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Robustness of Boundary Feedback Controls for a Flexible Beam with Respect to Perturbation*

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Abstract

In this paper we test the robustness of a nonlinear boundary feedback control for a flexible beam with respect to perturbation. We show that additional dynamics of perturbation at one of the ends of the beam, as long as they are strictly passive, will not destabilize the controlled beam.

Key Words: Flexible beam; robustness; stability; Lyapunov method.

AMS subject classification: 73K05, 35B35, 47H09.

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1 Introduction

It is well known that the flexible beam equation with a nonlinear boundary feedback control

$$u_{tt} + u_{xxxx} = 0 \qquad \text{in } (0,1)\times(0,\infty), \tag{1.1}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0 \qquad \text{in } (0,\infty), \tag{1.2}$$

$$u_{xxx}(1,t) = g_1(u_t(1,t)) \qquad \text{in } (0,\infty), \tag{1.3}$$

$$u(0) = u^0,\ u_t(0) = u^1 \qquad \text{in } (0,1) \tag{1.4}$$

is asymptotically stable (see, e.g., [5, 7, 8, 9, 10, 11, 18, 21, 24, 25, 27, 33, 34, 35, 38, 39, 40, 42, 46]). In (1.1)-(1.4), $u = u(x,t)$ denotes the transverse deflection of the beam, the subscripts denote the derivatives, $u(0)$, $u_t(0)$ denote the functions $x \to u(x,0)$, $x \to u_t(x,0)$, respectively, $u^0 = u^0(x)$, $u^1 = u^1(x)$ are intial data, and $g_1 = g_1(s)$ is continuous and increasing function on $\mathbb{R}$ with $g_1(0) = 0$.

In reality, all sensors, processors and actuators introduce time delay and perturbation into the controlled system. It has been proved that the feedback control is usually not robust with respect to the time dealy (see, e.g., [3, 13, 14, 15, 16, 17, 28]). The objective of this paper is to test the robustness with respect to the perturbation. Thus we introduce additional dynamics of perturbation $\eta = \eta(t)$ at the right end of the beam as follows

$$u_{xxx}(1,t) = g_1(u_t(1,t)) + \eta(t). \tag{1.5}$$

As long as the additional dynamics η is strictly passive, for instance, η satisfying

$$\eta_t = -g_2(\eta) + u_t(1,t), \tag{1.6}$$

where $g_2 = g_2(s)$ is continuous and increasing function on $\mathbb{R}$ with $g_2(0) = 0$, we can show that it will not destabilize system (1.1)-(1.4). Indeed, we shall show that the following closed-loop system

$$u_{tt} + u_{xxxx} = 0 \qquad \text{in } (0,1)\times(0,\infty), \tag{1.7}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0 \qquad \text{in } (0,\infty), \tag{1.8}$$

$$u_{xxx}(1,t) = g_1(u_t(1,t)) + \eta(t) \qquad \text{in } (0,\infty), \tag{1.9}$$

$$\eta_t = -g_2(\eta) + u_t(1,t) \qquad \text{in } (0,\infty), \tag{1.10}$$

$$u(0) = u^0,\ u_t(0) = u^1 \qquad \text{in } (0,1), \tag{1.11}$$

$$\eta(0) = \eta^0 \tag{1.12}$$

is globally asymptotically stable.

Since the above system is composed of a partial differential equation and an ordinary differential equation, it is often referred to as hybrid system in the literature (see, e.g., [35, Section 4.6] and [42]).

From the point of view of control engineering, feedback controls (1.9)–(1.10) can be explained in various ways. Substituting (1.9) into (1.10), we obtain

$$u_{xxxt}(1,t) = g_1'(u_t(1,t))u_{tt}(1,t) - g_2[u_{xxx}(1,t) - g_1(u_t(1,t))] + u_t(1,t). \tag{1.13}$$

This feedback control is of actuator dynamics. Further, (1.13) can be written as

$$u_{xxx}(1,t) = g_1(u_t(1,t)) + g_2^{-1}[g_1'(u_t(1,t))u_{tt}(1,t) + u_t(1,t) - u_{xxxt}(1,t)]. \tag{1.14}$$

Hence we obtain the feedback control which uses higher order derivative information as in [10, 34]. If $g_1(s) = \alpha ms$ and $g_2(s) = \alpha s$, where $\alpha, m > 0$ are constants, (1.14) is reduced to

$$-u_{xxx}(1,t) + mu_{tt}(1,t) = \frac{1}{\alpha}u_{xxxt}(1,t) - \frac{1+\alpha^2 m}{\alpha}u_t(1,t). \tag{1.15}$$

This is the feedback control for a flexible beam with a tip mass m, which was proposed in [10]. In addition, if $g_1(s) = g_2(s) = ks$ for some $k > 0$, we solve (1.10) and obtain

$$\eta = \eta^0 e^{-kt} + e^{-kt}\int_0^t e^{-ks}u_s(1,s)ds,$$

and then

$$u_{xxx}(1,t) = ku_t(1,t) + \eta^0 e^{-kt} + e^{-kt}\int_0^t e^{-ks}u_s(1,s)ds. \tag{1.16}$$

Thus, we obtain a feedback control which not only depends on the current state at time t but also takes into account the past memory $\int_0^t e^{-ks}u_s(1,s)ds$.

There has been extensive work on this topic and we have made efforts to try to collect all relevant references. We apologize to any we have accidentally missed to be enclosed.

We present our main results about the well-posedness, estimates of decay rate of energy of system (1.7)-(1.12) and a nonlinear mean ergodic theorem in Section 2. Using the theory of nonlinear semigroups and Lyapunov method, we prove them in Section 3.

Notation. Throughout the paper, $H^s(0,1)$ denotes the usual Sobolev space (see [1, 29]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. The norm of $L^2(0,1)$ is denoted by $\|\cdot\|$.

