}), \tag{3.21}$$

and

$$v = \varphi \in D(A_1^{1/2}). \tag{3.22}$$

By (3.19) and (3.21), we deduce

$$-A_2\theta_n \to B^*v + \xi \quad \text{in } H_2, \tag{3.23}$$

and consequently, it follows from (3.15) that

$$\theta_n \to \theta \quad \text{in } D(A_2). \tag{3.24}$$

It therefore follows from (3.19) and (3.24) that

$$\xi = -A_2\theta - B^*v, \quad \theta \in D(A_2). \tag{3.25}$$

By (3.16), (3.20) and (3.21), we deduce

$$w_n \to w \quad \text{in } H^1(g,(0,\infty),D(A_1^{1/2})), \tag{3.26}$$

and

$$z = v - w_s, \ \ w \in H^1(g,(0,\infty),D(A_1^{1/2})), \quad w(0) = 0. \tag{3.27}$$

In addition, it follows from (3.13), (3.16) and (3.24) that

$$-kA_1u_n - \int_0^\infty g(s)A_1w_n(s)ds - B\theta_n \tag{3.28}$$

$$\to -kA_1u - \int_0^\infty g(s)A_1w(s)ds - B\theta, \quad \text{in } (D(A_1^{1/2}))'. \tag{3.29}$$

It therefore follows from (3.18) and (3.29) that

$$\psi = -kA_1u - \int_0^\infty g(s)A_1w(s)ds - B\theta, \tag{3.30}$$

and consequently,

$$\kappa u + \int_0^\infty g(s)w(s)ds \in D(A_1), \tag{3.31}$$

since A_1 has an inverse $A_1^{-1} : H_1 \to D(A_1)$. Thus, by (3.22), (3.25), (3.27), (3.30) and (3.31), we deduce (3.12) and then $\mathcal{A}$ is closed. Therefore, to show that $I - \mathcal{A}$ is surjective, it is sufficient to show that the range of $I - \mathcal{A}$ is dense in $\mathcal{H}$. Thus, let us look at the problem

$$(I - \mathcal{A})(u, v, \theta, w) = (\varphi, \psi, \xi, \eta), \tag{3.32}$$

that is,

$$u - v = \varphi, \tag{3.33}$$

$$v + kA_1 u + \int_0^\infty g(s) A_1 w(s) ds + B\theta = \psi, \tag{3.34}$$

$$\theta + A_2\theta - B^* v = \xi, \tag{3.35}$$

$$w - v + w_s = \eta. \tag{3.36}$$

We may assume that $\eta(s)$ has compact support in $(0, \infty)$ and we seek a solution $(u, v, \theta, w) \in D(\mathcal{A})$. The solution of (3.36) is readily written down as

$$w(s) = (1 - e^{-s})v + e^{-s} \int_0^s e^t \eta(t) dt. \tag{3.37}$$

By substituting u and w into (3.34), we obtain

$$\begin{aligned} & v + [k + \int_0^\infty g(s)(1 - e^{-s}) ds] A_1 v + B\theta \\ & = \Psi, \end{aligned} \tag{3.38}$$

where

$$\Psi = \psi - kA_1\varphi - \int_0^\infty g(s) \int_0^s e^t A_1 \eta(t) dt ds. \tag{3.39}$$

Since we have assumed that $\eta(s)$ has compact support in $(0, \infty)$, it is easy to see that $\Psi \in (D(A_1^{1/2}))'$.

Define a linear operator $\mathcal{B}$ by

$$\mathcal{B}(v, \theta) = (v + [k + \int_0^\infty g(s)(1 - e^{-s}) ds] A_1 v + B\theta, \theta + A_2\theta - B^* v). \tag{3.40}$$

Obviously, to solve (3.33)-(3.36), it suffices to show that $\mathcal{B}$ maps $D(A_1^{1/2}) \times D(A_2^{1/2})$ onto $[D(A_1^{1/2})]' \times [D(A_2^{1/2})]'$. By Lax-Milgram theorem (see, e.g., [4, p.368]), it suffices to show $\mathcal{B}$ is coercive. This is true since, for $(v, \theta) \in D(A_1^{1/2}) \times D(A_2^{1/2})$, we have

$$\begin{aligned} \langle \mathcal{B}(v, \theta), (v, \theta) \rangle &= \langle v, v \rangle + [k + \int_0^\infty g(s)(1 - e^{-s}) ds] \langle A_1 v, v \rangle \\ &\quad + \langle \theta, \theta \rangle + \langle A_2\theta, \theta \rangle \\ &\geq \alpha(\|v\|^2_{D(A_1^{1/2})} + \|\theta\|^2_{D(A_2^{1/2})}), \end{aligned} \tag{3.41}$$

where

$$\alpha = \min\{1, \;\; k + \int_0^\infty g(s)(1 - e^{-s}) ds\}. \tag{3.42}$$

In the similar way, we can prove that $\mathcal{A}_d$ is an infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}$. We give here only a brief outline.

For any $(u, v, \theta, w) \in D(\mathcal{A}_d)$, we have

$$
\begin{aligned}
& (\mathcal{A}_d(u,v,\theta,w),(u,v,\theta,w))_{\mathcal{H}} \\
= \ & k(A_1^{1/2}v, A_1^{1/2}u) - (kA_1u + \int_0^\infty g(s)A_1w(s)ds + BA_2^{-1}B^*v, v) \\
& +(-A_2\theta + B^*v, \theta) + (v - w_s, w)_{L^2(g,(0,\infty),D(A_1^{1/2}))} \\
= \ & -\int_0^\infty g(s)(A_1^{1/2}w(s), A_1^{1/2}v)ds - \|A_2^{-1/2}B^*v\|^2 - \|A_2^{1/2}\theta\|^2 + (B^*v, \theta) \\
& + \int_0^\infty g(s)(A_1^{1/2}w(s), A_1^{1/2}v)ds - \int_0^\infty g(s)(A_1^{1/2}w_s(s), A_1^{1/2}w)ds \\
\leq \ & -\frac{1}{2}\|A_2^{1/2}\theta\|^2 - \frac{1}{2}\|A_2^{-1/2}B^*v\|^2 - \frac{1}{2}g(s)\|A_1^{1/2}w(s)\|^2\Big|_0^\infty + \frac{1}{2}\int_0^\infty g'(s)\|A_1^{1/2}w(s)\|^2ds \\
= \ & -\frac{1}{2}\|A_2^{1/2}\theta\|^2 - \frac{1}{2}\|A_2^{-1/2}B^*v\|^2 + \frac{1}{2}\int_0^\infty g'(s)\|A_1^{1/2}w(s)\|^2ds \ \ (\text{use} \quad \text{Lemma 3.2}) \\
\leq \ & 0.
\end{aligned} \tag{3.43}
$$

Thus, $\mathcal{A}_d$ is dissipative.

Replacing $B\theta$ by $BA_2^{-1}B^*v$ and repeating the above procedure for $\mathcal{A}$, we can prove that $\mathcal{A}_d$ is closed.

To prove that $I - \mathcal{A}_d$ is surjective, we define a linear operator $\mathcal{B}_d$ by

$$
\mathcal{B}_d(v,\theta) = (v + [k + \int_0^\infty g(s)(1 - e^{-s})ds]A_1v + BA_2^{-1}B^*v, \theta + A_2\theta - B^*v). \tag{3.44}
$$

The operator $\mathcal{B}_d$ is coercive since, for $(v, \theta) \in D(A_1^{1/2}) \times D(A_2^{1/2})$, we have

$$
\begin{aligned}
\langle \mathcal{B}(v,\theta),(v,\theta)\rangle \ = \ & \langle v,v\rangle + [k + \int_0^\infty g(s)(1-e^{-s})ds]\langle A_1v, v\rangle \\
& + \langle BA_2^{-1}B^*v, v\rangle + \langle \theta,\theta\rangle + \langle A_2\theta,\theta\rangle - \langle B^*v,\theta\rangle \\
\geq \ & \langle v,v\rangle + [k + \int_0^\infty g(s)(1-e^{-s})ds]\langle A_1v, v\rangle \\
& + \frac{1}{2}\|A_2^{-1/2}B^*v\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\|A_2^{1/2}\theta\|^2 \\
\geq \ & \alpha_1(\|v\|^2_{D(A_1^{1/2})} + \|\theta\|^2_{D(A_2^{1/2})}),
\end{aligned} \tag{3.45}
$$

where

$$
\alpha_1 = \min\{1/2, \ \ k + \int_0^\infty g(s)(1 - e^{-s})ds\}. \tag{3.46}
$$

□

4 Proof of Theorem 2.2

The proof of Theorem 2.2 is based on the following technical lemma of [7, Lemma 6].