Let X be a Banach space and $T > 0$. We denote by $C^n([0,T];X)$ the space of n times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

We further introduce other function spaces as follows:

$$H_{0^-}^2(0,1) = \{u \in H^2(0,1) \ : \ u(0) = u_x(0) = 0\}, \tag{1.17}$$

$$\mathcal{L} = H_{0^-}^2(0,1) \times L^2(0,1) \times \mathbb{R}, \tag{1.18}$$

$$\mathcal{H} = \{(\varphi_1, \varphi_2, r) \in (H_{0^-}^2(0,1) \cap H^4(0,1)) \times H_{0^-}^2(0,1) \times \mathbb{R} \ : \ \varphi_{1xx}(1) = 0, \ \varphi_{1xxx}(1) = g_1(\varphi_2(1)) + r\}. \tag{1.19}$$

The norm of $\mathcal{L}$ is defined by

$$\|(\varphi_1, \varphi_2, r)\|_{\mathcal{L}} = (\|\varphi_{1xx}\|^2 + \|\varphi_2\|^2 + |r|)^{1/2}, \quad \forall (\varphi_1, \varphi_2, r) \in \mathcal{L}. \tag{1.20}$$

2 Main Results

We define the energy functions by

$$E=\frac{1}{2}\int_0^1 \left(|u_t|^2+|u_{xx}|^2\right)dx, \tag{2.1}$$

and

$$V=E+\frac{1}{2}\eta^2. \tag{2.2}$$

The main results of this paper are as follows. First of all, we have the well-posedness theorem.

Theorem 2.1. *Assume that $g_1, g_2 \in C(\mathbb{R})$ are increasing (not necessarily strict) on $\mathbb{R}$ and satisfy that $g_1(0)=g_2(0)=0$.*

1. *For every initial condition $(u^0, u^1, \eta^0) \in \mathcal{L}$, problem (1.7)-(1.12) has a unique mild solution with*
$$(u, u_t, \eta) \in C([0,\infty); \mathcal{L}).$$
Moreover, for any two solutions (u_1, η_1) and (u_2, η_2) corresponding to initial conditions (u_1^0, u_1^1, η_1^0) and (u_2^0, u_2^1, η_2^0), respectively, we have
$$\|(u_1, u_{1t}, \eta_1)-(u_2, u_{2t}, \eta_2)\|_{\mathcal{L}} \le \|(u_1^0, u_1^1, \eta_1^0)-(u_2^0, u_2^1, \eta_2^0)\|_{\mathcal{L}}, \quad \forall t \ge 0. \tag{2.3}$$

2. *For every initial condition $(u^0, u^1, \eta^0) \in \mathcal{H}$, problem (1.7)-(1.12) has a unique classical solution with*
$$(u, u_t, \eta) \in C([0,\infty); \mathcal{H}).$$

We then have the nonlinear mean ergodic theorem.

Theorem 2.2. *Assume that $g_1, g_2 \in C(\mathbb{R})$ are increasing (not necessarily strict) on $\mathbb{R}$ and satisfy that $g_1(0)=g_2(0)=0$. Then, for every mild solution of problem (1.7)-(1.12) with initial condition $(u^0, u^1, \eta^0) \in \mathcal{L}$, the mean*

$$\frac{1}{T}\Big(\int_0^T u(x,t)dt, \int_0^T u_t(x,t)dt, \int_0^T \eta(t)dt\Big) \tag{2.4}$$

converges to zero in $\mathcal{L}$ as $T \to \infty$.

Remark 2.1. The functions g_1 and g_2 are not required to be strictly increasing on $\mathbb{R}$. In fact, we can take $g_1=g_2=0$. In this case, the above mean still converges to zero in $\mathcal{L}$ as $T \to \infty$ although $V(t) \equiv V(0)$, as we showed in [30] for the wave equation.

If g_1, g_2 satisfy additional conditions, then we can obtain stronger results than Theorem 2.2.

Theorem 2.3. *Assume that $g_1, g_2 \in C(\mathbb{R})$ satisfies the following conditions:*

1. *$g_1(0)=g_2(0)=0$;*

2. *g_1, g_2 are increasing on $\mathbb{R}$;*

3. *there are constants $c_1, c_2 > 0$ such that*
$$c_1^{-1}|s| \leq |g_i(s)| \leq c_2|s|, \quad \text{for } |s| \geq 1 \text{ and } i = 1, 2; \tag{2.5}$$

4. *there exists a strictly increasing positive function $h(s)$ defined on $[0, \infty)$ and a constant $c_3 > 0$ such that*
$$c_3^{-1}h(|s|) \leq |g_i(s)| \leq c_4 h^{-1}(|s|), \quad \text{for } |s| \leq 1 \text{ and } i = 1, 2, \tag{2.6}$$
where h^{-1} denotes the inverse of h and $c_4 = \max_{|s|\leq 1}\{|g_1(s)|,\ |g_2(s)|\}$;

5. *there exists an increasing, strictly positive and convex function $G = G(s)$ defined on $[0, \infty)$ and twice differentiable outside $s = 0$ such that $G(s^2) \leq h(|s|)|s|$ on $[-1, 1]$ and $G''(s)s$ is increasing on $[0, \infty)$.*

Then, for any initial condition $(u^0, u^1, \eta^0) \in \mathcal{L}$, the solution of problem (1.7)-(1.12) satisfies the following decay estimate
$$V(t) \leq 2V_\epsilon(t) \qquad \forall t \geq 0, \tag{2.7}$$
where V_ϵ is the solution of the following ordinary differential equation
$$V_\epsilon' \ = \ -\frac{\epsilon V_\epsilon}{3}G'\Big(\frac{2\delta V_\epsilon}{3}\Big) - \epsilon k_1 G\Big(\frac{2\delta V_\epsilon}{3}\Big). \tag{2.8}$$
Moreover, we have
$$\lim_{t\to\infty} V(t) = 0. \tag{2.9}$$
Furthermore, the constants ϵ and δ can be estimated as follows:
$$k_1 \ = \ 3 + c_4^2, \tag{2.10}$$
$$\delta \ = \ \frac{1}{2k_1}, \tag{2.11}$$
$$k_2 \ = \ \max\{2c_3,\ 2c_1 G'(\delta V(0))\}, \tag{2.12}$$
$$k_3 \ = \ \max\{c_3 + c_4,\ (c_1 + c_2)G'(\delta V(0))\}, \tag{2.13}$$
$$\epsilon \ = \ \min\Big\{\frac{1}{2G'(\delta V(0))},\ \frac{1}{\delta V(0)G''(\delta V(0)) + k_2},\ \frac{1}{\delta V(0)G''(\delta V(0)) + k_3}\Big\}. \tag{2.14}$$