Lemma 4.1. *([7, Lemma 6]) Let $S(t)$ be a C_0 semigroup in a Banach space X and M be a subset of $L^1([0,T];X)$. Then the set*

$$\{\int_0^t S(t-s)f(s) \ : \ f(s) \in M\} \tag{4.1}$$

is precompact in $C([0,T];X)$ if one of the following conditions holds:

(i) $\{f(s) \ : \ f \in M, \ 0 \le s \le T\}$ is precompact in X;

(ii) for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ and a compact set $K(\epsilon)$ of X such that $\int_0^\delta \|f(s)\|ds < \epsilon$ and $f(s)$ belongs to $K(\epsilon)$ for $\delta \le s \le T$ and $f \in M$.

Proof of Theorem 2.2. By the definition of compactness, we need to prove that

$$M = \{(e^{\mathcal{A}t} - e^{\mathcal{A}_d t})(u^0, u^1, \theta^0, w^0) \ : \ (u^0, u^1, \theta^0, w^0) \in B(0,1)\} \tag{4.2}$$

is precompact in $C([0,T];\mathcal{H})$, where $B(0,1)$ is the unit ball of $\mathcal{H}$. Let

$$(u, v, \theta, w) = e^{\mathcal{A}t}(u^0, u^1, \theta^0, w^0), \ \ (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}) = e^{\mathcal{A}_d t}(u^0, u^1, \theta^0, w^0). \tag{4.3}$$

Then by (2.1) and (2.15) we have

$$\begin{pmatrix} u(t) - \bar{u}(t) \\ v(t) - \bar{v}(t) \\ \theta(t) - \bar{\theta}(t) \\ w(t) - \bar{w}(t) \end{pmatrix} = \int_0^t e^{\mathcal{A}(t-s)} \begin{pmatrix} 0 \\ BA_2^{-1}B^*\bar{v}(s) - B\bar{\theta}(s) \\ 0 \\ 0 \end{pmatrix} ds. \tag{4.4}$$

By Lemma 4.1, it is sufficient to check that the set

$$M_0 = \{B\bar{\theta}(s) - BA_2^{-1}B^*\bar{v}(s) \ : \ (u^0, u^1, \theta^0, w^0) \in B(0,1)\} \tag{4.5}$$

satisfies one of the conditions of Lemma 4.1. To this end, by (2.15), we have

$$\begin{aligned}
& B\bar{\theta}(s) - BA_2^{-1}B^*\bar{v}(s) \\
= \ & Be^{-A_2 s}\theta^0 + B\int_0^s e^{-A_2(s-\tau)}B^*\bar{v}(\tau)d\tau - BA_2^{-1}B^*\bar{v}(s) \\
= \ & Be^{-A_2 s}\theta^0 + BA_2^{-1}\int_0^s A_2 e^{-A_2(s-\tau)}B^*\bar{v}(\tau)d\tau - BA_2^{-1}B^*\bar{v}(s) \\
= \ & Be^{-A_2 s}\theta^0 + BA_2^{-1}e^{-A_2(s-\tau)}B^*\bar{v}(\tau)\Big|_0^s \\
& -BA_2^{-1}\int_0^s e^{-A_2(s-\tau)}B^*\bar{v}'(\tau)d\tau - BA_2^{-1}B^*\bar{v}(s) \\
= \ & Be^{-A_2 s}\theta^0 - BA_2^{-1}e^{-A_2 s}B^*u^1 + kBA_2^{-1}\int_0^s e^{-A_2(s-\tau)}B^*A_1\bar{u}(\tau)d\tau \\
& +BA_2^{-1}\int_0^s e^{-A_2(s-\tau)}B^*[\int_0^\infty g(s)A_1\bar{w}(s)ds + BA_2^{-1}B^*\bar{v}(\tau)]d\tau.
\end{aligned} \tag{4.6}$$

We claim that the first two terms of the right hand side of (4.6) satisfy condition (ii) of Lemma 4.1 and the other terms satisfy condition (i) of Lemma 4.1. In fact, by Theorem 1.4.3

of [6, p.26], we have (the following c's denoting generic positive constants that may vary from line to line and that are independent of (u, v, θ, w))

$$\|A_2^{\delta} e^{-A_2 t}\| \leq c t^{-\delta}, \quad \delta > 0. \tag{4.7}$$

Thus we have

$$\|A_2^{\gamma} e^{-A_2 s}\theta_0\| \leq c\|\theta_0\| s^{-\gamma}. \tag{4.8}$$