Remark 2.2. The result of Theorem 2.3 is stronger than that of Theorem 2.2. In fact, if $\lim_{t\to\infty} V(t) = 0$, then we have
$$\begin{aligned}
&\lim_{T\to\infty}\Big\|\frac{1}{T}\Big(\int_0^T u(x,t)dt,\int_0^T u_t(x,t)dt,\int_0^T \eta(t)dt\Big)\Big\|_{\mathcal{L}} \\
&\leq \lim_{T\to\infty}\Big(\frac{2}{T}\int_0^T V(t)dt\Big)^{1/2} \\
&(\text{ We may as well assume that } \lim_{T\to\infty}\int_0^T V(t)dt = \infty) \\
&= \Big(\lim_{T\to\infty}\frac{2\frac{d}{dT}\int_0^T V(t)dt}{\frac{dT}{dT}}\Big)^{1/2} \\
&= 0.
\end{aligned} \tag{2.15}$$

Remark 2.3. As explained in [31, 32], there always exists the function G that satisfies the condition of Theorem 2.3 and the usual exponential, polynomial and logarithmic decay rates can be recovered from (2.8). Indeed, we can take

$$G(s) = \mathrm{conv}[s^{1/2}h(s^{1/2})], \tag{2.16}$$

where conv denotes the convex envelope of a function. Further, if $g_1(s) = g_2(s) = ks$, where k is a positive constant, then $h(s) = ks$ and $G(s) = ks$. Consequently, (2.8) becomes

$$V_\epsilon' = -\omega V_\epsilon, \tag{2.17}$$

where ω is a positive constant independent of $V(0)$. This gives exponential decay rate. If $g_1(s) = g_2(s) = k|s|^{p-1}s$ for $|s| \leq 1$ with $p > 1$ and $k > 0$, then $h(s) = ks^p$ and $G(s) = ks^{(p+1)/2}$. In this case, (2.8) becomes

$$V_\epsilon' = -\omega V_\epsilon^{(p+1)/2}, \tag{2.18}$$

where ω is a positive constant depending on $V(0)$. As usual, this implies the polynomial decay rate

$$V_\epsilon(t) \leq C(V(0))(1+t)^{-2/(p-1)}, \quad \forall t > 0. \tag{2.19}$$

If $g_1(s) = g_2(s) = s^3 e^{-\frac{1}{s^2}}$ for $|s| \leq 1$, then $h(s) = s^3 e^{-\frac{1}{s^2}}$ and $G(s) = s^2 e^{-\frac{1}{s}}$. Consequently, by droping out the first term in the right hand side of (2.8), (2.8) becomes

$$V_\epsilon'(t) \leq -\omega_1 V_\epsilon^2 e^{-\frac{\omega_2}{V_\epsilon}}, \tag{2.20}$$

which is the same as

$$\left(e^{\frac{\omega_2}{V_\epsilon}}\right)' \geq \omega_1\omega_2, \tag{2.21}$$

where ω_1, ω_2 are positive constants depending on $V(0)$. Solving the inequality, we obtain the logarithmic decay rate

$$V_\epsilon(t) \leq \frac{\sigma_1}{\log(\sigma_2 t + \sigma_3)}, \tag{2.22}$$

where $\sigma_1, \sigma_2, \sigma_3$ are positive constants depending on $V(0)$.

Remark 2.4. The dynamics η of perturbation has a significant impact on the original stabilized beam system. For example, if $g_1(s) = ks$ and no perturbation is involved, we know that the system is exponentially stable. However, if the dynamics η of perturbation is presented with $g_2(s) = k|s|^{p-1}s$ for $|s| \leq 1$ with $p > 1$ and $k > 0$, then the perturbed system is no longer exponentially stable, only polynomially stable.

Remark 2.5. There have been already extensive studies on the problem of describing the decay rate of energy when the nonlinear damping decays near the origin faster than any polynomial and many important results have been obtained, notably [26, 36, 37, 44], to mention a few. Indeed, to our knowledge, the original work in this aspect might be due to [26] and some ideas here are motivated by [26].

3 Proofs

We apply the theory of nonlinear semigroups to prove that problem (1.7)-(1.12) is well posed. Therefore, we formulate problem (1.7)-(1.12) as an abstract Cauchy problem. For this, we define the nonlinear operator $\mathcal{A}$

$$\mathcal{A}(\varphi_1, \varphi_2, r) = (\varphi_2, -\varphi_{1xxxx}, -g_2(r) + \varphi_2(1)) \tag{3.1}$$

with domain $D(\mathcal{A}) = \mathcal{H}$. Set

$$y = (u, u_t, \eta), \quad y^0 = (u^0, u^1, \eta^0).$$

Then problem (1.7)-(1.12) can be written as an abstract Cauchy problem

$$y_t = \mathcal{A}y, \tag{3.2}$$
$$y(0) = y^0. \tag{3.3}$$

Hence, to prove Theorem 2.1, by Theorems 1 and 2 of [6] (see also, e.g., [4, Chap.3], [45, p.121-122, Theorems 5.1 and 5.2]), it suffices to prove that $\mathcal{A}$ is m-dissipative. For the definition of m-dissipativeness, we refer to [4, p.71]. To achieve this, we need to consider the following nonlinear elliptic boundary value problem

$$\varphi + \varphi_{xxxx} = f, \qquad 0 < x < 1, \tag{3.4}$$
$$\varphi(0) = \varphi_x(0) = 0, \quad \varphi_{xx}(1) = a, \tag{3.5}$$
$$\varphi_{xxx}(1) = g_1(\varphi(1)) + (I + g_2)^{-1}(b + \varphi(1)), \tag{3.6}$$

where g_1 and g_2 are the functions given in Theorem 2.1, $(I+g_2)^{-1}$ denote the inverse function of $I + g_2$ and I denotes the function $I(x) = x$; f is a given function and a, b are arbitrary real numbers. Although it seems that the following Lemma 3.1 should be well known in the literature, we could not find it in the references we know. Therefore, for completeness, we present it here.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^2_{0^-}$ and $(H^2_{0^-})^*$.

Lemma 3.1. *Suppose that $g_1(s), g_2(s)$ are increasing and continuous functions and satisfy that $g_1(0) = g_2(0) = 0$. Then for every $f \in (H^2_{0^-}(0,1))^*$ and $a, b \in \mathbb{R}$, problem (3.4)-(3.6) has a unique weak solution $\varphi \in H^2_{0^-}(0,1)$ in the sense of distribution*

$$\int_0^1 \varphi\xi \, dx + \int_0^1 \varphi_{xx}\xi_{xx} \, dx + g_1(\varphi(1))\xi(1) + (I + g_2)^{-1}(b + \varphi(1))\xi(1) = a\xi_x(1) + \langle f, \xi \rangle \tag{3.7}$$

for any $\xi \in H^2_{0^-}(0,1)$. Moreover, if $f \in L^2(0,1)$, then $\varphi \in H^2_{0^-}(0,1) \cap H^4(0,1)$.

Proof. Define the nonlinear operator A and the functional f_a on $H^2_{0^-}$ by

$$\langle A\varphi, \xi \rangle = \int_0^1 \varphi\xi \, dx + \int_0^1 \varphi_{xx}\xi_{xx} \, dx + g_1(\varphi(1))\xi(1) + (I + g_2)^{-1}(b + \varphi(1))\xi(1), \tag{3.8}$$
$$\langle f_a, \xi \rangle = a\xi_x(1) \tag{3.9}$$

for any $\xi \in H^2_{0^-}(0,1)$. By the Sobolev embedding theorem (see, e.g., [1, p.97]), we can readily show that $A : H^2_{0^-} \to (H^2_{0^-})^*$ and $f_a \in (H^2_{0^-})^*$. Then (3.7) is equivalent to

$$A\varphi = f_a + f \quad \text{in } (H^2_{0^-})^*. \tag{3.10}$$

Therefore, to prove that problem (3.4)-(3.6) has a unique weak solution $\varphi \in H^2_{0^-}(0,1)$ for every $f \in (H^2_{0^-}(0,1))^*$ and $a, b \in \mathbb{R}$, it suffices to prove that A is onto and one-to-one. By Theorem 1.3 of [4, p.40], it suffices to show A is monotone, coercive and hemicontinuous.

For any $\varphi_1, \varphi_2 \in H^2_{0^-}(0,1)$, we have

$$\begin{aligned}
\langle A\varphi_1 - A\varphi_2, \varphi_1 - \varphi_2\rangle &= \int_0^1 (\varphi_1 - \varphi_2)^2\,dx + \int_0^1 (\varphi_{1xx} - \varphi_{2xx})^2\,dx \\
&\quad + [g_1(\varphi_1(1)) - g_1(\varphi_2(1))](\varphi_1(1) - \varphi_2(1)) + [(I+g_2)^{-1}(b + \varphi_1(1)) \\
&\quad - (I+g_2)^{-1}(b + \varphi_2(1))](\varphi_1(1) - \varphi_2(1)) \\
&= \int_0^1 (\varphi_1 - \varphi_2)^2\,dx + \int_0^1 (\varphi_{1xx} - \varphi_{2xx})^2\,dx \\
&\quad + [g_1(\varphi_1(1)) - g_1(\varphi_2(1))](\varphi_1(1) - \varphi_2(1)) + [(I+g_2)^{-1}(b + \varphi_1(1)) \\
&\quad - (I+g_2)^{-1}(b + \varphi_2(1))][(b + \varphi_1(1)) - (b + \varphi_2(1))] \\
&\geq 0,
\end{aligned} \tag{3.11}$$

since g_1, g_2 are increasing and $g_1(0) = g_2(0) = 0$. Therefore, A is monotone. On the other hand, by the continuity of g_1, g_2, we have for any $\varphi, \varphi_1, \varphi_2 \in H^2_{0^-}(0,1)$

$$\begin{aligned}
\lim_{t\to 0}\langle A(\varphi_1 + t\varphi_2), \varphi\rangle &= \lim_{t\to 0}\Big[\int_0^1 (\varphi_1 + t\varphi_2)\varphi\,dx + \int_0^1 (\varphi_{1xx} + t\varphi_{2xx})\varphi_{xx}\,dx \\
&\quad + g_1(\varphi_1(1) + t\varphi_2(1))\varphi(1) + (I+g_2)^{-1}(b + (\varphi_1(1) + t\varphi_2(1)))\Big] \\
&= \int_0^1 \varphi_1\varphi\,dx + \int_0^1 \varphi_{1xx}\varphi_{xx}\,dx \\
&\quad + g_1(\varphi_1(1))\varphi(1) + (I+g_2)^{-1}(b + \varphi_1(1)) \\
&= \langle A\varphi_1, \varphi\rangle,
\end{aligned} \tag{3.12}$$

which shows that A is hemicontinuous. Moreover, since g_2 is increasing, we have (see, e.g., [20, p.40])

$$|(I+g_2)^{-1}(s) - (I+g_2)^{-1}(r)| \leq |s - r| \quad \forall s, r \in \mathbb{R}.$$

It therefore follows that for any $\varphi \in H^2_{0^-}(0,1)$

$$\begin{aligned}
\langle A\varphi, \varphi\rangle &= \int_0^1 \varphi^2\,dx + \int_0^1 \varphi_{xx}^2\,dx + g_1(\varphi(1))\varphi(1) + (I+g_2)^{-1}(b+\varphi(1))\varphi(1) \\
&= \int_0^1 \varphi^2\,dx + \int_0^1 \varphi_{xx}^2\,dx + g_1(\varphi(1))\varphi(1) \\
&\quad + (I+g_2)^{-1}(b+\varphi(1))(b+\varphi(1)) - b(I+g_2)^{-1}(b+\varphi(1)) \\
&\geq \int_0^1 \varphi^2\,dx + \int_0^1 \varphi_{xx}^2\,dx - |b(I+g_2)^{-1}(b+\varphi(1))| \\
&\geq \int_0^1 \varphi^2\,dx + \int_0^1 \varphi_{xx}^2\,dx - |b(b+\varphi(1))|, \qquad (3.13)
\end{aligned}$$

which shows that A is coercive.

Furthermore, if $f \in L^2(0,1)$, then $\varphi_{xxxx} = \varphi - f \in L^2(0,1)$. Hence $\varphi \in H^2_{0^-}(0,1) \cap H^4(0,1)$. $\square$

We now prove that $\mathcal{A}$ is m-dissipative on $\mathcal{L}$.