Using the boundedness of $A_2^{-1/2}B^*$ and (4.7), we deduce that

$$\|A_2^{\gamma-1} e^{-A_2 s} B^* u^1\| = \|A_2^{\gamma-\frac{1}{2}} e^{-A_2 s} A_2^{-1/2} B^* u^1\| \leq c\|u^1\| s^{\frac{1}{2}-\gamma}. \tag{4.9}$$

Since $BA_2^{-\gamma}$ is compact and $Be^{-A_2 s}\theta^0 - BA_2^{-1}e^{-A_2 s}B^*u^1 = BA_2^{-\gamma}[A_2^{\gamma}e^{-A_2 s}\theta^0 - A_2^{\gamma}A_2^{-1}e^{-A_2 s}B^*u^1]$, it follows from (4.8) and (4.9) that $Be^{-A_2 s}\theta^0 - BA_2^{-1}e^{-A_2 s}B^*u^1$ satisfies condition (ii) of Lemma 4.1. Furthermore, using (4.7), the boundedness of $A_2^{-1}B^*A_1^{1/2}$ and the inequality

$$\|(\bar{u}(t), \bar{v}(t), \bar{\theta}(t), \bar{w}(t))\|_{\mathcal{H}} \leq \|(u^0, u^1, \theta^0, w^0)\|_{\mathcal{H}}, \tag{4.10}$$

we deduce

$$\begin{aligned}
& \|A_2^{\gamma} A_2^{-1} \int_0^s e^{-A_2(s-\tau)} B^* A_1 \bar{u}(\tau) d\tau\| \\
= \; & \| \int_0^s A_2^{\gamma} e^{-A_2(s-\tau)} A_2^{-1} B^* A_1^{1/2} A_1^{1/2} \bar{u}(\tau) d\tau\| \\
\leq \; & c \int_0^s (s-\tau)^{-\gamma} \|A_1^{1/2}\bar{u}(\tau)\| d\tau \\
\leq \; & c\|(u^0, u^1, \theta^0, w^0)\|_{\mathcal{H}} \int_0^s (s-\tau)^{-\gamma} d\tau \\
\leq \; & \frac{c s^{1-\gamma}\|(u^0, u^1, \theta^0, w^0)\|_{\mathcal{H}}}{1-\gamma},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
& \|A_2^{\gamma} A_2^{-1} \int_0^s e^{-A_2(s-\tau)} \int_0^{\infty} g(s) B^* A_1 \bar{w}(\tau, s) ds d\tau\| \\
= \; & \| \int_0^s A_2^{\gamma} e^{-A_2(s-\tau)} \int_0^{\infty} g(s) A_2^{-1} B^* A_1 \bar{w}(\tau, s) ds d\tau\| \\
\leq \; & c \int_0^s (s-\tau)^{-\gamma} \int_0^{\infty} g(s) \|A_1^{1/2} \bar{w}(\tau, s)\| ds d\tau \\
\leq \; & c \int_0^s (s-\tau)^{-\gamma} (\int_0^{\infty} g(s) ds)^{1/2} (\int_0^{\infty} g(s) \|A_1^{1/2} \bar{w}(\tau, s)\|^2 ds)^{1/2} d\tau \\
\leq \; & c\|(u^0, u^1, \theta^0, w^0)\|_{\mathcal{H}} \int_0^s (s-\tau)^{-\gamma} d\tau \\
\leq \; & \frac{c s^{1-\gamma}\|(u^0, u^1, \theta^0, w^0)\|_{\mathcal{H}}}{1-\gamma},
\end{aligned} \tag{4.12}$$

and

$$
\begin{aligned}
& \|A_2^{\gamma}A_2^{-1}\int_0^s e^{-A_2(s-\tau)}B^*BA_2^{-1}B^*\bar{v}(\tau)d\tau\| \\
= & \ \|\int_0^s A_2^{\gamma-\frac{1}{2}}e^{-A_2(s-\tau)}A_2^{-1/2}B^*BA_2^{-1}B^*\bar{v}(\tau)d\tau\| \\
\leq & \ c\int_0^s (s-\tau)^{\frac{1}{2}-\gamma}\|\bar{v}(\tau)\|d\tau \\
\leq & \ \frac{cs^{1+\frac{1}{2}-\gamma}\|(u^0,u^1,\theta^0,w^0)\|_{\mathcal{H}}}{1+\frac{1}{2}-\gamma}. \qquad (4.13)
\end{aligned}
$$

By the compactness of $BA_2^{-\gamma}$, it follows from (4.11), (4.12) and (4.13) that the last three terms of the right hand side of (4.6) satisfy condition (i) of Lemma 4.1. This completes the proof of Theorem 2.2. □

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