Lemma 3.2. *Suppose that $g_1(s), g_2(s)$ are increasing and continuous functions and satisfy that $g_1(0) = g_2(0) = 0$. Then the operator $\mathcal{A}$ defined by* (3.1) *is m-dissipative on $\mathcal{L}$, that is, $\mathcal{A}$ is dissipative and $(I-\mathcal{A})(D(\mathcal{A})) = \mathcal{L}$. Moreover, $(I-\mathcal{A})^{-1}$ is compact.*

Proof. For any $(\varphi_1, \varphi_2, r)$, $(\psi_1, \psi_2, s) \in D(\mathcal{A})$, we have

$$\begin{aligned}
&\langle \mathcal{A}(\varphi_1,\varphi_2,r) - \mathcal{A}(\psi_1,\psi_2,s), (\varphi_1,\varphi_2,r) - (\psi_1,\psi_2,s)\rangle \\
&= \int_0^1 (\varphi_{2xx} - \psi_{2xx})(\varphi_{1xx} - \psi_{1xx}) - (\varphi_{1xxxx} - \psi_{1xxxx})(\varphi_2 - \psi_2))\,dx \\
&\quad - (r-s)(g_2(r) - g_2(s) - \varphi_2(1) + \psi_2(1)) \\
&= -(\varphi_{1xxx}(1) - \psi_{1xxx}(1))(\varphi_2(1) - \psi_2(1)) - (r-s)(g_2(r) - g_2(s) - \varphi_2(1) + \psi_2(1)) \\
&= -(g_1(\varphi_2(1)) - g_1(\psi_2(1)) + r - s)(\varphi_2(1) - \psi_2(1)) \\
&\quad - (r-s)(g_2(r) - g_2(s) - \varphi_2(1) + \psi_2(1)) \\
&= -(g_1(\varphi_2(1)) - g_1(\psi_2(1)))(\varphi_2(1) - \psi_2(1)) - (r-s)(g_2(r) - g_2(s)) \\
&\leq 0, \qquad (3.14)
\end{aligned}$$

since g_1 and g_2 are increasing and $g_1(0) = g_2(0) = 0$. Therefore, $\mathcal{A}$ is dissipative. On the other hand, for any $(f_1, f_2, s) \in \mathcal{L}$, let us consider the equation

$$(\varphi_1, \varphi_2, r) - \mathcal{A}(\varphi_1, \varphi_2, r) = (f_1, f_2, s),$$

that is,

$$\varphi_1 - \varphi_2 = f_1, \qquad (3.15)$$
$$\varphi_2 + \varphi_{1xxxx} = f_2, \qquad (3.16)$$
$$r + g_2(r) - \varphi_2(1) = s, \qquad (3.17)$$
$$\varphi_1(0) = \varphi_{1x}(0) = \varphi_{1xx}(1) = 0, \qquad (3.18)$$
$$\varphi_{1xxx}(1) = g_1(\varphi_2(1)) + r. \qquad (3.19)$$

Substituting the first equation into the second and fifth equations and the third equation into the fifth equation gives

$$\varphi_1 + \varphi_{1xxxx} = f_1 + f_2, \tag{3.20}$$
$$\varphi_1(0) = \varphi_{1x}(0) = \varphi_{1xx}(1) = 0, \tag{3.21}$$
$$\varphi_{1xxx}(1) = g_1(\varphi_1(1) - f_1(1)) + (I + g_2)^{-1}(s + (\varphi_1(1) - f_1(1))). \tag{3.22}$$

Set

$$\xi(x) = \varphi_1(x) - f_1(1)x^2.$$

Then (3.20)-(3.22) is transformed into

$$\xi + \xi_{xxxx} = f_1 + f_2 - f_1(1)x^2, \tag{3.23}$$
$$\xi(0) = \xi_x(0) = 0, \quad \xi_{xx}(1) = -2f_1(1), \tag{3.24}$$
$$\xi_{xxx}(1) = g_1(\xi(1)) + (I + g_2)^{-1}(s + \xi(1)). \tag{3.25}$$

Lemma 3.1 shows that problem (3.23)-(3.25) has a solution $\xi \in H^2_{0-}(0,1) \cap H^4(0,1)$ and then problem (3.15)-(3.19) has a solution

$$\varphi_1 = \xi + f_1(1)x^2, \tag{3.26}$$
$$\varphi_2 = \xi + f_1(1)x^2 - f_1, \tag{3.27}$$
$$r = (I + g_2)^{-1}(s + \xi(1)). \tag{3.28}$$

Obviously, we have $(\varphi_1, \varphi_2, r) \in D(\mathcal{A})$. Therefore, we have $(I - \mathcal{A})(D(\mathcal{A})) = \mathcal{L}$. Moreover, $(I - \mathcal{A})^{-1}$ is compact since the embedding of $D(\mathcal{A})$ into $\mathcal{L}$ is compact. □

Lemma 3.3. *Suppose that $g_1(s)$ is an increasing and continuous function and $g_1(0) = 0$. Then $D(\mathcal{A})$ is dense in $\mathcal{L}$.*

Proof. Set

$$D_0 = \{(\varphi_1, \varphi_2, r) \in (H^2_{0-}(0,1) \cap H^4(0,1)) \times H^2_0(0,1) \times \mathbb{R} \ : \ \varphi_{1xx}(1) = 0, \varphi_{1xxx}(1) = r\}. \tag{3.29}$$

Since $g_1(0) = 0$, it is clear that $D_0 \subset D(\mathcal{A})$. Therefore, to prove that $D(\mathcal{A})$ is dense in $\mathcal{L}$, it suffices to prove that D_0 is dense in $\mathcal{L}$ and then it suffices to prove that

$$W = \{(\varphi, r) \in (H^2_{0-}(0,1) \cap H^4(0,1)) \times \mathbb{R} \ : \ \varphi_{xx}(1) = 0, \varphi_{xxx}(1) = r\} \tag{3.30}$$

is dense in $H^2_{0-}(0,1) \times \mathbb{R}$. For any fixed $(\varphi, r) \in H^2_{0-}(0,1) \times \mathbb{R}$, there exists $\varphi_n \in H^2_{0-}(0,1) \cap H^4(0,1)$ with $\varphi_{nxx}(1) = \varphi_{nxxx}(1) = 0$ such that φ_n converges to $\varphi - \frac{1}{6}rx^3 + \frac{1}{2}rx^2$ in $H^2_{0-}(0,1)$. Set

$$\psi_n = \varphi_n + \frac{1}{6}rx^3 - \frac{1}{2}rx^2.$$

It is clear that $(\psi_n, r) \in W$ and (ψ_n, r) converges to (φ, r) in $H^2_{0-}(0,1) \times \mathbb{R}$. □

We are now ready to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Theorem 2.1 follows simply from Lemmas 3.2 and 3.3 and Theorems 5.1 and 5.2 of [45, p.121-122]. □

Proof of Theorem 2.2. By Lemmas 3.2 and 3.3 and Theorems 5.1 and 5.2 of [45, p.121-122], we deduce that $\mathcal{A}$ generates a contraction semigroups $S(t)$. It is clear that the fixed point of $S(t)$ is 0. Further, it follows from Theorem 3 of [12] and Lemma 3.2 that the orbit $\gamma(u^0, u^1, \eta^0) = \{S(t)(u^0, u^1, \eta^0) \ : \ t \geq 0\}$ is precompact. Therefore Theorem 2.2 readily follows from Theorem 3.1 of [43] (or Theorem 7 of [41]). $\square$

We now turn to the proof of Theorem 2.3. The proof is based on the construction of an appropriate Lyapunov function and the generalized Young's inequality (see, e.g., [2, p. 64]) as in [31, 32], which were originally suggested by Zuazua.

Proof of Theorem 2.3. Let V be the energy function defined by (2.2). Then we have

$$\begin{aligned} V' &= \int_0^1 (-u_t u_{xxxx} + u_{xx} u_{xxt})\, dx + \eta\eta_t \\ &= -u_t(1,t)(g_1(u_t(1,t)) + \eta) + \eta(-g_2(\eta) + u_t(1,t)) \\ &= -u_t(1,t) g(u_t(1,t)) - \eta g_2(\eta). \end{aligned} \tag{3.31}$$

If $V(t_0) = 0$ for some $t_0 \geq 0$, we have $V(t) \equiv 0$ for $t \geq t_0$ and then the theorem holds. Therefore, we may assume that $V(t) > 0$ for $t \geq 0$. This assumption ensures that, in the following proof, $G''(\delta V(t))$ makes sense as we have assumed that $G(s)$ is twice differentiable outside $s = 0$.

For any $\epsilon > 0$, we define the Lyapunov function V_ϵ by

$$V_\epsilon = V + \epsilon\psi(V)F, \tag{3.32}$$

where F is defined by

$$F = \int_0^1 x u_x u_t\, dx + \frac{1}{2}\eta^2,$$

and $\psi(s)$ will be determined in the proof, satisfying that $\psi(s)$ and $\psi'(s)s$ are positive and increasing functions on $(0, +\infty)$. It is clear that

$$[1 - \epsilon\psi(V(0))]V \leq V_\epsilon \leq [1 + \epsilon\psi(V(0))]V. \tag{3.33}$$

Since

$$\begin{aligned} F' &= \int_0^1 x(u_{xt}u_t - u_x u_{xxxx})\, dx + \eta\eta' \\ &= \frac{1}{2}u_t^2(1,t) - \frac{1}{2}\int_0^1 u_t^2\, dx - u_x(1,t)u_{xxx}(1,t) - \frac{3}{2}\int_0^1 u_{xx}^2\, dx + \eta\eta' \\ &= -V + \frac{1}{2}u_t^2(1,t) - u_x(1,t)(\eta + g_1(u_t(1,t))) + \frac{\eta^2}{2} - \eta g_2(\eta) + \eta u_t(1,t) - \int_0^1 u_{xx}^2\, dx \\ &\quad (\text{ note that } u_x^2(1,t) \leq \textstyle\int_0^1 u_{xx}^2\, dx) \\ &\leq -V + u_t^2(1,t) + g_1^2(u_t(1,t)) + 2\eta^2 - \eta g_2(\eta), \end{aligned} \tag{3.34}$$

we deduce

$$
\begin{aligned}
V_\epsilon' &= V' + \epsilon\psi'(V)V'F + \epsilon\psi(V)F' \\
&\le -\epsilon V\psi(V) + [1 - \epsilon V(0)\psi'(V(0))]V' + \epsilon\psi(V)[u_t^2(1,t) + g_1^2(u_t(1,t)) + 2\eta^2] \\
&= -\epsilon V\psi(V) - [1 - \epsilon V(0)\psi'(V(0))][u_t(1,t)g_1(u_t(1,t)) + \eta g_2(\eta)] \\
&\quad + \epsilon\psi(V)[u_t^2(1,t) + g_1^2(u_t(1,t)) + 2\eta^2]. \qquad (3.35)
\end{aligned}
$$

We now estimate the third term. If $|u_t(1,t)| \ge 1$, then by (2.5), we have

$$
\begin{aligned}
\psi(V)u_t^2(1,t) &\le c_1\psi(V(0))u_t(1,t)g_1(u_t(1,t)), \qquad (3.36)\\
\psi(V)g_1^2(u_t(1,t)) &\le c_2\psi(V(0))u_t(1,t)g_1(u_t(1,t)). \qquad (3.37)
\end{aligned}
$$

Let G^* denote the dual of G in the sense of Young (see [2, p. 64] for the definition). Then, by (2.6) and Young's inequality [2, p. 64], we deduce for $|u_t(1,t)| \le 1$

$$
\begin{aligned}
\psi(V)u_t^2(1,t) &\le G^*(\psi(V)) + G(u_t^2(1,t)) \\
&\le G^*(\psi(V)) + h(|u_t(1,t)|)|u_t(1,t)| \\
&\le G^*(\psi(V)) + c_3u_t(1,t)g_1(u_t(1,t)), \qquad (3.38)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(V)g_1^2(u_t(1,t)) &= c_4^2\psi(V)c_4^{-2}g_1^2(u_t(1,t)) \\
&\le c_4^2G^*(\psi(V)) + c_4^2G(c_4^{-2}g_1^2(u_t(1,t))) \\
&\le c_4^2G^*(\psi(V)) + c_4^2h(|c_4^{-1}g_1(u_t(1,t))|)|c_4^{-1}g_1(u_t(1,t))| \\
&\le c_4^2G^*(\psi(V)) + c_4^2|u_t(1,t)||c_4^{-1}g_1(u_t(1,t))| \\
&= c_4^2G^*(\psi(V)) + c_4u_t(1,t)g_1(u_t(1,t)). \qquad (3.39)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\psi(V)\eta^2 &\le c_1\psi(V(0))\eta g_2(\eta) \qquad \text{for } |\eta| \ge 1, \qquad (3.40)\\
\psi(V)\eta^2 &\le G^*(\psi(V)) + c_3\eta g_2(\eta) \qquad \text{for } |\eta| \le 1. \qquad (3.41)
\end{aligned}
$$

Set

$$
\begin{aligned}
k_1 &= 3 + c_4^2, \qquad (3.42)\\
k_2 &= \max\{2c_3,\ 2c_1\psi(V(0))\}, \qquad (3.43)\\
k_3 &= \max\{c_3 + c_4,\ (c_1 + c_2)\psi(V(0))\}. \qquad (3.44)
\end{aligned}
$$

It therefore follows from (3.35)-(3.44) that

$$
\begin{aligned}
V_\epsilon' &\le -\epsilon V\psi(V) + k_1G^*(\psi(V)) + [\epsilon V(0)\psi'(V(0)) + \epsilon k_2 - 1]\eta g_2(\eta) \\
&\quad + [\epsilon V(0)\psi'(V(0)) + \epsilon k_3 - 1]u_t(1,t)g_1(u_t(1,t)) \\
&\le -\epsilon V\psi(V) + \epsilon k_1G^*(\psi(V)), \qquad (3.45)
\end{aligned}
$$

if

$$
\epsilon \le \min\left\{\frac{1}{V(0)\psi'(V(0)) + k_2},\ \frac{1}{V(0)\psi'(V(0)) + k_3}\right\}. \qquad (3.46)
$$

By the definition of the dual function in the sense of Young $G^*(s)$ of the convex function $G(s)$, $G^*(t)$ is the Legendre transform of $G(s)$, which is given by (see [2, p. 61-62])

$$G^*(t) = tG'^{-1}(t) - G[G'^{-1}(t)]. \tag{3.47}$$

Thus, we have

$$G^*(\psi(V)) = \psi(V)G'^{-1}(\psi(V)) - G[G'^{-1}(\psi(V))]. \tag{3.48}$$

This motivates us to make the choice

$$\psi(s) = G'(\delta s) \tag{3.49}$$

so that

$$G^*(\psi(V)) = \delta G'(\delta V)V - G(\delta V), \tag{3.50}$$

where the constant δ will be determined later. By condition (5), $\psi(s)$ satisfies the requirement we set at the beginning of the proof, that is, ψ and $\psi'(s)s$ are positive and increasing on $(0, +\infty)$. Therefore, we deduce from (3.45) and (3.50) that

$$\begin{aligned} V_\epsilon' &\le -\epsilon k_1 G(\delta V) + \epsilon(\delta k_1 - 1)VG'(\delta V) \\ &= -\epsilon k_1 G(\delta V) - \frac{\epsilon}{2} VG'(\delta V) \end{aligned} \tag{3.51}$$

with

$$\delta = \frac{1}{2k_1}. \tag{3.52}$$

Since $G(s)$ and $G'(s)$ are positive and increasing on $(0, \infty)$, it follows from (3.33) that

$$\begin{aligned} V_\epsilon' &\le -\frac{\epsilon V_\epsilon}{2[1+\epsilon G'(\delta V(0))]} G'\Big(\frac{\delta V_\epsilon}{1+\epsilon G'(\delta V(0))}\Big) - \epsilon k_1 G\Big(\frac{\delta V_\epsilon}{1+\epsilon G'(\delta V(0))}\Big) \\ &\le -\frac{\epsilon V_\epsilon}{3} G'\Big(\frac{2\delta V_\epsilon}{3}\Big) - \epsilon k_1 G\Big(\frac{2\delta V_\epsilon}{3}\Big), \end{aligned} \tag{3.53}$$

if

$$\epsilon = \min\Big\{\frac{1}{2G'(\delta V(0))}, \frac{1}{\delta V(0)G''(\delta V(0)) + k_2}, \frac{1}{\delta V(0)G''(\delta V(0)) + k_3}\Big\}. \tag{3.54}$$

Since the solution of differential inequality (3.53) is less than the solution of differential equation (2.8) (see, e.g., [19, p.31]), decay estimate (2.7) holds.

It remains to prove (2.9). We argue by contradiction. Suppose that $V(t)$ doesn't tend to zero as $t \to \infty$. Since $V(t)$ is decreasing on $[0, \infty)$, we have

$$V(0) \ge V(t) \ge \sigma, \quad \forall t \ge 0 \tag{3.55}$$

for some $\sigma > 0$ and then, by (3.33), we have

$$2V(0) \ge V_\epsilon(t) \ge \beta, \quad \forall t \ge 0 \tag{3.56}$$

for some $\beta > 0$. Thus there exists $\gamma > 0$ such that

$$G'\Big(\frac{2\delta V_\epsilon}{3}\Big) \ge \gamma, \quad \forall t \ge 0. \tag{3.57}$$

It therefore follows from (3.53) that

$$V'_\epsilon(t) \leq -\frac{\gamma\epsilon V_\epsilon}{3}, \ \ \forall t \geq 0, \tag{3.58}$$

which is in contradiction with (3.56). $\square$

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References

[1] R. Adams, *Sobolev Spaces.* Academic Press, New York (1975).

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York (1989).

[3] G. Avalos, I. Lasiecka and R. Rebarber, Lack of time-delay robustness for stabilization of a structural acoustics model. *SIAM J. Control Optim.* 37 (1999), no. 5, 1394–1418.

[4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, Leyden, The Netherlands (1976).

[5] G. Chen, M. C. Delfour, A. M. Krall and G. Payre, Modeling, stabilization and control of serially connected beams. *SIAM J. Control Optim.* 25 (1987), no. 3, 526–546.

[6] M.G. Crandall and T.M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces. *Amer. J. Math.* 93 (1971) 265-298.

[7] F. Conrad, Stabilization of beams by pointwise feedback control. *SIAM J. Control Optim.* 28 (1990), no. 2, 423–437.

[8] F. Conrad Stabilization of Euler-Bernoulli beams with nonlinear dissipation on the boundary. In *Progress in partial differential equations: the Metz surveys,* 69–87, Pitman Res. Notes Math. Ser., 249, Longman Sci. Tech., Harlow, 1991.

[9] F. Conrad, J. Leblond and J.-P. Marmorat, Energy decay estimates for a beam with nonlinear boundary feedback. In *Stabilization of flexible structures* (Montpellier, 1989), 46–61, Lecture Notes in Control and Inform. Sci., 147, Springer, Berlin, 1990.

[10] F. Conrad and O. Morgul, On the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.* 36 (1998) 1962-1986.

[11] J.-M. Coron and B. d'Andréa-Novel, Stabilization of a rotating body beam without damping. *IEEE Trans. Automat. Control* 43 (1998) 608-618.

[12] C. M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups. *J. Funct. Analysis* 13 (1973) 97-106.

[13] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM J. Control Optim.* 26 (1988), no. 3, 697–713.

[14] R. Datko, The destabilizing effect of delays on certain vibrating systems. *Advances in computing and control* (Baton Rouge, LA, 1988), 324–330, Lecture Notes in Control and Inform. Sci., 130, Springer, Berlin-New York, 1989.

[15] R. Datko, Two examples of ill-posedness with respect to small time delays in stabilized elastic systems. *IEEE Trans. Automat. Control* 38 (1993), no. 1, 163–166.

[16] R. Datko, J. Lagnese and M.P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* 24 (1986), no. 1, 152–156.

[17] R. Datko and Y.C. You, Some second-order vibrating systems cannot tolerate small time delays in their damping. *J. Optim. Theory Appl.* 70 (1991), no. 3, 521–537.

[18] M. Grobbelaar-Van Dalsen and A. Van der Merwe, Boundary stabilization for the extensible beam with attached load. *Math. Models Methods Appl. Sci.* 9 (1999), no. 3, 379–394.

[19] J.K. Hale, *Ordinary Differential Equations.* Wiley-interscience, New York (1969).

[20] A. Haraux, *Semi-Linear Hyperbolic Problems in Bounded Domains.* Harwood Academic Publishers, Chur (1987).

[21] B. B. King and R. E. Fennell, Modeling and boundary stabilization of a multiple beam system. *Numer. Funct. Anal. Optim.* 16 (1995), no. 7-8, 939–963.

[22] V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method.* John Wiley & Sons, Masson, Paris (1994).

[23] M. Krstić, I. Kanellakopoulos and P. Kokotović, *Nonlinear and Adaptive Control Design.* John Wiley & Sons, Inc., New York (1995).

[24] J. Lagnese and G. Leugering, Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. *J. Differential Equations* 91 (1991), no. 2, 355–388.

[25] I. Lasiecka, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary. *J. Differential Equations* 79 (1989) 340-381.

[26] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential & Integral Equations* 6 (1993) 507–533.

[27] H. Laousy, C. Z. Xu and G. Sallet, Boundary feedback stabilization of a rotating body-beam system. *IEEE Trans. Automat. Control* 41 (1996), no. 2, 241–245.

[28] X.J. Li and K.S. Liu, The effect of small time delays in the feedbacks on boundary stabilization. *Sci. China Ser.* A 36 (1993), no. 12, 1435–1443.

[29] J.L. Lions and E. Magenes, *Non-homogeneous Boundary value Problems and Applications, Vol.I.* Springer-Verlag, Berlin, Heidelberg, New York (1972).

[30] W.J. Liu and K.K. Tan, Mean ergodic theorems for C_0 semigroups of continuous linear operators. Submitted.

[31] W.J. Liu and E. Zuazua, Uniform stabilization of the higher dimensional system of thermoelasticity with a nonlinear boundary feedback. *Quarterly Appl. Math.* (to appear).

[32] W.J. Liu and E. Zuazua, Decay rates for dissipative wave equations. *Ricerche Mat.* 48 (1999), 61–75.

[33] Z.H. Luo, Exponential stabilization of an Euler-Bernoulli beam with a tip body using strain feedback. *Trans. Inst. Systems Control Inform. Engrs.* 8 (1995), no. 5, 189–195.

[34] Z.H. Luo, Feedback stabilization of Euler-Bernoulli beam equations using higher-order derivative information. *Systems Control Inform.* 41 (1997), no. 9, 343–350.

[35] Z.H. Luo, B.Z. Guo and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications.* Springer, London (1999).

[36] P. Martinez, A new method to obtain decay rate estimates for dissipative systems with localized damping. *Revista Matemática Complutense* (to appear).

[37] P. Martinez, A new method to obtain decay rate estimates for dissipative systems. *ESAIM:COCV* (to appear).

[38] O. Morgul, Control and stabilization of a flexible beam attached to a rigid body. Internat. J. Control 51 (1990), no. 1, 11–31.

[39] O. Morgul, Orientation and stabilization of a flexible beam attached to a rigid body: planar motion. *IEEE Trans. Automat. Control* 36 (1991), no. 8, 897, 953–962.

[40] O. Morgul, Dynamic boundary control of an Euler-Bernoulli beam. IEEE Trans. Automat. Control 37 (1992), no. 5, 639–642.

[41] H. Oka, Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings. *Nonlinear Anal. TMA* 18 (1992) 619-635.

[42] B. Rao, A compact perturbation method for the boundary stabilization of the Rayleigh beam equation. *Appl. Math. Optim.* 33 (1996), no. 3, 253–264.

[43] K.-K. Tan and H.K. Xu, An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces. *Nonlinear Anal. TMA* 19 (1992) 805-813.

[44] J. Vascontenoble, Optimalité d'estimations d'énergie pour une équation des ondes. *C. R. Acad. Sci. Paris* (to appear).

[45] J.A. Walker, *Dynamical Systems and Evolution Equations, Theory and Applications.* Plemun Press, New York (1980).

[46] C. Z. Xu and J. Baillieul, Stabilizability and stabilization of a rotating body-beam system with torque control. *IEEE Trans. Automat. Control* 38 (1993), no. 12, 1754–1765.